Mathemaical Analysis of Hexagonal Tiling in Medieval Islamic Art

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Abstract

Many strapworks seen in medieval Islamic arts can be made by tessellating several kinds of girih tiles. The present paper studies tilings with the hexagonal girih tile. First we show that there exists a unique tiling with five-fold rotational symmetry. Next tilings that contain tree-like sub-patterns are introduced. Remarkably then can contain at most two tree-like sub-patterns. Finally all periodic tilings are determined. We show further that a similar way of making such periodic tilings can generate uncountably many aperiodic tilings.

Keyword: tiling, girih tile, periodicity, rotational symmetry

1 Introduction

It has long been thought that strapworks seen in medieval Islamic Architecture were designed as a network of zigzag lignes and drawed directly with a ruler and a compass. However P.J.Lu and P.J.Steinhardt show that by 13th century AD a breakthrough occurred in which such strapworks were re-designed as by-products of tilings with some kinds of polygons.

These polygons are called "girih tiles" ("girih" comes from a Persian word meaning "knot"). They are equilateral polygons with all internal angles being multiples of 36°. Although every tile is decorated with simple lines, a tessellation of these tiles generates a surprisingly complex zigzag lines.

The above authors, examing a lot of medieval Islamic arts, have found five kinds of girih tiles. They claim that any periodic patterns in Islamic arts can be generated by tiling these girih tiles . Furthermore they claim that a certain tiling can generate even a quasi-crystaline Penrose pattern.

The present paper asks if the above claims are correct or not. Although our final object concerns tilings with five kinds of girih tiles, in the present study we confine ourselves within tiling with a particular kind of girih tile, i.e. a hexagonal tile.

In the section 2, after a hexagonal tile isdefined exactly, a way of specifining position and orientation of a tile is proposed. In the section 3 we enumerate ways to place hexagonal tilesrounda point. In the

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section 4 a pattern of five-fold rotational symmetry will be made. Such a pattern is unique. In the section 5 we show patterns that have no symmetry, in particular, aperiodic. On the other hand, in the section 6, we show many periodic patterns as well as aperiodic patterns. As is easily seen, uncountably many aperiodic strapworks exist.

2 Lattice generated by an angle 36°

Our problem is to place Girih tiles so that they cover the whole plane without any gap and any overlap. Every Girih tile has the following two properties:

(1) any interior angles are multiples of $\beta := 36^\circ$. And, since $5\beta = 180^\circ$, any exterior angles are also multiples of β.

(2) every tile is equilateral, i.e. lengthes of any edges are equal each other. In the below we always asuume that the common length equals 1.

From the properties of Girih tiles, it follows that any vertices of tiles can be represented as

$$\sum_{k=0}^{4} a_k \mathbf{b}_k,$$

where $a_k \in \mathbb{Z}$, and

$$\mathbf{b}_k = \begin{pmatrix} \cos k\beta \\ \sin k\beta \end{pmatrix}.$$

In other words every vertex can be representes as a linear combination of bases $\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4$ with integral coefficients. Viewing such forms of representation, we call them 'lattice points'. Thus lattice points can be specified by 'coordinates' $(a_0, a_1, a_2, a_3, a_4)$.

bf Remark. A note to lattice points. The set of all lattice points is not $\mathbb{Z}^5$. In fact five vectors $\mathbf{b}_k$ are linearly dependent over $\mathbb{Q}$.

To prove the fact, we suppose that $\sum_{k=0}^{4} a_k \mathbf{b}_k = \mathbf{0}$. First we compute x-coordinates:

$$\begin{aligned} 0 &= a_0 + a_1 \cos\beta + a_2 \cos 2\beta + a_3 \cos 3\beta + a_4 \cos 4\beta \\ &= a_0 + a_1 \cdot \frac{\sqrt{5}+1}{4} + a_2 \cdot \frac{\sqrt{5}-1}{4} + a_3 \cdot \left(-\frac{\sqrt{5}-1}{4}\right) + a_4 \cdot \left(-\frac{\sqrt{5}+1}{4}\right) \\ &= \frac{1}{4}\left[(4a_0 + a_1 - a_2 - a_3 - a_4) + (a_1 + a_2 - a_3 - a_4)\sqrt{5}\right] \end{aligned}$$

Accordingly we see that

$$4a_0 + a_1 - a_2 + a_3 - a_4 = 0 \quad \text{and} \quad a_1 + a_2 - a_3 - a_4 = 0.$$

Next we compute y-coordinates:

$$\begin{aligned} 0 &= a_0 \sin 0 \cdot \beta + a_1 \sin\beta + a_2 \sin 2\beta + a_3 \sin 3\beta + a_4 \sin 4\beta \\ &= (a_1 + a_4)\sin\beta + (a_2 + a_3)\sin 2\beta = \sin\beta\,[(a_1 + a_4) + (a_2 + a_3)\cdot 2\cos\beta] \\ &= \sin\beta\left[(a_1 + a_4) + (a_2 + a_3)\cdot\frac{\sqrt{5}+1}{2}\right] = \frac{\sin\beta}{2}\left[2(a_1 + a_4) + (a_2 + a_3) + (a_2 + a_3)\sqrt{5}\right] \end{aligned}$$

Hence it follows that

$$a_1 + a_4 = 0 \quad \text{and} \quad a_2 + a_3 = 0.$$

From these four relations we an easily deduce $a_1 = a_3 = -a_0,\ a_2 = a_4 = a_0$. Therefore the relation $\mathbf{b}_0 - \mathbf{b}_1 + \mathbf{b}_2 - \mathbf{b}_3 + \mathbf{b}_4 = \mathbf{0}$ holds. And no other relations hold. Since $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4$ are linearly dependent over $\mathbb{Q}$, there exists a one-to-one correspondence between the set of all lattice points and $\mathbb{Z}^4$.

3 Hexagonal Girih tile

One kind of five Girih tiles is a hexagon $\mathrm{P}_1\mathrm{P}_2\mathrm{P}_3\mathrm{P}'_1\mathrm{P}'_2\mathrm{P}'_3$ with the following properties

(1) $\mathrm{P}_1\mathrm{P}_2 = \mathrm{P}_2\mathrm{P}_3 = \mathrm{P}_3\mathrm{P}'_1 = \mathrm{P}'_1\mathrm{P}'_2 = \mathrm{P}'_2\mathrm{P}'_3 = \mathrm{P}'_3\mathrm{P}_1 = 1$

(2) $\angle\mathrm{P}_1 = \angle\mathrm{P}'_1 = 2\beta, \angle\mathrm{P}_2 = \angle\mathrm{P}'_2 = 4\beta, \angle\mathrm{P}_3 = \angle\mathrm{P}'_3 = 4\beta$.

Let us make a tiling by such hexagonal tiles in general. Then we can assume that their vertices lie at lattice points (defined in the previous section). If we consider a tile $\mathrm{P}_1\mathrm{P}_2\mathrm{P}_3\mathrm{P}'_1\mathrm{P}'_2\mathrm{P}'_3$, its position in the entire plane can be specified by the position of vertex P_1 and the direction of the vector $\overrightarrow{\mathrm{P}_1\mathrm{P}'_1}$, which is the same as that of $\overrightarrow{\mathrm{P}_2\mathrm{P}_3}$. Thus the tile can be known by the coordinates $(a_0, a_1, a_2, a_3, a_4)$ of P_0 and the orirntation of the index k such that $\mathbf{b}_k = \overrightarrow{\mathrm{P}_2\mathrm{P}_3}$. Note that k should be taken as $\mod 10$. We will denote the tile simply by $(a_0, a_1, a_2, a_3, a_4 : k)$.

Now we place such hexagonal tiles around a point O. Then, for each tile, there arise three possibilities: (i) P_1 or P'_1 coinsides with O; (ii P_2 or P'_2 coinsides with O; (iii) P_3 or P'_3 coinsides with O. Let denote the tile by H_1 or H_2 or H_3 respectively according as the first or the second or the third possibility occurs.

Suppose that the number k of tiles are placed around a point O. Then the sum of k interior angles, one angle coming from one tile, must be equal to $10\beta = 360°$. Accordingly three patterns of sums $4+4+2, 4+2+2+2, 2+2+2+2+2$ are permissible. Further each pattern of sum can be realized in several differentare ways

- $4+4+2$: $H_1H_2H_2, H_1H_2H_3, H_1H_3H_2$, and $H_1H_3H_3$.
- $4+2+2+2$: $H_1H_1H_1H_2, H_1H_1H_1H_3$.
- $2+2+2+2+2$: $H_1H_1H_1H_1H_1$.

It is easy to see that patterns $H_1H_2H_2$ and $H_1H_3H_3$ are symmetric to each other, and also patterns $H_1H_1H_1H_2$ and $H_1H_1H_1H_3$ symmetric to each other. Thus, neglecting $H_1H_3H_3$ and $H_1H_1H_1H_3$, we call patterns $H_1H_2H_2, H_1H_3H_2, H_1H_2H_3, H_1H_1H_1H_2, H_1H_1H_1H_1H_1$ by $P_\alpha, P_\beta, P_\gamma, P_\delta, P_\epsilon$ respectively.

4 Tiling generated by P_ϵ

Let us place five tiles around the lattice point O. These tiles can be represented by

$$(0,0,0,0,0:1), (0,0,0,0,0:3), (0,0,0,0,0:5), (0,0,0,0,0:7), (0,0,0,0,0:9).$$

The following lemma can be seen easily by placing tiles 'by hand'.

Lemma 1. *A pair of tiles* $(0,0,0,0,0:1),(0,0,0,0,0:-1)$ *necessarily generates a tree-like pattern of tiles where*

(i) the upper boudary consists of tiles $(n,n,0,0,0:1), n=1,2,\ldots$

(ii) the lower boudary consists of tiles $(n,0,0,0,-n:-1), n=1,2,\ldots$

(iii) the surrounded area between two boudarys consists of tiles $(m+n+1,m,0,0,-n:0), m,n=1,2,\ldots$

Fig 1 shows the tree stated in Lemma 1.

From Lemma 1 the following Proposition immediately follows.

Proposition 1. *Make a tiling from a pattern* (P_ϵ)*. Then the tiling is unique. It has a five-fold rotational symmetry but it is not periodic.*

Fig 2 below shows a part (in the entire plane) of the tiling. Here boundaries of tiles are colored by magenta. Blue lines shows the strapwork derived from the tiling.

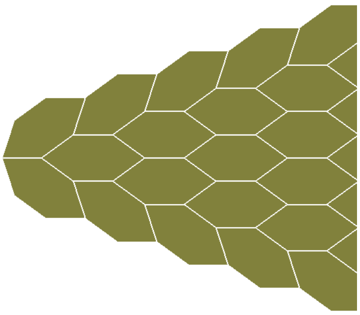

Fig 1

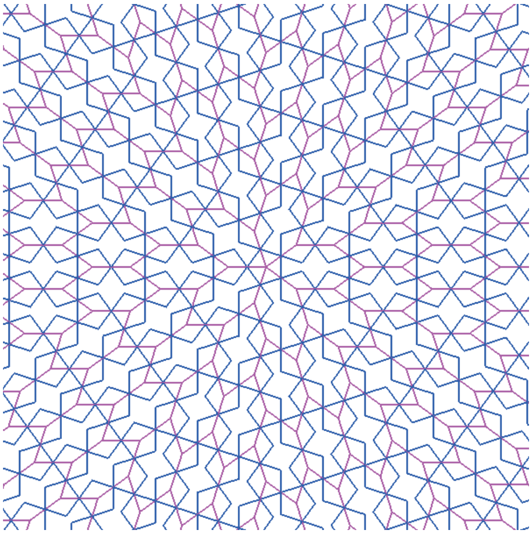

Fig 2

5 Tiling generated by P_δ

Let us place four tiles around the lattice point O. These tiles can be represented by

$$(0,0,0,0,0:0),(0,0,0,0,0:2),(0,0,0,0,0:-2),(0,-1,-1,0,0:-3).$$

Here the three tiles from the first are of H_1.

Lemma 2. *A triplet of tiles* $(0,0,0,0,0:0),(0,0,0,0,0:2),(0,0,0,0,0:-2)$ *necessarily generates a pattern of union of several trees and two lalf-lines.*

We call the pattern in Lemma 2 a fat tree. A fat tree is shown in Fig 3 where brown tiles are a uion of several trees, and orange tiles are unions of half-lines.

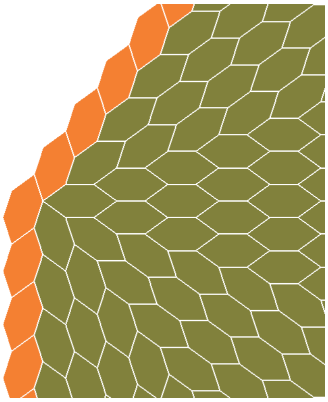

Fig 3

Proposition 2. *If a tiling has the number k of fat trees, it is necessary that* $k \leqq 2$.

Fig 4 shows a tiling that consists of 2 fat trees and 2 trees. Fig 5 shows a strapwork that derives from the tiling in Fig 4.

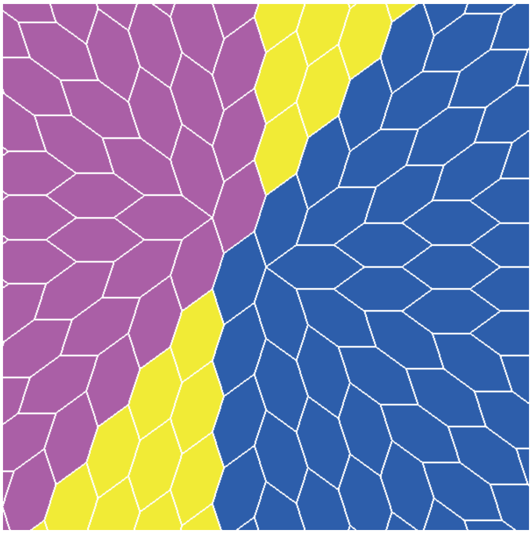

Fig 4

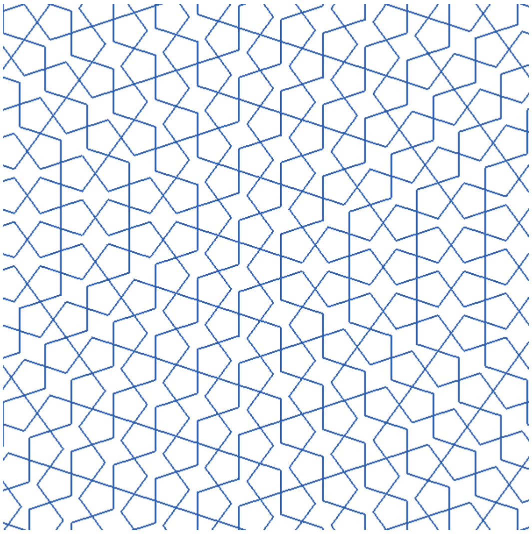

Fig 5

6 Tiling that contain neither P_δ nor P_ϵ

It is easy to prove the following result.

Lemma 3. *If a tiling contains* P_γ, then it necessarily contains P_δ *or* P_ϵ.

Thus it remains to study tilings generated by P_α or those by P_β. We only investigate the former because investigation on the latter can go similarly.

Place three tiles $H_1H_2H_2$ are placed around O. It can be realized by setting $H1 = (-1,-1,0,1,0:0)$, the former $H_2 = (0,-1,-1,0,0:1)$, and the latter $H_2 = (0,0,0,0,1:0)$. To simplify statement we change name of the latter H_2 to H_2' while naming of the former H_2 is unchaged.

It is easy to see that two infinite sequences of tiles automatically will arise: one sequence is $H_1, T^{-1}H_1, T^{-2}H_1, \cdots$ and the other is $H_2, T^{-1}H_2, T^{-2}H_2, \cdots$, where T denotes the translation operator defined by a vector $\mathbf{b}_0 + \mathbf{b}_1$ and T^{-1} denotes its inverse. Note that these infinite sequences continue leftward from the initial H_1 and H_2. We ask if they can continue rightward too. The answer is given by the folowing lemma.

Lemma 4. *A tiling generated* by P_α *is the union of infinite sequences* $\{T^nH_1 : n \in \mathbb{Z}\}$ *and* $\{T^nH_2 : n \in \mathbb{Z}\}$.

We call these infinite sequences by bands, and denote them by B_1, B_2 respectively.

Proposition 3. *If we stack two bands periodically, a periodic tiling occurs. On the contary, by stackin two bands aperiodically, we get an aperiodic tiling.*

Fig 6 shows an example of aperiodic tilings. Fig 7 is a strapwork that derives from the tiling in Fig 6.

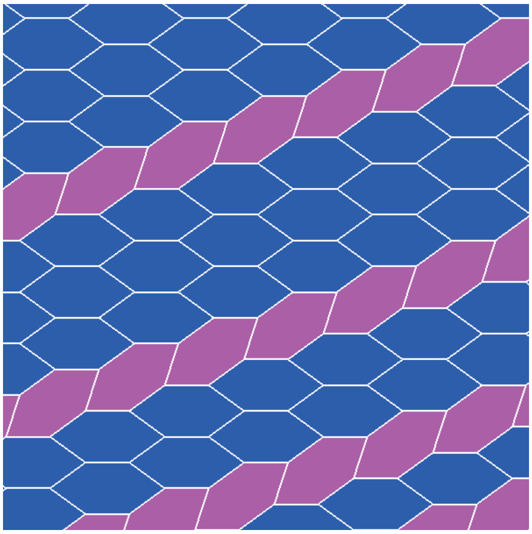

Fig 6

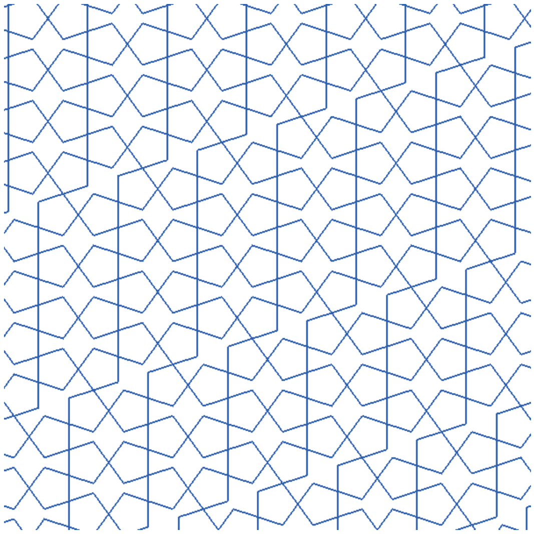

Fig 7

Corollary 1. *The number of periodic tilings is countably infinite. On the contary, The number of aperiodic tilings uncountably infinite.*

References

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