Original Article

An Analogue of an Old Quadrature Problem by Archimedes

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Abstract

Consider a solid made by intersecting two congruent cones while assuming that axes of these cones intersect perpendicularly. The paper studies quadrature of both surface area S and volume V of the solid, and gives explicit exressions for S and V. In a limitting case where cones tend to cylinders, these explicit expressions reduce to the oldest result by Archimedes. In the final section we investigate relationship between S and V, and find a relation which states that volume V of our solid is equal to that of a cone with base being equal to S and height (after being appropriately defined). This relation is similar to the greatest discovery by Archimedes for spheres.

Keyword : Cone, Suface area, Volume, Archimedes

1 Introduction

One of the oldest quadrature problems is the following proposed by Archimedes ([1], [2]).

ἐὰν εἰς κύβον κύλινδρος ἐγγράφῇ τὰς μὲν βάσεις ἔχων πρός τοῖς κατεναντίον παραλληλογάμμοις
τὴν δὲ ἐπιφάνειαν τῶν λοιπῶν τεσσάρων ἐπιπέδων ἐφαπτόμενος,
ἐγγράφῇ δὲ καὶ ἄλλος κύλινδρος εἰς τὸν αὐτὸν κύβον τὰς μὲν βασέις ἐν ἄλλοις παραλληλογράμμοις
τὴν δὲ ἐπιφάνειαν τῶν λοιπῶν τεσσάρων ἐπιπέδων ἐφαπτόμενος,
τὸ περιληφθὲν σχῆμα ὑπὸ τῶν ἐπιφανειῶν τῶν κύλινδρων, ὅ ἐστιν ἀμφοτέροι τοῖς κυλίνδροις,
δ΄μοιρον ἐστι τοῦ ὅλου κύβου.

If in a cube a cylinder be inscribed which has its bases in the opposite parallelograms and touches with its surface the remaining four planes, and if there also be inscribed in the same cube another cylinder which has its bases in other parallelograms and touches with its surface the remaining four planes, then the figure bounded by the surfaces of the cylinders, which is within both cylinders, is two-thirds of the whole cube

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This famous problem is treated in many modern textbooks of calculus, and in fact it has be well-known in 'Wasan' (mathematics in Japan's Edo era （[3])。Needless to say, mathematcians in Edo era have invented and solved the problem, independently of Archimedes.

To make the problem easier to understand, we will rephrase it with modern notations.

> Consider two cylinders; one with base being a circle of radius r and with axis lying on the x axis, and another with base being also a circle of radius r and with axis lying on the y axis. To find volume of the intersection of these cylinders, where we assume that heights of these cylinder are infinitely long.

The solution is $16/3r^3$. It is mysterious that the solution does not contain π although the problem comes from cylinder (a solid related to a circle). Archimedes seemed to have the same impression.

By the way, there are many quadrature problems concerning volumes of solids. On the other hand, it seems that there are fewer quadrature problems concerning surfaces of solids, because it is generally more difficult to quadrature surfaces. However, in case of the intersection of two cylinders solid, quadrature of the surface is easy, and results in $16r^2$.

In this paper, instead of the intersection of two congruent cylinders, we consider the intesection of two congruent cones. We study qudrature of its surface and its volume. To state the problem with notation, the problem is as follows:

> Cnsider two congruent cones: one is with its axis lying on the x axis and with infinite height, and another with its axis lying on the y axis and also with infinite height. It is assumed that the apex of the former has x-coordinates a, and the apex of the latter has y-coordinates a. Now let β denote half apex angle (angle between any generator and the axis) of both cones. We assume that $\beta < 45°$. Let $\mathbb{V}$ be the intersection of two cones. Our problem is to quadrature the surface area and the volume of $\mathbb{V}$.

In the section 2, we first investigate shape of the solid $\mathbb{V}$, and then we find its surface area by developping surface on a plane.

In the section 3, we first study shape of the section made by cutting $\mathbb{V}$ by a plane which passes through two apices of cones. It will be seen that shape of the section is like a kite. Finnaly integration of areas of the kites will provide the volume of $\mathbb{V}$.

In the section 4 our thought will be on relationship between the surface area and the volume of $\mathbb{V}$. By the way, one of the most famous discoveries (and proofs) by Archimedes is the relationship between the surface area and the volume of the sphere. Our finding is an analogue of the relationship discoverd by Archimedes.

2 Surface area

2.1 Shape of $\mathbb{V}$

Figure 1 shows two cones when $\beta = 15°$, and Figure 2 shows their intersection $\mathbb{V}$. Yellow bars denote the positive parts of the x axis and of y axis.

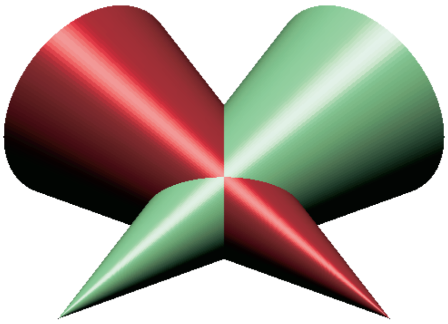

Figure 1

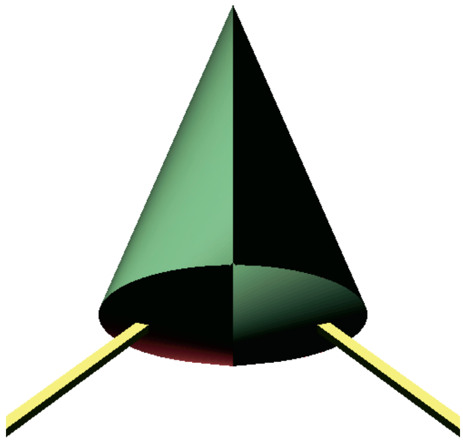

Figure 2

Surface of $\mathbb{V}$ consists of four parts, which we will call by "leaves". On the surface of the Earth, These four leaves are arranged as though the following four parts are arranged on the surface of the Earth; a part with the the west longitude $180 \sim 90$, a part with the the west longitude $90 \sim 0$, a part with the the east longitude $0 \sim 90$, and a part with the the east longitude $90 \sim 180$.

Adjacent leaves (green ones and red ones in Figure2) lie on surfaces of different cones, and are divided by intersection curves that are made from surfaces of two cones.

What is the equation of these intersection curves ? Surfaces of two cones respectively have equations

$$y^2 + z^2 = \tau^2(a-x)^2,\ x^2 + z^2 = \tau^2(a-y)^2,$$

where $\tau = \tan\beta$. Subtracting one from the other, we have

$$y^2 - x^2 = \tau^2\{(a-x)^2 - (a-y)^2\},$$

and then we see

$$x = y \quad \text{or} \quad x + y = \frac{2a\tau^2}{1+\tau^2}. \tag{2.1}$$

These represent equations of planes. We call the former π_1 and the latter π_2. Thus the intersection curves lie on planes π_1 or π_2.

Four leaves intersect simultaneously at two points N and $\S$, as though on the Earth four parts intersect simultaneously at the North and the South pole. An easy computation shows that

$$\mathrm{N} = \left(\frac{a\tau^2}{1+\tau^2}, \frac{a\tau^2}{1+\tau^2}, \frac{a\tau\sqrt{1-\tau^2}}{1+\tau^2}\right),\ \S = \left(\frac{a\tau^2}{1+\tau^2}, \frac{a\tau^2}{1+\tau^2}, -\frac{a\tau\sqrt{1-\tau^2}}{1+\tau^2}\right).$$

Note that these two points do not lie on the z axis.

2.2 develop the surface of $\mathbb{V}$

Denote by K the cone with axis lying on the x axis. Let us consider a part of the surface of $\mathbb{V}$ that is contained in the surface of K. Cutting this cone by plane $x = 0$ we have a circle C. Any point P on

C can be expressed as $\mathrm{P} = (0, a\tau\cos\varphi, a\tau\sin\varphi)$. Since the apex of the cone is $\mathrm{A} = (a, 0, 0)$, equation of the generator AP is given by

$$x = a - t\cos\beta,\ y = t\sin\beta\cos\varphi,\ z = t\sin\beta\sin\varphi \quad (-\infty < t < \infty).$$

Suppose that the generator AP intersects the plane π_1 when $t = f_1(\varphi)$, and the plane π_2 when $t = f_2(\varphi)$. An easy computstion derives

$$f_1(\varphi) = \frac{a}{\cos\beta + \sin\beta\cos\varphi}, \quad f_2(\varphi) = \frac{a(\cos^2\beta - \sin^2\beta)}{\cos\beta - \sin\beta\cos\varphi}.$$

Furthermore it can be easily seen that $f_1(\varphi) \leqq f_2(\varphi) \Leftrightarrow \cos\varphi \geqq \tau$. That is, if we define an angle φ_0 by $\cos\varphi_0 = \tau, 0 < \varphi_0 < \pi/2$, we have $f_1(\varphi) \leqq f_2(\varphi) \Leftrightarrow |\varphi| \leqq \varphi_0$.

Let g be the generator which passes through a point $(y, z) = (-a\tau, 0)$ on the circle C. Now, cutting the surface of the cone K by g, we develop the surface. Then C will be developped into an arc of a fan with $a/\cos\beta$. (The center of the fan coincides with the apex A of K.) Since the length of the circumference C is equal to the length of the arc of the fan, and it equals $2\pi a\tau$, we see that the angle at the center of the fan equals $2\pi\sin\beta$.

Suppose thst a developped figure lies on $\xi\eta$ plane, and the apex A is developped onto the origin $(\xi, \eta) = (0, 0)$, and the generator g onto the ξ axis. Then we consider a generator that passes through a point $(y, z) = (a\tau\cos\varphi, a\tau\sin\varphi)$ on C. If the generator is developped onto a straight line $\eta = \xi\tan\theta$, we see that

$$\theta = \sin\beta\varphi.$$

Now we use the polar coordinates (ρ, θ) on the $\xi\eta$ plane. By observation steted in the above, developped figure of a leaf coincides with a region bounded by two curves

$$\rho = \frac{a}{\cos\beta + \sin\beta\cos(\sin\beta\theta)},\ \rho = \frac{a(\cos^2\beta - \sin^2\beta)}{\cos\beta - \sin\beta\cos(\sin\beta\theta)}.$$

Figure 3 shows the region. Blue part denotes a leaf that lies on the cone K, and green part denotes another leaf that lies on the same cone. Green part is separated into two parts on the developped figure, but in fact two parts are connected on the surface of K.

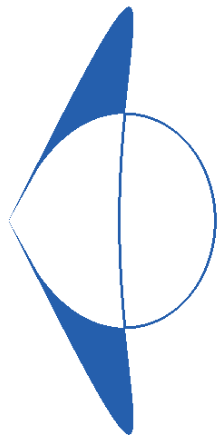

Figure 3

2.3 Surface area

1. The surface of the solid $\mathbb{V}$ consists of four leaves. Two of them (which we denote by L_1, L_2) lie on the surface of the cone K, and the other two (which we denote by L_3, L_4) lie on the surface of the other cone. By change of notations L_3, L_4 if necessary, we may assume that L_3, L_4 are congruent to L_1, L_2 respectively. Accordingly the surface area of $\mathbb{V}$ (which we denote by S) is equal to two times multiple of the surface area of $L_1 \cup L_2$. Thus our problem is to find areas of both blue region and green region in Figure 3, and double the sum of them. That is, if S_{blue} and S_{green} stand for area of blue region and green region respectively, we have $S = 2S_{blue} + 2S_{green}$.

2. S_{blue}, S_{green} can be expressed as the following integrals:

$$S_{blue} = \int_{|\varphi| \leqq \varphi_0} \left(\frac{1}{2} f_2(\varphi)^2 \, d\theta - \frac{1}{2} f_1(\varphi)^2 \, d\theta \right) = \sin\beta \int_0^{\varphi_0} (f_2(\varphi)^2 - f_1(\varphi)^2) d\varphi$$
$$S_{green} = \int_{|\varphi| \leqq \varphi_0} \left(\frac{1}{2} f_1(\varphi)^2 \, d\theta - \frac{1}{2} f_2(\varphi)^2 \, d\theta \right) = \sin\beta \int_{\varphi_0}^{\pi} (f_1(\varphi)^2 - f_2(\varphi)^2) d\varphi$$

Now we introduce a primitive function

$$F_1(\varphi) := \int_0^{\varphi} f_1(\varphi)^2 d\varphi, \quad F_2(\varphi) := \int_0^{\varphi} f_2(\varphi)^2 d\varphi,$$

we can write

$$S_{blue} = \sin\beta(F_2(\varphi_0) - F_1(\varphi_0))$$
$$S_{green} = \sin\beta(F_1(\pi) - F_2(\pi)) + \sin\beta(F_2(\varphi_0) - F_1(\varphi_0)).$$

3. It is necessary to find an explicit expression for the primitive function. This task is elementary but needs bulky computation. Thus we use a computer algebra system (maxima). A result is as follows:

$$F_1(\varphi)/k_1 = \frac{2}{(1-\tau^2)^{3/2}} \arctan\left(\sqrt{\frac{1-\tau}{1+\tau}} h(\varphi) \right) - \frac{2\tau}{1-\tau^2} \cdot \frac{h(\varphi)}{(1-\tau)h(\varphi)^2 + (1+\tau)},$$
$$F_2(\varphi)/k_2 = \frac{2}{(1-\tau^2)^{3/2}} \arctan\left(\sqrt{\frac{1+\tau}{1-\tau}} h(\varphi) \right) + \frac{2\tau}{1-\tau^2} \cdot \frac{h(\varphi)}{(1+\tau)h(\varphi)^2 + (1-\tau)},$$

where we define

$$h(\varphi) = \frac{\sin\varphi}{1+\cos\varphi}; \quad k_1 = \left(\frac{a}{\cos\beta} \right)^2, \; k_2 = \left(\frac{a(\cos^2\beta - \sin^2\beta)}{\cos\beta} \right)^2.$$

Since

$$h(\pi) = \lim_{\varphi \uparrow \pi} \frac{\sin\varphi}{1+\cos\varphi} = +\infty,$$

we see

$$F_1(\pi)/k_1 = \frac{2}{(1-\tau^2)^{3/2}} \arctan(+\infty) = \frac{2}{(1-\tau^2)^{3/2}} \cdot \frac{\pi}{2} = \frac{\pi}{(1-\tau^2)^{3/2}}$$

and

$$F_2(\pi)/k_2 = \frac{\pi}{(1-\tau^2)^{3/2}}.$$

Moreover, as

$$h(\varphi_0) = \sqrt{\frac{1-\tau}{1+\tau}},$$

we have

$$\arctan\left(\sqrt{\frac{1-\tau}{1+\tau}}\,h(\varphi_0)\right) = \arctan\left(\frac{1-\tau}{1+\tau}\right).$$

Then the addition formula of tan shows that

$$\tan\left(\frac{\pi}{4}-\beta\right) = \frac{\tan\frac{\pi}{4} - \tan\beta}{1+\frac{\pi}{4}\tan\beta} = \frac{1-\tau}{1+\tau},$$

from which follows

$$\arctan\left(\frac{1-\tau}{1+\tau}\right) = \frac{\pi}{4} - \beta.$$

Consequently we see

$$\arctan\left(\sqrt{\frac{1-\tau}{1+\tau}}\,h(\varphi_0)\right) = \frac{\pi}{4} - \beta.$$

Further it can be immediately seen that

$$\arctan\left(\sqrt{\frac{1+\tau}{1-\tau}}\,h(\varphi_0)\right) = \frac{\pi}{4}.$$

4. These results for primitive function imply

$$\begin{aligned} S_{blue} &= \sin\beta \cdot \frac{2a^2}{(1-\tau^2)^{3/2}(1+\tau^2)}\left\{-\pi\tau^2 + \beta(1+\tau^2)^2 + \tau(1-\tau^2)\right\} \\ S_{green} &= \sin\beta \cdot \frac{4a^2\pi\tau^2}{(1-\tau^2)^{3/2}(1+\tau^2)} + \sin\beta \cdot \frac{2a^2}{(1-\tau^2)^{3/2}(1+\tau^2)}\left\{-\pi\tau^2 + \beta(1+\tau^2)^2 + \tau(1-\tau^2)\right\} \\ &= \sin\beta \cdot \frac{2a^2}{(1-\tau^2)^{3/2}(1+\tau^2)}\left\{\pi\tau^2 + \beta(1+\tau^2)^2 + \tau(1-\tau^2)\right\}. \end{aligned}$$

Therefore, substituting $\tau = \tan\beta$, we obtain the follwing proposition.

Proposition 1

Area of blue leaf:

$$S_{blue} = \frac{a^2}{2} \cdot \frac{\sin 2\beta}{(\cos 2\beta)^{3/2}}\left(-\frac{\pi}{2}(\sin 2\beta)^2 + 2\beta + \sin 2\beta\cos 2\beta\right)$$

Are of green leaf:

$$S_{green} = \frac{a^2}{2} \cdot \frac{\sin 2\beta}{(\cos 2\beta)^{3/2}}\left(\frac{\pi}{2}(\sin 2\beta)^2 + 2\beta + \sin 2\beta\cos 2\beta\right)$$

Surface area of $\mathbb{V}$:

$$S = 2a^2 \cdot \frac{\sin 2\beta}{(\cos 2\beta)^{3/2}}\left(2\beta + \sin 2\beta\cos 2\beta\right)$$

Corollary 1

Let a tend to ∞ and β tend to 0 with keeping $a\tan\beta = r =$ a constant. Then S tendsto $16a^2\beta^2 = 16r^2$, which is the surface area of the intersection of two cylinders (base radius r). Moreover, the ratio S_{blue}/S_{green} goes to 1.

Proof of the corollary. First, by using $a\tan\beta = r$, we eliminate a in the expression for S of Proposition 1. To simplify expressions, we write $x = 2\beta, g(x) = S/(2r^2)$. Then we have

$$g(x) = \frac{(1+\cos x)(x+\sin x\cos x)}{(\cos x)^{3/2}}.$$

Further we can confirm that $g(x)$ is strictly increasing and convex in an interval $0 < x < \pi/2$, and $g(+0) = 0, g(\pi/2 - 0) = +\infty$.

Corollary 2

Let increase β upto $\pi/4$ while keeping a being a constant. That is, let the apex of a cone go near to the surface of the other cone. Then S tends to ∞ (in detail $S_{green} \to +\infty$ and $S_{blue} \to 0$).

It is an interesting problem to find shape of $\mathbb{V}$ when $\beta = \pi/4$. Then blue leaves on both cones vanishes (empty sets). On the other hand, a green leaf on the surface of the cone $K : y^2 + z^2 = (a-x)^2$ coinsides with a part of the surface which is made by cutting by plane $x = y$ and contain the apex. That is, the surface of the cone is cut by a plane which is perpendicular to a generator; the surface of the cone divides into two parts; boundary curve of each of two parts is a parabola; a green leaf is a part that contains the apex.

3 Volume

3.1 Cut $\mathbb{V}$ by rotating planes

Let $\mathrm{A} = (a,0,0), \mathrm{B} = (0,a,0)$ be the apices of two cones respectively. Let us rotate the xy plane about a straight line AB and consider a resulting plane. When the plane makes an angle θ against the xy plane, we call it π_θ.

Consider a $\xi\eta$ coordinates system on the plane π_θ. Suppose that the origin of the coordinates system locates at the midpoint of AB, the ξ axis is on the straight line AB, and the η axis is perpendicular to AB. In the $\xi\eta$ coordinates system, A has coordinates $(\xi,\eta) = (-a/\sqrt{2}, 0)$, and B has coordinates $(\xi,\eta) = (a/\sqrt{2}, 0)$.

If a point (ξ,η) on the plane π_θ has coordinates (x,y,z), we have

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a/2 \\ a/2 \\ 0 \end{pmatrix} + \xi \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} + \eta \begin{pmatrix} -1/\sqrt{2}\cdot\cos\theta \\ -1/\sqrt{2}\cdot\cos\theta \\ \sin\theta \end{pmatrix}.$$

That is,

$$\sqrt{2}x = \left(\frac{a}{\sqrt{2}} - \xi\right) - \eta\cos\theta,\ \sqrt{2}y = \left(\frac{a}{\sqrt{2}} + \xi\right) - \eta\cos\theta,\ z = \eta\sin\theta.$$

Now we study an intersection of the plane π_θ with the surface of the cone K (the cone whose axis lies on the x axis). For this purpose, in the equation of the surface $y^2 + z^2 = (\tau(a-x))^2$, we eliminate x, y, z. Then we obtain the following

$$(1-\tau^2)\left(\xi + \frac{a}{\sqrt{2}}\right)^2 - 2(1+\tau^2)\cos\theta\left(\xi + \frac{a}{\sqrt{2}}\right)\eta + (2-(1+\tau^2)\cos^2\theta)\eta^2 = 0. \qquad (3.1)$$

It can be seen as a quadratic equation of an unknown variable $\eta/(\xi + \frac{a}{\sqrt{2}})$.

To compute its determinant D,

$$D = 2(1+\tau^2)\left(\cos^2\theta - \frac{1-\tau^2}{1+\tau^2}\right).$$

If $D \leqq 0$, the quadratic equation (3.1) holds only when $\xi + \frac{a}{\sqrt{2}} = 0, \eta = 0$. That is, the case that A is the unique intersection of the plane with the cone surface. Otherwise, that is, if $|\cos\theta| > \sqrt{\dfrac{1-\tau^2}{1+\tau^2}}$, the intersection of the plane with the cone surface consists of two straight lines. Let k_1, k_2 be solutions of the quadratic equation (3.1). Then we have

$$k_1 + k_2 = \frac{2(1+\tau^2)\cos\theta}{2-(1+\tau^2)\cos^2\theta},\ k_1k_2 = \frac{1-\tau^2}{2-(1+\tau^2)\cos^2\theta}.$$

We see immediately $k_1 > 0, k_2 > 0$. In the below we assume $k_1 < k_2$.

Cutting $\mathbb{V}$ by π_θ, we ask what shape the section has. If we use the $\xi\eta$ coordinates system on the plane π_θ, the section of the first cone by the plane is a region between two straight lines

$$k_1(\xi + a/\sqrt{2}) \leqq \eta \leqq k_2(\xi + a/\sqrt{2}).$$

Similarly the section of the second cone by the plane is a region between two straight lines

$$-k_1(\xi - a/\sqrt{2}) \leqq \eta \leqq -k_2(\xi - a/\sqrt{2}).$$

Consequently the section of $\mathbb{V}$ is a region bounded by these four straight lines. This is a kite.

3.2 Infinitesimally thin plate of kite shape

A kite defined in the above obviously depends on θ. Thus we denote it by K_θ. A kite K_θ is a region bounded by four lines

$$\eta = k_1(\xi + d),\ \eta = k_2(\xi + d),\ \eta = -k_1(\xi - d),\ \eta = -k_2(\xi - d),$$

(where $d = a/\sqrt{2}$) . These four lines make four intersections (points) that have coordinates

$$(0, k_1d),\ (\sigma, k_1(\sigma + d)),\ (0, k_2d),\ (-\sigma, k_1(\sigma + d)),$$

where $\sigma = (k_2 - k_1)/(k_1 + k_2)d$.

Now we consider an infinitesimally thin plate sandwiched by K_θ and $K_{\theta+d\theta}$, and compute its volume $dV(\theta)$. When $0<\xi<\sigma$, the lowest vertex of K_θ has coordinate $\eta_1 = k_1(\xi+d)$, and the highest vertex has coordinate $\eta_2 = -k_2(\xi-d)$. Accordingly

$$dV(\theta) = 2\int_0^\sigma \left(\frac{1}{2}\eta_2^2 d\theta - \frac{1}{2}\eta_1^2 d\theta\right) d\xi = d\theta \int_0^\sigma (\eta_2^2-\eta_1^2)d\xi.$$

We have

$$\begin{aligned}\int_0^\sigma (\eta_2^2-\eta_1^2)d\xi &= \int_0^\sigma \{(-k_2(\xi-d))^2-(k_1(\xi+d))^2\}d\xi\\ &= (k_2^2-k_1^2)\frac{\sigma^3}{3}-(k_2^2+k_1^2)d\sigma^2+(k_2^2-k_1^2)d^2\\ &= \frac{d^3}{3}\cdot\frac{(k_2-k_1)^2}{(k_1+k_2)^2}\,(k_1^2+4k_1k_2+k_2^2)\\ &= \frac{a^3}{6\sqrt{2}}\cdot\frac{(k_2-k_1)^2}{(k_1+k_2)^2}\,(k_1^2+4k_1k_2+k_2).\end{aligned}$$

Introducing $\lambda = 2/(1+\tau^2)$, we can write

$$k_1+k_2 = \frac{2\cos\theta}{\lambda-\cos^2\theta},\ \ k_1k_2 = \frac{\lambda-1}{\lambda-\cos^2\theta}.$$

Thus, if a function g is defined as

$$\begin{aligned}g(\theta) &= \frac{[\cos^2\theta-(\lambda-1)][(3-\lambda)\cos^2\theta+\lambda(\lambda-1)]}{\cos^2\theta(\lambda-\cos^2\theta)^2}\\ &= \frac{(3-\lambda)\cos^4\theta+(\lambda-1)(2\lambda-3)\cos^2\theta-\lambda(\lambda-1)^2}{\cos^2\theta(\lambda-\cos^2\theta)^2},\end{aligned}$$

it can be seen that

$$\frac{(k_2-k_1)^2}{(k_1+k_2)^2}\,(k_1^2+4k_1k_2+k_2) = 2\lambda\, g(\theta).$$

Therefore we obtain

$$dV(\theta) = \frac{a^3\lambda}{3\sqrt{2}}\cdot g(\theta)d\theta.$$

3.3 Volume

Our volume is

$$V = 2\int_0^{\theta_1} dV(\theta),$$

where θ_1 is defined by $\cos^2\theta_1 = (1-\tau^2)/(1+\tau^2), 0<\theta_1<\pi/2$. If a primitive function

$$G(\theta) = \int_0^\theta g(\theta)d\theta$$

is introduced, it can be expressed as

$$V = \frac{\sqrt{2}a^3\lambda}{3}\,G(\theta_1).$$

Integration using a computer algebra system (maxima) gives

$$G(\theta) = \frac{2-\lambda}{(\lambda-1)\sqrt{\lambda(\lambda-1)}}\ \arctan\left(\sqrt{\frac{\lambda}{\lambda-1}}\ \tan\theta\right) + \frac{1}{\lambda(\lambda-1)}\cdot\frac{\tan\theta}{\lambda\tan^2\theta+\lambda-1} - \frac{(\lambda-1)^2}{\lambda}\ \tan\theta.$$

Then by an elentary computation we have

$$\begin{aligned} G(\theta_1) &= \frac{2\sqrt{2}\tau^2(1+\tau^2)}{(1-\tau^2)^{3/2}}\cdot\beta + \frac{2\sqrt{2}\tau^3}{(1+\tau^2)\sqrt{1-\tau^2}} \\ &= 2\sqrt{2}\,\frac{\sin^2\beta}{\cos\beta(\cos 2\beta)^{3/2}}\cdot\beta + 2\sqrt{2}\,\frac{\sin^3\beta}{(\cos 2\beta)^{1/2}} \\ &= \sqrt{2}\,\frac{\sin^2\beta}{\cos\beta(\cos 2\beta)^{3/2}}\,(2\beta+\sin 2\beta\cos 2\beta) \end{aligned}$$

Therefoew we obtain the following result.

Proposition 2

$$V = \frac{2a^3}{3}\,\frac{\sin\beta\sin 2\beta}{(\cos 2\beta)^{3/2}}\,(2\beta+\sin 2\beta\cos 2\beta)$$

4 Relation between surface area and volume

Proposition 33 and Proposition 34 in the book by Archimedes are monumental achievements in history of mathematics （[4],[5]）. They are stated as follows.

Proposition 33 of Archimedes

Πάσης σφαίρας ἡ ἐπιφάνεια τετραπλασία ἐστὶ τοῦ μεγίστου κύκλου τῶν ἐν αὐτῇ.

(The surface of any sphere is equal to four times the greatest circle in it.)

Proposition 34 of Archimedes

Πᾶσα σφα˜σιρα τετραπρασία ἐστὶ κώου τοῦ βάσιν μὲν ἔχοντος ἴσην τῷ μεγίστῳ κύκλῳ τῶν ἐν τῇ σφαίρᾳ, ὕξος δὲ τὴν ἐκ τοῦ κέντρου τῆς σφαίρας.

(Any sphere is equal to four times the cone which has its base equal to the greatest circle in the sphere and its height equal to the radius of the sphere.)

From these Propositions we can deduce imediately that the volume of any sphere is equal to that of a cone with base being equal to surface area of the sphere and with height being equal to radius of the sphere. To interest us, a similar result holds for $\mathbb{V}$. To state it precisely we prepare a concept.

First, for a point P and a smooth surface Σ, we define the distance between them by $d(\mathrm{P},\Sigma) = \inf_{\mathrm{Q}\in\Sigma} d(\mathrm{P},\mathrm{Q})$. Next, for a solid, assume that its surface consits of a finite number of smooth surfaces $\Sigma_1, \Sigma_2, \ldots, \Sigma_k$, and there exists a point P such that all of distances $d(\mathrm{P},\Sigma_1), d(\mathrm{P},\Sigma_2), \ldots, d(\mathrm{P},\Sigma_k)$ are of the same value. Then we say P a center of the solid, the common value of distances the radius of the solid.

Then the following is true. Proof is easy.

Proposition 3

The volume of solid $\mathbb{V}$ is equal to that of a cone with base being equal to surface area of the solid and with height being equal to the radius of the solid.

References

[1] J.L.Heberg "Eine Neue Archimedeshandschrift", *Hermes* 42. Bd., H. 2 (1907), pp. 235-303, Franz Steiner Verlag

[2] T.L.Heath *THE METHOD OF ARCHIMEDES Recently Discovered by HEiberg* (1912), Cambridge University Press.

[3] Hasegawa Zenzaemon *Sanpou Kyuseki Tuukou* (Consistent Tretise on Quadrature) (1874)、, http://www.wasan.jp/archive.html.

[4] J.L.Heberg "ΠΕΡΙ ΣΦΑΙΡΑΣ ΚΑΙ ΚΥΛΙΝΔΡΟΥ Α′", *ARCHIMEDIS OPERA OMNIA* Vol.I.

[5] T.L.Heath "On Sphere and Cylinder", *WORKS OF ARCHIMEDES*.