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A note on upper continuity properties of relations

Hitoshi FURUSAWA[1)*], Satoru YOSHIDA[2)]

[1)] Department of Mathematics and Computer Science, Kagoshima University
[2)] General Education Center, Tottori University of Environmental Studies

*furusawa@sci.kagoshima-u.ac.jp

Abstract: This note introduces a notion of upper continuity of relations and provides 5 equivalent conditions with the upper continuity. As the case of continuous functions, each condition characterlises upper continuous relations by respectively using open sets, neighbourhoods, interiors, closures, and closed sets. A notion of sequential continuity is also studied. We show a relevance between upper continuous relations and sequentially upper continuous relations which is, again, analogous to the case of functions.

Keywords: relations, topological spaces, upper continuity, sequential upper continuity

1. Introduction

This note introduces a notion of upper continuity of relations and provides quite fundamental results on the upper continuity. There are various definitions of continuous relations. For example, Brattka and Hertling [2] investigated relevance between computability and the following continuity of relations:

a relation is continuous iff preimage of each open set under the relation is open in the domain of it.

Generalising well-known characterisations of continuous functions, [3] has provided equivalent conditions with the continuity. Berge [1] introduced lower and upper semi-continuity of relations:

- a relation is lower semi-continuous iff preimage of each open set under it is open;
- a relation is upper semi-continuous iff upper preimage of each open set under it is open and image of each singleton set is compact.

Then, in [1], both lower and upper semi-continuous relations are called continuous. The continuity plays a key rôle in a branch of mathematical economics [4]. In this note we introduce a generalisation of Berge's upper semi-continuity. The generalisation does not require compactness but concerns about the domain of relations like the notion adopted by Brattka and Hertling.

Notion and properties of continuous functions are very fundamental and studied very well. Everyone learns the following facts in math classes.

Fact 1. Let $f\colon X \to Y$ be a function. Then the following 6 conditions are equivalent.

1. f is continuous.
2. For each open set $V \subseteq Y$ the set $f^{-1}(V)$ is open in X.
3. For each point $x \in X$ the set $f^{-1}(N(f(x)))$ is a neighbourhood of x for each neighbourhood $N(f(x))$ of $f(x)$.
4. $f^{-1}(B^\circ) \subseteq (f^{-1}(B))^\circ$ for each subset $B \subseteq Y$.
5. $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ for each subset $B \subseteq Y$.
6. For each closed set $F \subseteq Y$ the set $f^{-1}(F)$ is closed in X.

Fact 2. Let $f\colon X \to Y$ be a function.

1. If f is continuous, then it is sequentially continuous.
2. If X is a first-countable space and f is sequentially continuous, then it is continuous.

This note provides analogous properties of the above facts based on the generalisation of Berge's upper semi-continuity.

2. Images and preimages of relations

This section recalls images and preimages of relations which will be used later.

Let $R \subseteq X \times Y$ be a relation from a set X to a set Y. For $x \in X$, $A \subseteq X$, $y \in Y$ and $B \subseteq Y$ we define

$$[x]R = \{y \in Y \mid (x, y) \in R\} \qquad [A]R = \bigcup_{a \in A}[a]R$$
$$R[y] = \{x \in X \mid (x, y) \in R\} \qquad R[B] = \bigcup_{b \in B} R[b],$$

and respectively call $[A]R$ and $R[B]$ *image* of A and *preimage* (or, equivalently, *inverse image*) of B under R. The domain of R is denoted by $\mathrm{dom}(R)$, that is, $\mathrm{dom}(R) = R[Y]$. For a subset $B \subseteq Y$ *upper preimage* $R[\![B]\!]$ of B under R is defined by

$$R[\![B]\!] = \{x \in X \mid [x]R \subseteq B\}.$$

A relation $R \subseteq X \times Y$ is called *univalent* (or a *partial function* in other words) if $(x, y) \in R$ and $(x, y') \in R$ implies $y = y'$ for all $x \in X$ and $y, y' \in Y$. If a relation R satisfies $\mathrm{dom}(R) = X$, we call it *total*. If a relation R is univalent and total, we call it a *function* (or a *map* in other words) and is written by $R\colon X \to Y$. For a function $f\colon X \to Y$, $f(x)$ and $f(A)$ denote the value at x of f and the image of A under f, respectively. Of course, $\{f(x)\} = [x]f$ and $f(A) = [A]f$. Notations $f^{-1}(y)$ and $f^{-1}(B)$ for preimages $f[y]$ and $f[B]$ of $y \in Y$ and $B \subseteq Y$ under a function f are also used. Note that $f[B] = f^{-1}(B) = f[\![B]\!]$ for any $B \subseteq Y$ since

$$x \in f[B] \iff x \in f^{-1}(B) \iff f(x) \in B \iff [x]f \subseteq B \iff x \in f[\![B]\!].$$

The next lemma will be used only in Appendix.

Lemma 2.1. *Let $R \subseteq X \times Y$ be a relation. Then the following two inclusions hold for any subsets $A \subseteq X$ and $B \subseteq Y$.*

$$A \subseteq R[\![[A]R]\!] \qquad [R[\![B]\!]]R \subseteq B$$

Proof. $A \subseteq R[\![[A]R]\!]$ holds by

$$a \in A \implies [a]R \subseteq [A]R \iff a \in R[\![[A]R]\!].$$

Also $[R[\![B]\!]]R \subseteq B$ holds by

$$\begin{aligned} y \in [R[\![B]\!]]R &\iff \exists x \in X.\, x \in R[\![B]\!] \text{ and } y \in [x]R \\ &\iff \exists x \in X.\, [x]R \subseteq B \text{ and } y \in [x]R \\ &\implies y \in B. \end{aligned}$$

□

The next lemma connects preimages and upper preimages, and will be used to prove Proposition 3.5, 3.7 and also in Appendix. For a set S we denote the complement of S by S^{C}.

Lemma 2.2. *Let $R \subseteq X \times Y$ be a relation. Then the following two equations hold for each subset $B \subseteq Y$.*

$$R[B] = (R[\![B^{\mathsf{C}}]\!])^{\mathsf{C}} \qquad R[\![B]\!] = (R[B^{\mathsf{C}}])^{\mathsf{C}}$$

Proof. $R[B] = (R[\![B^{\mathsf{C}}]\!])^{\mathsf{C}}$ holds by

$$\begin{aligned} x \in R[B] &\iff \exists y \in Y.\, y \in B \text{ and } y \in [x]R \\ &\iff \exists y \in Y.\, y \notin B^{\mathsf{C}} \text{ and } y \in [x]R \\ &\iff [x]R \not\subseteq B^{\mathsf{C}} \\ &\iff x \notin R[\![B^{\mathsf{C}}]\!] \\ &\iff x \in (R[\![B^{\mathsf{C}}]\!])^{\mathsf{C}} \end{aligned}$$

Also $R[\![B]\!] = (R[B^{\mathsf{C}}])^{\mathsf{C}}$ holds by $R[\![B]\!] = ((R[\![(B^{\mathsf{C}})^{\mathsf{C}}]\!])^{\mathsf{C}})^{\mathsf{C}} = (R[B^{\mathsf{C}}])^{\mathsf{C}}$. □

The interior and closure of a subset S of a topological space are respectively denoted by S° and $\overline{S}$. Lemma 2.2 and well-known one-to-one correspondences

$$S^\circ = \overline{S^{\mathsf{C}}}^{\mathsf{C}} \qquad \overline{S} = ((S^{\mathsf{C}})^\circ)^{\mathsf{C}}$$

between interiors and closures induce the following property. It will be used to prove Proposition 3.5 and also in Appendix.

Lemma 2.3. *Let X and Y be topological spaces and $R \subseteq X \times Y$ a relation. Then the following four equations hold for each $B \subseteq Y$.*

$$R[B^\circ] = (R[\![\overline{B^{\mathsf{C}}}]\!])^{\mathsf{C}} \qquad R[\![B^\circ]\!] = (R[\overline{B^{\mathsf{C}}}])^{\mathsf{C}} \qquad R[\overline{B}] = (R[\![(B^{\mathsf{C}})^\circ]\!])^{\mathsf{C}} \qquad R[\![\overline{B}]\!] = (R[(B^{\mathsf{C}})^\circ])^{\mathsf{C}}$$

3. Point-wise upper continuity and its equivalent conditions

Point-wise upper continuity of relations is defined as follows. Throughout of the section we assume X and Y to be topological spaces.

Definition 3.1. Let $R \subseteq X \times Y$ be a relation.

1. Given a point $(x, y) \in R$, R is *upper continuous at* (x, y) if for each open neighbourhood $N(y)$ of y satisfying $[x]R \subseteq N(y)$ there exists a neighbourhood $N(x)$ of x such that $[x']R \subseteq N(y)$ for all x' in $N(x)$.
2. R is *(point-wise) upper continuous* if R is upper continuous at all points (x, y) in R.

A function $f\colon X \to Y$ is continuous at $x \in X$ iff f is upper continuous at $(x, f(x))$ since $\mathrm{dom}(f) = X$ and $(x, y) \in f$ is equivalent to $y = f(x)$ for any $x \in X$. Thus the upper continuity of relations is a generalisation of the (point-wise) continuity of functions.

In the rest of this section we provide generalisations of the items 2–5 from Fact 1.

Proposition 3.2. *A relation $R \subseteq X \times Y$ is upper continuous iff for each open set $V \subseteq Y$ the set $R[\![V]\!] \cap \mathrm{dom}(R)$ is open in $\mathrm{dom}(R)$.*

Proof. Let $x \in R[\![V]\!] \cap \mathrm{dom}(R)$. Then, by $x \in \mathrm{dom}(R)$ there exists $y_0 \in Y$ such that $(x, y_0) \in R$. Also by $x \in R[\![V]\!]$ V is an open neighbourhood of y_0 satisfying $[x]R \subseteq V$. Thus there exists a neighbourhood $N(x)$ of x such that $[x']R \subseteq V$ for all $x' \in N(x)$ since R is upper continuous. So $N(x) \cap \mathrm{dom}(R)$ is a neighbourhood of x with respect to $\mathrm{dom}(R)$ satisfying $N(x) \cap \mathrm{dom}(R) \subseteq R[\![V]\!] \cap \mathrm{dom}(R)$. Therefore $R[\![V]\!] \cap \mathrm{dom}(R)$ is open in $\mathrm{dom}(R)$. Next, let $(x, y) \in R$ and $N(y)$ an open neighbourhood of y satisfying $[x]R \subseteq N(y)$. Then, $R[\![N(y)]\!] \cap \mathrm{dom}(R)$ is an open neighbourhood of x with respect to $\mathrm{dom}(R)$. Thus there exists a neighbourhood $N(x)$ of x such that $N(x) \cap \mathrm{dom}(R) = R[\![N(y)]\!] \cap \mathrm{dom}(R)$. Therefore $[x']R \subseteq N(y)$ for all $x' \in N(x)$ since $x' \in R[\![N(y)]\!]$ if $x' \in \mathrm{dom}(R)$, and $[x']R = \emptyset$ iff $x' \notin \mathrm{dom}(R)$. □

Recalling the definition of neighbourhoods, we have the following property. Its consequence is a generalisation of the third item of Fact 1.

Proposition 3.3. *Let $R \subseteq X \times Y$ be a relation such that for each open set $V \subseteq Y$ the set $R[\![V]\!] \cap \mathrm{dom}(R)$ is open in $\mathrm{dom}(R)$. Then for each point $(x, y) \in R$ the set $R[\![N(y)]\!]$ is a neighbourhood of x for each neighbourhood $N(y)$ of y satisfying $[x]R \subseteq N(y)$.*

Proof. Let $(x, y) \in R$ and $N(y)$ be a neighbourhood of y satisfying $[x]R \subseteq N(y)$. Then there exists an open set $V \subseteq Y$ such that $V \subseteq N(y)$. So, there exists an open set $O \subseteq X$ such that $O \cap \mathrm{dom}(R) = R[\![V]\!] \cap \mathrm{dom}(R)$ since $R[\![V]\!] \cap \mathrm{dom}(R)$ is open in $\mathrm{dom}(R)$. This open set O is contained by $R[\![N(y)]\!]$ since $x' \in O$ implies $x' \in R[\![N(y)]\!]$ if $x' \in \mathrm{dom}(R)$ and $[x']R = \emptyset$ iff $x' \notin \mathrm{dom}(R)$. Also, $[x]R \subseteq N(y)$ iff $x \in R[\![N(y)]\!]$. Therefore $R[\![N(y)]\!]$ is a neighbourhood of x. □

Recalling the definition of interiors, we have the following property. Its consequence is a generalisation of the fourth item of Fact 1.

Proposition 3.4. *Let $R \subseteq X \times Y$ be a relation such that for each point $(x, y) \in R$ the set $R\llbracket N(y)\rrbracket$ is a neighbourhood of x for each neighbourhood $N(y)$ of y satisfying $[x]R \subseteq N(y)$. Then $R\llbracket B^\circ\rrbracket \cap \mathrm{dom}(R) \subseteq (R\llbracket B\rrbracket)^\circ \cap \mathrm{dom}(R)$ holds for each subset $B \subseteq Y$.*

Proof. Let $x \in R\llbracket B^\circ\rrbracket \cap \mathrm{dom}(R)$. Then there exists $y_0 \in Y$ such that $(x, y_0) \in R$ since $x \in \mathrm{dom}(R)$. Also $y_0 \in B^\circ$ since $x \in R\llbracket B^\circ\rrbracket$. Thus there exists a neighbourhood $N(y_0)$ such that $N(y_0) \subseteq B$. Now set $V = N(y_0) \cup [x]R$. Then V is a neighbourhood of y_0 satisfying $[x]R \subseteq V$. Hence $R\llbracket V\rrbracket$ is a neighbourhood of x. V also satisfies $V \subseteq B$ since $N(y_0) \subseteq B$ and $[x]R \subseteq B^\circ \subseteq B$. Hence $R\llbracket V\rrbracket \subseteq R\llbracket B\rrbracket$ holds. So $x \in (R\llbracket B\rrbracket)^\circ$. Therefore $x \in (R\llbracket B\rrbracket)^\circ \cap \mathrm{dom}(R)$. □

The consequence of the following proposition is a generalisation of the fifth item of Fact 1.

Proposition 3.5. *Let $R \subseteq X \times Y$ be a relation satisfying $R\llbracket B^\circ\rrbracket \cap \mathrm{dom}(R) \subseteq (R\llbracket B\rrbracket)^\circ \cap \mathrm{dom}(R)$ for each subset $B \subseteq Y$. Then R also satisfies $\overline{R[B]} \cap \mathrm{dom}(R) \subseteq R[\overline{B}]$ for each $B \subseteq Y$.*

Proof. Assume that $R\llbracket B^\circ\rrbracket \cap \mathrm{dom}(R) \subseteq (R\llbracket B\rrbracket)^\circ \cap \mathrm{dom}(R)$ for each subset $B \subseteq Y$. Then we have

$$\begin{aligned}
\overline{R[B]} \cap \mathrm{dom}(R) &= \overline{(R\llbracket B^{\mathsf{C}}\rrbracket)^{\mathsf{C}}} \cap \mathrm{dom}(R) && \text{by Lemma 2.2}\\
&= ((R\llbracket B^{\mathsf{C}}\rrbracket)^\circ)^{\mathsf{C}} \cap \mathrm{dom}(R) &&\\
&\subseteq ((R\llbracket B^{\mathsf{C}}\rrbracket)^\circ \cap \mathrm{dom}(R))^{\mathsf{C}} \cap \mathrm{dom}(R) &&\\
&\subseteq (R\llbracket (B^{\mathsf{C}})^\circ\rrbracket \cap \mathrm{dom}(R))^{\mathsf{C}} \cap \mathrm{dom}(R) && \text{by assumption}\\
&= ((R\llbracket (B^{\mathsf{C}})^\circ\rrbracket)^{\mathsf{C}} \cup \mathrm{dom}(R)^{\mathsf{C}}) \cap \mathrm{dom}(R) &&\\
&= (R\llbracket (B^{\mathsf{C}})^\circ\rrbracket)^{\mathsf{C}} \cap \mathrm{dom}(R) &&\\
&= R[\overline{B}] \cap \mathrm{dom}(R) && \text{by Lemma 2.3}\\
&= R[\overline{B}]. &&
\end{aligned}$$

□

The consequence of the following proposition is a generalisation of the sixth item of Fact 1.

Proposition 3.6. *Let $R \subseteq X \times Y$ be a relation satisfying $\overline{R[B]} \cap \mathrm{dom}(R) \subseteq R[\overline{B}]$ for each $B \subseteq Y$. Then for each closed set $F \subseteq Y$ the set $R[F]$ is closed in $\mathrm{dom}(R)$.*

Proof. For each closed set F we have $\overline{R[F]} \cap \mathrm{dom}(R) = R[F]$ by $R[F] = R[F] \cap \mathrm{dom}(R) \subseteq \overline{R[F]} \cap \mathrm{dom}(R)$ and $\overline{R[F]} \cap \mathrm{dom}(R) \subseteq R[\overline{F}] = R[F]$. Thus $R[F]$ is closed in $\mathrm{dom}(R)$. □

Proposition 3.7. *Let $R \subseteq X \times Y$ be a relation. Then the following two conditions are equivalent:*

1. *For each closed set $F \subseteq Y$ the set $R[F]$ is closed in $\mathrm{dom}(R)$.*
2. *For each open set $V \subseteq Y$ the set $R\llbracket V\rrbracket \cap \mathrm{dom}(R)$ is open in $\mathrm{dom}(R)$.*

Proof. 1 implies 2 since

$$R\llbracket V\rrbracket \cap \mathrm{dom}(R) = R[V^{\mathsf{C}}]^{\mathsf{C}} \cap \mathrm{dom}(R)$$

by Lemma 2.2 and $R[V^{\mathsf{C}}]$ is closed in $\mathrm{dom}(R)$. Also 2 implies 1 since

$$\begin{aligned}
R[F] &= (R\llbracket F^{\mathsf{C}}\rrbracket)^{\mathsf{C}} && \text{by Lemma 2.2}\\
&= (R\llbracket F^{\mathsf{C}}\rrbracket \cap (\mathrm{dom}(R) \cup \mathrm{dom}(R)^{\mathsf{C}}))^{\mathsf{C}} &&\\
&= ((R\llbracket F^{\mathsf{C}}\rrbracket \cap \mathrm{dom}(R)) \cup (R\llbracket F^{\mathsf{C}}\rrbracket \cap \mathrm{dom}(R)^{\mathsf{C}}))^{\mathsf{C}} &&\\
&= ((R\llbracket F^{\mathsf{C}}\rrbracket \cap \mathrm{dom}(R)) \cup \mathrm{dom}(R)^{\mathsf{C}})^{\mathsf{C}} && \text{by } \mathrm{dom}(R)^{\mathsf{C}} \subseteq R\llbracket F^{\mathsf{C}}\rrbracket\\
&= (R\llbracket F^{\mathsf{C}}\rrbracket \cap \mathrm{dom}(R))^{\mathsf{C}} \cap \mathrm{dom}(R) &&
\end{aligned}$$

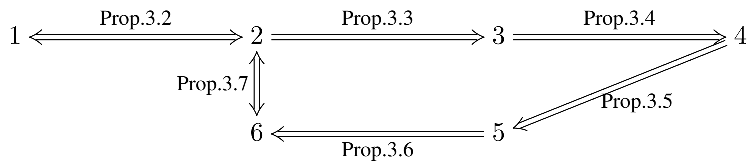
and $R\llbracket F^{\mathsf{C}}\rrbracket \cap \mathrm{dom}(R)$ is open in $\mathrm{dom}(R)$. □

Summing up the discussion above, we have the following theorem.

Theorem 3.8. *Let $R \subseteq X \times Y$ be a relation. Then the following conditions are equivalent:*

1. *R is upper continuous.*
2. *For each open set $V \subseteq Y$ the set $R[\![V]\!] \cap \mathrm{dom}(R)$ is open in $\mathrm{dom}(R)$.*
3. *For each point $(x, y) \in R$ the set $R[\![N(y)]\!]$ is a neighbourhood of x for each neighbourhood $N(y)$ of y satisfying $[x]R \subseteq N(y)$.*
4. *$\overline{R[\![B^\circ]\!]} \cap \mathrm{dom}(R) \subseteq (R[\![B]\!])^\circ \cap \mathrm{dom}(R)$ for each subset $B \subseteq Y$.*
5. *$\overline{R[B]} \cap \mathrm{dom}(R) \subseteq R[\overline{B}]$ for each $B \subseteq Y$.*
6. *For each closed set $F \subseteq Y$ the set $R[F]$ is closed in $\mathrm{dom}(R)$.*

Proof.



□

Now Fact 1 is a corollary of Theorem 3.8.

Corollary 3.9. *Let $f\colon X \to Y$ be a function. Then the following conditions are equivalent:*

1. *f is continuous.*
2. *For each open set $V \subseteq Y$ the set $f^{-1}(V)$ is open in X.*
3. *For each point $x \in X$ the set $f^{-1}(N(f(x)))$ is a neighbourhood of x for each neighbourhood $N(f(x))$ of $f(x)$.*
4. *$\overline{f^{-1}(B^\circ)} \subseteq (f^{-1}(B))^\circ$ for each subset $B \subseteq Y$.*
5. *$\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ for each subset $B \subseteq Y$.*
6. *For each closed set $F \subseteq Y$ the set $f^{-1}(F)$ is closed in X.*

4. Sequential upper continuity and point-wise upper continuity

In this section, we study sequential upper continuity of relations. Again, throughout of the section we assume X and Y to be topological spaces.

Definition 4.1. Let $R \subseteq X \times Y$ be a relation.

1. Given a point $(x, y) \in R$, R is *sequentially upper continuous at* (x, y) if for each sequence $\{x_n\}$, whenever $\{x_n\}$ converges to x, then, for each open neighbourhood $N(y)$ of y satisfying $[x]R \subseteq N(y)$, there exists $M \in \mathbb{N}$ such that $[x_m]R \subseteq N(y)$ for any $m \geq M$.
2. R is *sequentially upper continuous* if R is sequentially upper continuous at all points (x, y) in R.

As the case of point-wise continuity, a function $f\colon X \to Y$ is sequentially continuous at $x \in X$ iff f is sequentially upper continuous at $(x, f(x))$ since $\mathrm{dom}(f) = X$ and $(x, y) \in f$ is equivalent to $y = f(x)$ for any $x \in X$.

The upper continuity of relations implies sequential upper continuity of them.

Proposition 4.2. *If a relation $R \subseteq X \times Y$ is upper continuous, then it is sequentially upper continuous.*

Proof. Let $(x, y) \in R$, $\{x_n\}$ be a sequence conversing to x in X, and $N(y)$ an open neighbourhood of y satisfying $[x]R \subseteq N(y)$. Since R is upper continuous, there is a neighbourhood $N(x)$ of x such that $[x']R \subseteq N(y)$ for all x' in $N(x)$. Also, since $\{x_n\}$ converges to x, there exists $M \in \mathbb{N}$ such that $x_m \in N(x)$ for any $m \geq M$. Thus $[x_m]R \subseteq N(y)$ holds for $m \geq M$. □

Obviously, the converse of Proposition 4.2 does not holds since it is known that there exists a sequentially continuous function which is not continuous. However, the following holds as the case of functions.

Proposition 4.3. *A sequentially upper continuous relation $R \subseteq X \times Y$ is upper continuous if X is a first-countable space.*

Proof. We prove contraposition. Suppose that R is not upper continuous. Then, by Theorem 3.8, for some point $(x_0, y_0) \in R$ there exists a neighbourhood $N(y_0)$ of y_0 such that $[x_0]R \subseteq N(y_0)$ and $R[\![N(y_0)]\!]$ is not a neighbourhood in $\mathrm{dom}(R)$. Also, since X is a first-countable space, there exists a countable fundamental neighbourhood system $\{N_n(x_0)\}$ of x_0 such that $N_k(x_0) \supseteq N_{k+1}(x_0)$ for each $k \in \mathbb{N}$. Now, since $R[\![N(y_0)]\!]$ is not a neighbourhood, $N_k(x_0) \not\subseteq R[N(y_0)]$ for each $k \in \mathbb{N}$. Thus it is able to take a sequence $\{x_n\}$ such that $x_k \in N_k(x_0)$ and $x_k \notin R[\![N(y_0)]\!]$ for each $k \in \mathbb{N}$. Then, though the sequence $\{x_n\}$ converges to x_0, $[x_k]R \not\subseteq N(y_0)$ for each $k \in \mathbb{N}$. Therefore R is not sequentially upper continuous. □

Now the 2 items of Fact 2 are corollaries of Proposition 4.2 and 4.3.

Corollary 4.4. *Let $f\colon X \to Y$ be a function.*

1. *If f is continuous, then it is sequentially continuous.*
2. *If X is a first-countable space and f is sequentially continuous, then it is continuous.*

5. Concluding remarks

In this note we have generalised well-known properties Fact 1 and 2 of continuous functions to ones of upper continuous relations. It is also well-known that the following condition is equivalent to a function $f\colon X \to Y$ being continuous, but we have not yet found its generalisation to upper continuous relations:

$f(\overline{A}) \subseteq \overline{f(A)}$ for each subset $A \subseteq X$.

We are still looking for it.

Though in [3] we obtained similar results of continuous relations in the sense of [2], we did not consider analogous conditions of the fourth and fifth item of Fact 1. See Appendix for them.

Acknowledgement

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References

[1] Claude Berge. Topological Spaces: including a treatment of multi-valued functions, vector spaces and convexity (translated by E. M. Patterson). Oliver and Boyd, 1963

[2] Vasco Brattka and Peter Hertling. Continuity and computability of relations, Informatik Berichte **164**, FernUniversität in Hagen (1994)

[3] Hitoshi Furusawa and Satoru Yoshida. A note on continuity properties of relations, Reports of the Faculty of Science, Kagoshima University 52 (2019) 1–6 (available at `http://hdl.handle.net/10232/00030848`)

[4] Erwin Klein and Anthony C. Thompson. Theory of Correspondences, Including Applications to Mathematical Economics. Wiley, 1984

Appendix

Let X and Y be topological spaces and $R \subseteq X \times Y$. We consider the following 3 conditions.

1. $\overline{R[B^\circ]} \subseteq (R[B] \cup \mathrm{dom}(R)^{\mathsf{C}})^\circ$ for each $B \subseteq Y$.
2. $\overline{R[\![B]\!] \cap \mathrm{dom}(R)} \subseteq R[\![\overline{B}]\!]$ for each $B \subseteq Y$.
3. $[\overline{A}]R \subseteq \overline{[A]R}$ for each $A \subseteq \mathrm{dom}(R)$.

Note that [3] has shown that R is continuous in the sense of [2] iff R satisfies 3. Showing these 3 conditions are equivalent, we ensure that 1 and 2 are respectively generalisations of the fourth and fifth item of Fact 1 for the continuity of relations.

Suppose that R satisfies 1. Then we have

$$\begin{aligned}
\overline{R\llbracket B\rrbracket \cap \mathrm{dom}(R)} &= (((R\llbracket B\rrbracket \cap \mathrm{dom}(R))^{\mathsf{C}})^{\circ})^{\mathsf{C}} && \\
&= (((R\llbracket B\rrbracket)^{\mathsf{C}} \cup \mathrm{dom}(R)^{\mathsf{C}})^{\circ})^{\mathsf{C}} && \\
&= ((R[B^{\mathsf{C}}] \cup \mathrm{don}(R)^{\mathsf{C}})^{\circ})^{\mathsf{C}} && \text{by Lemma 2.2} \\
&\subseteq R[(B^{\mathsf{C}})^{\circ}]^{\mathsf{C}} && \text{by 1} \\
&= R\llbracket \overline{B}\rrbracket. && \text{by Lemma 2.3}
\end{aligned}$$

Conversely, suppose that R satisfies 2. Then we have

$$\begin{aligned}
R[B^{\circ}] &= (R\llbracket \overline{B^{\mathsf{C}}}\rrbracket)^{\mathsf{C}} && \text{by Lemma 2.3} \\
&\subseteq \overline{R\llbracket B^{\mathsf{C}}\rrbracket \cap \mathrm{dom(R)}}^{\mathsf{C}} && \text{by 2} \\
&= (R\llbracket B^{\mathsf{C}}\rrbracket \cap \mathrm{dom(R)}^{\mathsf{C}})^{\circ} && \\
&= ((R\llbracket B^{\mathsf{C}}\rrbracket)^{\mathsf{C}} \cup \mathrm{dom}(R)^{\mathsf{C}})^{\circ} && \\
&= (R[B] \cup \mathrm{dom}(R)^{\mathsf{C}})^{\circ} && \text{by Lemma 2.2}
\end{aligned}$$

Thus 1 and 2 are equivalent. Again, suppose that R satisfies 2. Then we have

$$\begin{aligned}
[\overline{A}]R &\subseteq [\overline{R\llbracket [A]R\rrbracket \cap \mathrm{dom}(R)}]R && \text{by Lemma 2.1 and } A \subseteq \mathrm{dom}(R) \\
&\subseteq [R\llbracket \overline{[A]R}\rrbracket]R && \text{by 2} \\
&\subseteq \overline{[A]R}. && \text{by Lemma 2.1}
\end{aligned}$$

Conversely, suppose that R satisfies 3. Then we have

$$\begin{aligned}
\overline{R\llbracket B\rrbracket \cap \mathrm{dom}(R)} &\subseteq R\llbracket [\overline{R\llbracket B\rrbracket \cap \mathrm{dom}(R)}]R\rrbracket && \text{by Lemma 2.1} \\
&\subseteq R\llbracket \overline{[R\llbracket B\rrbracket \cap \mathrm{dom}(R)]R}\rrbracket && \text{by 3} \\
&\subseteq R\llbracket \overline{[R\llbracket B\rrbracket]R}\rrbracket && \\
&\subseteq R\llbracket \overline{B}\rrbracket. && \text{by Lemma 2.1}
\end{aligned}$$

Thus 2 and 3 are equivalent. Therefore these 3 conditions are equivalent.