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How to join two circles with one circle inside the other

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Abstract

Spiral segments have several advantages of containing neither inflection points, singularities nor curvature extrema. The object of this note is to consider how to join two circles with one circle inside the other by use of T-cubic and PH quintic spiral segments. It enables us to rediscover all the results on how to join a straight line to a circle; two circles with a broken back; two circles with an S; two non-parallel straight lines.

Key words: PH quintic spiral, spiral segments, spline, T-cubic spiral

1 Introduction

Meek & Walton have considered a possibility of a PH (Pythagorean hodograph) quintic spiral to join (i) a straight line to a circle, (ii) two circles with a broken back C, (iii) two circles with an S, (iv) two non-parallel straight lines and also (v) two circles with one circle inside the other ([4]).

The object of this paper is to first consider a possibility of a T-cubic spiral to the case (v), next to examine a PH quintic spiral to the last case (v) treated without a sufficient analysis and finally to show that the case (i) is the limiting case of (iv) with the radius ($\to \infty$) of the larger circle. Now we consider two circles Ω_0, Ω_1 centered at C_0, C_1 with radii r_0, r_1, and such that Ω_1 is completely contained inside Ω_0. By d, we mean the distance between the centers of the two circles.

2 T-cubic spirals

We assume that the a T-cubic spline meets Ω_0 at $t = 0$ and Ω_1 at $t = 1$ and that the larger circle Ω_0 is tangent to the X-axis at the origin, i.e., its center is $(0, r_0)$. The required T-cubic spline $\boldsymbol{z}(t) = (x(t), y(t)), 0 \le t \le 1$ is given with $k = (r_0/r_1)^{1/4} (> 1)$ by

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$$x(t) = \frac{2kr_1 t}{3}\left\{(k^2 - 2k\cos\frac{\theta}{2} + \cos\theta)t^2 - 3k(k - \cos\frac{\theta}{2})t + 3k^2\right\}\sin\frac{\theta}{2} \tag{2.1}$$
$$y(t) = \frac{2kr_1 t^2}{3}\left\{-2(k - \cos\frac{\theta}{2})t + 3k\right\}\sin^2\frac{\theta}{2}$$

where

$$\text{(i)}\quad \boldsymbol{z}(0) = (0,0), \quad \text{(ii)}\quad \kappa(0) = 1/r_0, \qquad \text{(iii)}\quad \kappa(1) = 1/r_1 \tag{2.2}$$
$$\text{(iv)}\quad z'(0) \parallel (1,0), \quad \text{(v)}\quad z'(1) \parallel (\cos\theta, \sin\theta)$$

Now, the curvature $\kappa(t)$ is

$$\kappa(t) = \frac{1}{r_1\{(k^2 - 2kz + 1)t^2 - 2k(k - z)t + k^2\}^2}, \quad 0 < z(= \cos\frac{\theta}{2}) < 1 \tag{2.3}$$

In addition, the center (p_1, q_1) of the smaller circle Ω_1 is given by

$$p_1 = \frac{2r_1}{3}(k^3 + k^2\cos\frac{\theta}{2} + k\cos\theta - 3\cos\frac{\theta}{2})\sin\frac{\theta}{2} \tag{2.4}$$
$$q_1 = \frac{r_1}{3}\{(1 - \cos\theta)k^2 + (\cos\frac{\theta}{2} - \cos\frac{3\theta}{2})k + 3\cos\theta\}$$

Find a root of $\kappa'(t) = 0$ to obtain a condition for the T-cubic spline (2.1) to be a spiral of monotone increasing curvature:

$$\frac{k(k - z)}{k^2 - 2kz + 1} \geq 1 \Rightarrow z \in [1/k, 1) \tag{2.5}$$

Then Ω_1 is completely contained in Ω_0 since

$$9\{(r_0 - r_1)^2 - d^2\} = (8r_1^2k^2)(z^2 - 1)\{2k^2z^2 - 2k(k^2 + 1)z - k^4 + 4k^2 - 1\} > 0 \tag{2.6}$$

where

$$9d^2 = r_1^2\{-16k^4z^4 + 16k^3(k^2 + 1)z^3 + 8k^2(k^2 - 1)^2z^2 \tag{2.7}$$
$$-16k^3(k^2 + 1)z + 9k^8 - 8k^6 + 14k^4 - 8k^2 + 9\}$$

Here the derivative of $9d^2$ with respect to z is equal to

$$16k^2r_1^2(k - z)(kz - 1)(k^2 + 4kz + 1)\ (\geq 0), \quad z \in [1/k, 1) \tag{2.8}$$

In practical determination of the cubic spiral (2.1), we have to solve (2.8) for z (i.e.,θ) for a given distance d between $C_i, i = 0, 1$. Then, we have the following theorem:

Theorem 2.1 *If*

$$\frac{\sqrt{9k^4 + 10k^2 + 17}}{3(k^2 + 1)}(r_0 - r_1) \leq d < (r_0 - r_1), \quad k = (r_0/r_1)^{1/4}, \tag{2.9}$$

then the T-cubic spline (2.1) is a spiral of monotone increasing curvature joining the two circles with one circle inside the other.

As in [4], require $0 < \theta < \pi/2$ to replace (2.10) for $k \geq \sqrt{2}$ with

$$\frac{\sqrt{9k^8 - 4k^6 + 2k^4 - 4k^2 + 9 - 4\sqrt{2}k^3(k^2+1)}}{3(k^4-1)}(r_0 - r_1) < d < r_0 - r_1 \tag{2.10}$$

Figure 1.1 is a numerical example of the two circles and the cubic spiral segment with $(r_0, r_1, d) = (4, 1, 2.9)$ where $\theta \approx 1.03645 (< \pi/2)$ and the spiral meets the smaller circle approximately at $(1.74559, 0.729055)$. Figure 1.2 is a numerical example of the two circles and the cubic spiral segment with $(r_0, r_1, d) = (9, 1, 7.55)$ where $\theta \approx 1.7811 (> \pi/2)$ and the spiral meets the smaller circle approximately at $(2.50566, 1.87791)$.

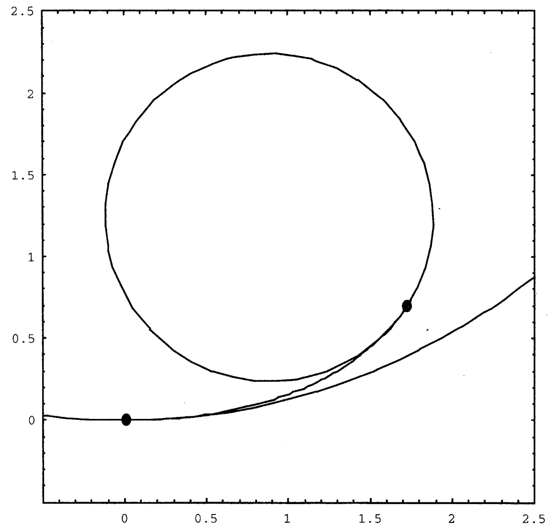

Figure 1.1. Circle to circle transition with T-cubic spiral

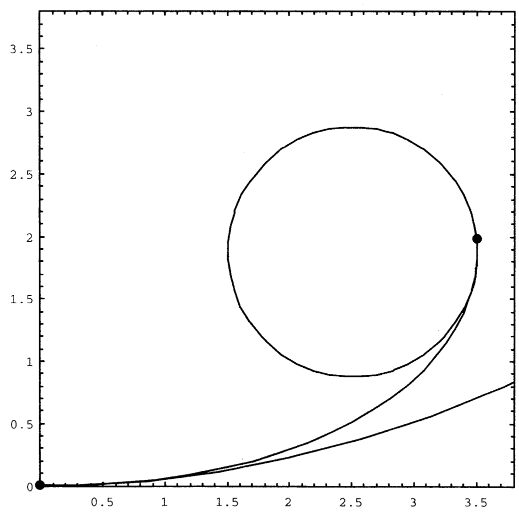

Figure 1.2. Circle to circle transition with T-cubic spiral

3 PH quintic spirals

The PH quintic segment $\boldsymbol{z}(t) = (x(t), y(t))$ is given by

$$x'(t) = U(t)^2 - V(t)^2, \quad y'(t) = 2U(t)V(t) \tag{3.1}$$

with

$$U(t) = a_0(1-t)^2 + 2a_1(1-t)t + a_2t^2, \quad V(t) = b_0(1-t)^2 + 2b_1(1-t)t + b_2t^2 \tag{3.2}$$

We require it to satisfy the following 6 conditions:

$$\begin{aligned} &\text{(i)} \quad (x(0), y(0)) = (0,0), \quad \text{(ii)} \quad \boldsymbol{z}'(0) \parallel (1,0), \quad \text{(iii)} \quad \kappa(0) = 0 \\ &\text{(iv)} \quad \kappa(1) = 1/r_1, \qquad \text{(v)} \quad \kappa'(1) = 0, \qquad \text{(vi)} \quad \boldsymbol{z}'(0) \parallel (\cos\theta, \sin\theta) \end{aligned} \tag{3.3}$$

Since it has eight parameters, we choose $a_0 = a_1$ as an additional one for a unique determination of it to have

$$\begin{aligned} x(t) &= \frac{7r_1 t \sin\theta \{3(32\cos^2\theta - 24\cos\theta - 7)t^4 + 70(4\cos\theta - 3)t^2 + 735\}}{960(1+\cos\theta)^2} (= r_1 f_1(t)) \\ y(t) &= \frac{7r_1 t^3 (1-\cos\theta)\{3(4\cos\theta - 3)t^2 + 35\}}{120(1+\cos\theta)} (= r_1 f_2(t)) \end{aligned} \tag{3.4}$$

Then, the curvature $\kappa(t)$ is given by

$$\kappa(t) = \frac{1024t(1+\cos\theta)^2}{r_1\{(25-24\cos\theta)t^4 + 14(4\cos\theta - 3)t^2 + 49\}^2} \tag{3.5}$$

where

$$\kappa'(t) = \frac{7168(1+\cos\theta)^2(1-t^2)\{(25-24\cos\theta)t^2 + 7\}}{r_1\{(25-24\cos\theta)t^4 + 14(4\cos\theta-3)t^2 + 49\}^3} \quad (\geq 0), \quad 0 \leq t \leq 1 \tag{3.6}$$

Thus the quintic segment (3.4) is always a spiral of monotone increasing curvature. In addition, the center (p_1, q_1) of the smaller circle Ω_1 is given by

$$\begin{aligned} p_1 &= \frac{r_1 \sin\theta(321 - 58\cos\theta - 36\cos^2\theta)}{120(1+\cos\theta)^2} (= r_1 g_1(\theta)) \\ q_1 &= \frac{r_1(91 + 11\cos\theta + 18\cos^2\theta)}{60(1+\cos\theta)} (= r_1 g_2(\theta)) \end{aligned} \tag{3.7}$$

Here, we assume that the segment (3.4) meets the larger circle Ω_0 at $t = m$ $(0 \leq m < 1)$ with the angle ψ from the X-axis to the tangent vector $\boldsymbol{z}(m)/\|\boldsymbol{z}(m)\|$. Then, (3.4) gives

$$\begin{aligned} &(8(4\cos\theta - 3)m^4 + 56m^2, (32cos^2\theta - 24\cos\theta - 7)m^4 + 14(4\cos\theta - 3)m^2 + 49) \\ &\parallel (\cos\psi, \sin\psi) \end{aligned} \tag{3.8}$$

from which follows

$$m = \sqrt{\frac{7\sin\frac{\psi}{2}}{4\sin(\theta-\frac{\psi}{2})+3\sin\frac{\psi}{2}}} \tag{3.9}$$

With help of *Mathematica* (if necessary),

$$\kappa(m)(=\frac{1}{r_0}) = \frac{1}{r_1}\sqrt{\frac{\sin\frac{\psi}{2}}{\sin\frac{\theta}{2}}}\left\{\frac{4\sin(\theta-\frac{\psi}{2})+3\sin\frac{\psi}{2}}{7\sin\frac{\theta}{2}}\right\}^{7/2} \tag{3.10}$$

or

$$\frac{r_1}{r_0} = \sqrt{\frac{\sin\frac{\psi}{2}}{\sin\frac{\theta}{2}}}\left\{\frac{4\sin(\theta-\frac{\psi}{2})+3\sin\frac{\psi}{2}}{7\sin\frac{\theta}{2}}\right\}^{7/2} \tag{3.11}$$

With $\alpha = m^{7/4}/k$, $\beta = (7/m^2-3)/4$, a combination of (3.7) and (3.9) gives

$$\sin\frac{\theta}{2} = \frac{1}{2}\sqrt{\frac{4-\alpha^2(1+\beta)^2}{1-\alpha^2\beta}}, \quad \sin\frac{\psi}{2} = \alpha\sin\frac{\theta}{2} \tag{3.12}$$

from which we have the implicit formulas for θ, ψ in terms of m on $m \in (m(k), 1)$ with $m(k$ (a unique root of $m^2+7 = 8km^{1/4}$ in $(0,1)$) as:

$$\begin{aligned} \theta &= 2\arccos\left[\frac{7(1-m^2)}{4m^{1/4}\sqrt{4k^2-7m^{3/2}+m^{7/2}}}\right] \\ \psi &= 2\arccos\left[\frac{8k^2-7m^{3/2}-m^{7/2}}{4k}\sqrt{4k^2-7m^{3/2}+m^{7/2}}\right] \end{aligned} \tag{3.13}$$

Then, the center (p_0, q_0) of Ω_0 is given by

$$\begin{aligned} p_0 &= \frac{7mr_1\sin\theta}{960(1+\cos\theta)^2}\Big\{3(32\cos^2\theta-24\cos\theta-7)m^4 \\ &\qquad +70(4\cos\theta-3)m^2+735\Big\} - r_0\sin\psi \\ q_0 &= \frac{7m^3r_1(1-\cos\theta)}{120(1+\cos\theta)}\Big\{3(4\cos\theta-3)m^2+35\Big\} + r_0\cos\psi \end{aligned} \tag{3.14}$$

Here, we consider the distance d between the centers (p_0, q_0) and (p_1, q_1) for $m \in (m(k), 1)$.
Case 1 ($m \to 1$): Letting $m = 1+u$, then

$$p_0 - p_1 = \sum_{i=0}^{5} a_i u^i, \quad q_0 - q_1 = \sum_{i=0}^{5} b_i u^i \tag{3.15}$$

where

$$\begin{aligned} a_0 &= r_0\sin\psi - r_1\sin\theta \\ a_1 &= \frac{7r_1}{4\cos\frac{\theta}{2}}(\sin\frac{\theta}{2} - \sin\frac{3\theta}{2}) \end{aligned}$$

$$
\begin{aligned}
a_2 &= \frac{7r_1}{8\cos\frac{\theta}{2}}(11\sin\frac{\theta}{2} - 4\sin\frac{3\theta}{2}) \\
a_3 &= \frac{7r_1}{96\cos^3\frac{\theta}{2}}(-14\sin\frac{\theta}{2} + 23\sin\frac{3\theta}{2} - 12\sin\frac{5\theta}{2}) \\
a_4 &= \frac{7r_1}{128\cos^3\frac{\theta}{2}}(-21\sin\frac{\theta}{2} + 20\sin\frac{3\theta}{2} - 8\sin\frac{5\theta}{2}) \\
a_5 &= \frac{1}{5}a_4; \\
b_0 &= r_1\cos\psi - r_0\cos\psi \\
b_1 &= -7r_1\sin^2\frac{\theta}{2} \\
b_2 &= \frac{-7r_1}{8}(8\cos\theta + 1)\tan^2\frac{\theta}{2} \\
b_3 &= \frac{7r_1}{24}(-24\cos\theta + 11)\tan^2\frac{\theta}{2} \\
b_4 &= \frac{7r_1}{8}(-4\cos\theta + 3)\tan^2\frac{\theta}{2} \\
b_5 &= \frac{1}{5}b_4
\end{aligned}
$$

Since

$$\tan\frac{\theta}{2} = \frac{\sqrt{64k^2\sqrt{m} - (m^2+7)^2}}{7(1-m^2)}, \tag{3.16}$$

we obtain

$$u\tan\frac{\theta}{2} \to -\frac{4\sqrt{k^2-1}}{7} \quad (m \to 1) \tag{3.17}$$

Hence we have

$$p_0 - p_1 \to \frac{8r_1}{3}(k^2-1)^{3/2}, \quad q_0 - q_1 \to r_1(-k^4 + 4k^2 - 3) \quad (m \to 1) \tag{3.18}$$

from which follows

$$d \to \frac{r_1}{3}\sqrt{9k^8 + 10k^2 + 17} \quad (m \to 1) \tag{3.19}$$

Case 2 ($m \to m(k)$): Letting $m \to m(k)$ gives $(p_0 - p_1, q_0 - q_1) \to (0, r_0 - r_1)$, i.e., $d \to r_0 - r_1$ $(m \to m(k))$. Since d is a continuous function of m on $(m(k), 1)$, the intermediate value theorem assures the existence of the solution m for a given d and so

Theorem 3.1 *If*

$$\frac{\sqrt{9k^8 + 10k^2 + 17}}{3(k^4-1)}(r_0 - r_1) < d < (r_0 - r_1), \quad k = (r_0/r_1)^{1/4}, \tag{3.20}$$

then the PH quintic spiral (3.4) of monotone increasing curvature joins the two circles with one circle inside the other.

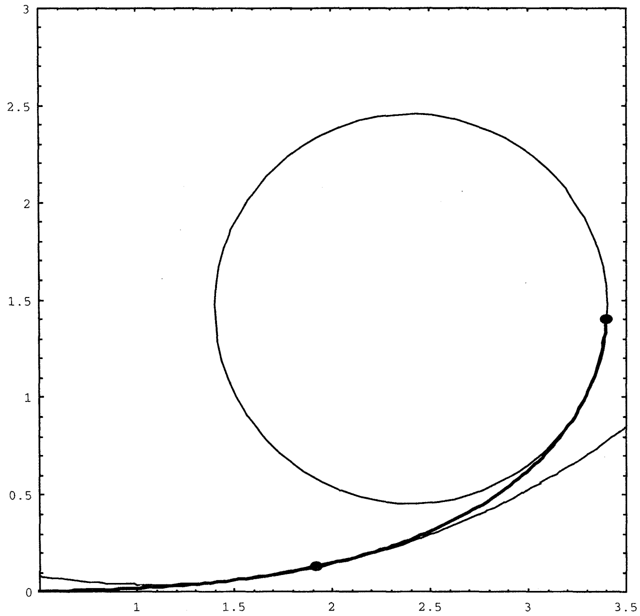

Figure 2. Circle to circle transition with PH quintic spiral.

In practical determination of the PH quintic spiral, we have to solve $d = \rho$ (ρ is a given distance satisfying (3.16)) for m by an iterative method, for example, bisection one. Figure 2 is a numerical example of a PH quintic spiral transition joining two circles with $(r_0, r_1) = (4, 1)$ when $d = 2.9$ requires
$(m, \theta, \psi) \approx (0.41086, 1.52119, 0.205919)$, $(p_1, q_1; p_0, q_0) \approx (2.40284, 1.45438; 1.08671, 4.03852)$, and the spiral and the circles meet approximately at $(1.90457, 0.123028), (3.40161, 1.40480)$.

Corollary 3. 1 *([4]). The curve segment $z_1(t), 0 \le t \le 1$ is a transit spiral joining the straight line (the positive part of the X-axis) and the circle centered (p_1, q_1) by with a radius r_1.*

This Corollary is of much use to the analysis of all the remaining cases (ii)-(iv) in [4]. Given another circle Ω_0 centered C_0 in the second or third quadrant with a radius r_0, then

(i) for two circles with a broken back C: $z_0(t) = r_0(-f_1(t), f_2(t))$ is a spiral joining the negative part of the X-axis (at the origin) and Ω_0 with centered $r_0(-g_1(\theta), g_2(\theta))$;

(ii) for two circles with an S: $z_0(t) = -r_0(f_1(t), f_2(t))$ is a spiral joining the negative part of the X-axis (at the origin) and Ω_0 centered $-r_0(g_1(\theta), g_2(\theta))$.

For the above cases (i) and (ii), we have the same determining equations in θ with the given distance d between the centers of the two circles, respectively as

$$(3.21) \qquad (r_1 + r_0)^2 g_1(\theta)^2 + (r_1 - r_0)^2 g_2(\theta)^2 = d^2, \ (r_1 + r_0)^2 (g_1(\theta)^2 + g_1(\theta)^2) = d^2$$

Next, given two non-parallel straight lines (which are the X-axis and $y = -(x - m)\tan\alpha, m > 0, 0 < \alpha < \pi$), we consider a spiral:

$$(3.22) \qquad z_3(t) = \begin{bmatrix} a \\ b \end{bmatrix} + r \begin{bmatrix} \cos\alpha & -\sin\alpha \\ -\sin\alpha & -\cos\alpha \end{bmatrix} \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}, \quad b = -(a - m)\tan\alpha$$

where for the curvature $\kappa_i(t), i = 1,3$ of $z_i(t)$,

$$\text{(3.23)} \quad \begin{aligned} &\text{(i)} \quad \kappa_i(0) = 0, \kappa_i'(1) = 0, i = 1,3; \quad \kappa_1(1) = -\kappa_3(1)(= 1/r) \\ &\text{(ii)} \quad z_1'(0) \parallel (1,0), \quad z_3'(0) \parallel (\cos\alpha, -\sin\alpha) \\ &\text{(iii)} \quad z_1'(1) \parallel (\cos\theta, \sin\theta), \quad z_3'(1) \parallel (\cos(\theta+\alpha), -\sin(\theta+\alpha)) \end{aligned}$$

As in [4], let $\theta = (\pi - \alpha)/2$ to have $z_1'(1) \parallel -z_3'(1)$. Therefore, the desired spiral (a pair of spirals z_1 and z_3) joining the non-parallel straight lines is obtained by $z_3(1) = z_1(1)$ from which

$$\text{(3.24)} \quad \frac{m}{r}(= f_1(1) + f_2(1)\tan\theta) = \frac{7\{75\sin(3\theta/2) - 23\sin(\theta/2)\}}{480\cos\theta\cos^3(\theta/2)}$$

Here, a value of r (or a parameter m) is left for a curve designer to use.

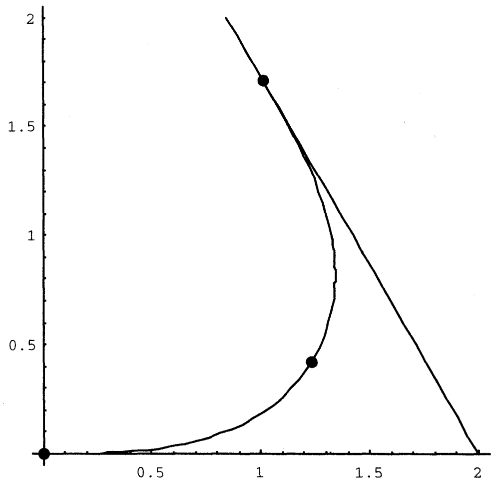

Figure 3. Straight line to straight line transition.

Figure 3 is an example for straight line to straight line transition with $\alpha = \pi/3$ where $m = 2$ and then $r \approx 0.70139$, (a, b) (the starting point of the spiral z_3) $\approx (1, 1.73205)$ and the join of the two spirals is about $(1.2440, 0.436422)$.

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