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A Note on an Asymptotic Behaviour of Coefficients of Loss in Limited Input Poisson Queueing Systems

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We consider queueing systems with Poisson input and exponential servicing times, which called servicing problems of machines with several repairmen. If the number of machines and repairmen increase under the condition that the ratio of machines to repairmen is constant, either one of a coefficient of loss for repairmen, loss for machines or both coefficients of loss decrease to zero compared with the constant and the servicing factor in the system.

§ 1 Introduction.

We start with the equations of the birth-death process. On specializing the coefficients of the equations in §2, we obtain queueing systems with Poisson input and exponential servicing times. As the steady-state solutions or limit probabilities are exist, we define the coefficient of loss for machines, for repairmen. Also we illustrate interesting numerical examples. In §3, we expose an asymptotic behaviour of coefficient of loss for machines, for reparimen when the number of machines and repairmen increase under the condition that the ratio of machines to repairmen is constant.

In conclusion, there are three cases under that condition;

1° *coefficient of loss for reparimen tends to zero, but that of loss for machines decreases to some positive value,*

2° *opposite case of* 1°,

3° *both coefficients of loss tend to zero.*

From these results, it is not always advantageous to both sides concerning about the coefficients to manage many machines by many repairmen cooperatively.

Palm (1958), Malcolm (1955), Fetter (1955) charge costs on a machine working, being serviced, idling, etc., and compute the table for the economically optimal number of machines per repairmen. But we do not refer to the costs here.

$\{E_n; n\geq 0\}$ denote states in the system. Let $P_n(t)$ be probability that at time t the system is in the state E_n. We postulate the following (a), (b), (c);

(a) the system changes only through trasitions from E_n to E_{n+1} or E_{n-1} if $n\geq 1$, but from E_0 to E_1 only,

(b) at any time t the system is in state E_n, the probability that during $(t, t+h)$ the transition $E_n \rightarrow E_{n+1}$ occurs equals $\lambda_n h+o(h)$, and that of $E_n \rightarrow E_{n+1}$ $(n\geq 1)$ equals $\mu_n h+o(h)$,

(c) the probability that during $(t, t+h)$ more than one change occurs is $o(h)$.

Accordingly we get differential equations called the birth-death equations;

$$(1) \qquad P_n'(t) = -(\lambda_n+\mu_n)P_n(t)+\lambda_{n-1}P_{n-1}(t)+\mu_{n+1}P_{n+1}(t) \quad (n\geq 1),$$

$$(2) \qquad P_0'(t) = -\lambda_0 P_0(t)+\mu_1 P_1(t).$$

In the next section, we specialize λ_n, μ_n in the above equation and obtain an appropriate description of servicing problems.

§ 2 Servicing Problem

We consider automatic machines which normally require no human care. However, at any time a machine may break down and call for service. The time required for servicing the machine is taken as an independent random variable with an exponential distribution. Machines are serviced by several repairmen. A machine which break down is serviced immediately unless the reparimen are servicing another machines, in which case a waiting line is formed. Let

m; total number of machines,
r ; number of repairmen,
λ ; arrival rate, i.e. $1/\lambda$ is the mean of the distribution of running times,
μ ; servicing rate, i.e. $1/\mu$ is the mean of the distribution of service times.

The machines working independently are characterized by λ and μ with the following properties. If at time t a machine is in working state, the probability that it will call for service before time $t+h$ is $\lambda h+o(h)$. Conversely, if at time t a machine is being serviced, the probability that the servicing time terminates before $t+h$ and the machine reverts to the working state is $\mu h+o(h)$. The ratio λ/μ is called the servicing factor.

We say that the system is in state E_n if n machines are not working, i.e. being serviced, which $r-n$ repairmen are idle for $n\leq r$. But for $n>r$ the state E_n signifies that r machines are being serviced and $n-r$ machines are in waiting line. Hence this situation is described by the birth-death equations (1), (2) with

$$(3) \qquad \lambda_n = (m-n)\lambda\,, \quad \mu_n = \begin{cases} n\mu & (n\leq r)\,, \\ r\mu & (r<n)\,. \end{cases}$$

It can be shown by Karlin and McGregor (1955) that the limits $\lim_{t\to\infty} P_n(t)=p_n$ exist. They satisfy the system of linear equations;

$$(4) \qquad m\lambda p_0 = \mu p_1\,,$$

$$(5) \qquad \{(m-n)\lambda+n\mu\}\,p_n = (m-n+1)\lambda p_{n-1}+(n+1)\mu p_{n+1} \quad (1\leq n<r)\,,$$

$$(6) \qquad \{(m-n)\lambda+r\mu\}\,p_n = (m-n+1)\lambda p_{n-1}+r\mu p_{n+1} \quad (r\leq n\leq m)\,.$$

Let us define a coefficient of loss for machines by

$$(7) \qquad \frac{w}{m} = \frac{\text{average number of machines in waiting line}}{\text{number of machines}} = \frac{\sum_{n=r}^{m}(n-r)p_n}{m}$$

and a coefficient of loss for repairmen by

$$(8)\quad \frac{\rho}{r}=\frac{\text{average number of repairmen idle}}{\text{number of repairmen}}=\frac{\sum_{n=0}^{r}(r-n)p_n}{r}$$

Numerical examples reveal surprising facts.

EXAMPLE 1 (cf. Feller (1970), Palm (1958)). Probabilities for $\lambda/\mu=0.1$, $m=6$, $r=1$.

n	Machines in Waiting Line	p_n
0	0	0.4845
1	0	.2907
2	1	.1454
3	2	.0581
4	3	.0174
5	4	.0034
6	5	.0003

EXAMPLE 2 (cf. Feller (1970), Palm (1958)). Probabilities for $\lambda/\mu=0.1$, $m=20$, $r=3$.

n	Machines in Waiting Line	p_n
0	0	0.1363
1	0	.2725
2	0	.2589
3	0	.1553
4	1	.0880
5	2	.0469
6	3	.0235
7	4	.0110
8	5	.0047
9	6	.0019
10	7	.0007
11	8	.0002

	Example 1	Example 2
Number of Machines	6	20
Number of Repairmen	1	3
Coefficient of Loss for Repairmen	0.48451	0.40420
Coefficient of Loss for Machines	0.05494	0.01694

A comparison of numerical example 1 and 2 proves that for machines with $\lambda/\mu=0.1$ 20 machines per 3 repairmen are ever so much economical than 6 machines per a repairman. Because the number of machines per repairmen has increased from 6 to $6^2/_3$, but at the same time, the machines are serviced more efficiently. Generally, one of the coefficients of loss decreases to zero, the other to some positive value. But in a critical case both decrease to zero. We will prove this in theorem 4, §3.

§ 3 Asymptotic behaviour of coefficients of loss

From equations (5), (6), ρ/r and w/m can be rewritten as follows.

LEMMA 1.

$$\frac{\rho}{r}=\left(1-\frac{\lambda}{\lambda+\mu}\frac{m}{r}\right)\sum_{n=0}^{r-1}p_n+\frac{\mu}{\lambda+\mu}p_r, \tag{9}$$

$$\frac{w}{m}=\left(1-\frac{\lambda+\mu}{\lambda}\frac{r}{m}\right)\sum_{n=r+1}^{m}p_n+\left(1-\frac{r}{m}\right)p_r, \tag{10}$$

and for a relation between ρ/r and w/m,

$$\frac{w}{m}-\frac{\lambda+\mu}{\lambda}\frac{r}{m}\frac{\rho}{r}=1-\frac{\lambda+\mu}{\lambda}\frac{r}{m} \tag{11}$$

or

$$\frac{\rho}{r}-\frac{\lambda}{\lambda+\mu}\frac{m}{r}\frac{w}{m}=1-\frac{\lambda}{\lambda+\mu}\frac{m}{r}.$$

Immediately lemma 1 follows that

THEOREM 2. *If $m/r>(\lambda+\mu)/\lambda$, then $\rho/r<w/m$,*
$m/r<(\lambda+\mu)/\lambda$, then $\rho/r>w/m$,
and if $m/r=(\lambda+\mu)/\lambda$, then $\rho/r=w/m$.

Next we consider the case that m, r increase to infinity. Preparations are needed for it.

LEMMA 3. *Let $m, r\to\infty$ with $m/r=c$ is constant. If the constant c satisfies* $(c>1)$

$$c>(\lambda+\mu)/\lambda, \tag{12}$$

then

$$\frac{\sum_{n=0}^{r-1}\binom{cr}{n}\left(\frac{\lambda}{\mu}\right)^n}{\sum_{n=r+1}^{cr}\frac{n!}{r^{n-r}r!}\binom{cr}{n}\left(\frac{\lambda}{\mu}\right)^n}\downarrow 0. \tag{13}$$

(14) *If $c<(\lambda+\mu)/\lambda$, then the left of* (13) *increases to infinity.*

(proof) Set the function of r;

$$T_1(r)=\sum_{n=0}^{r-1}\binom{cr}{n}\binom{cr}{r}^{-1}\left(\frac{\lambda}{\mu}\right)^{n-r}, \tag{15}$$

$$T_2(r)=\sum_{n=r+1}^{cr}\frac{n!}{r^{n-r}r!}\binom{cr}{n}\binom{cr}{r}^{-1}\left(\frac{\lambda}{\mu}\right)^{n-r} \tag{16}$$

with $c=m/r$, respectively. It follows easily that, letting $a=(c-1)\lambda/\mu$,

$$(17) \qquad T_1(r) = \sum_{n=0}^{r-1} \frac{1}{a^{r-n}} \frac{\left(1-\frac{1}{r}\right)\cdots\left(1-\frac{r-n-1}{r}\right)}{\left(1+\frac{r-n}{r}\frac{\lambda}{a\mu}\right)\cdots\left(1+\frac{1}{r}\frac{\lambda}{a\mu}\right)},$$

$$(18) \qquad T_2(r) = \sum_{n=r+1}^{cr} a^{n-r}\left(1-\frac{1}{r}\frac{\lambda}{a\mu}\right)\cdots\left(1-\frac{n-r-1}{r}\frac{\lambda}{a\mu}\right).$$

To prove the lemma, let us see how $T_1(r)$ and $T_2(r)$ behave as $r\uparrow\infty$.

1° If $a>1$, then $\limsup_{r\to\infty} T_1(r) \leq \frac{1}{a-1}$.

Because of (17), $T_1(r) < \sum_{n=0}^{r-1} \frac{1}{a^{r-n}} = \frac{1}{a-1}\frac{a^r-1}{a^r}$ for $r\geq 1$.

2° If $a<1$, then $\limsup_{r\to\infty} T_2(r) \leq \frac{a}{1-a}$.

Because of (18), $T_2(r) < \sum_{n=r+1}^{cr} a^{n-r} = \frac{a}{1-a}(1-a^{(c-1)r})$ for $r\geq 1$.

3° If $a\geq 1$, then $\lim_{r\to\infty} T_2(r)=\infty$.

Because, for each $1\leq k\leq (c-1)r$,

$$T_2(r) > a + a^2\left(1-\frac{1}{r}\frac{\lambda}{a\mu}\right) + \cdots + a^k\left(1-\frac{1}{r}\frac{\lambda}{a\mu}\right)\cdots\left(1-\frac{k-1}{r}\frac{\lambda}{a\mu}\right).$$

Letting $r\uparrow\infty$ with fix k, we obtain

$$\liminf_{r\to\infty} T_2(r) \geq a + a^2 + \cdots + a^k.$$

This holds for arbitrary k, hence $\lim_{r\to\infty} T_2(r)=\infty$.

4° If $a\leq 1$, then $\lim_{r\to\infty} T_1(r)=\infty$.

It can be shown similar to 3°.

Now we prove the lemma using the results of 1°~4°. Note that the left of (13) equals $T_1(r)/T_2(r)$ and the condition (12) is equivalent to $a>1$. From 1°, 3°,

$$0 \leq \limsup_{r\to\infty} T_1(r)/T_2(r) \leq \limsup_{r\to\infty} T_1(r)/\liminf_{r\to\infty} T_2(r) = 0.$$

Consequently $\lim_{r\to\infty} T_1(r)/T_2(r) = 0$, i.e. (13) holds.

Similarly since $c<\frac{\lambda+\mu}{\lambda}$ is equivalent to $a<1$,

$$\liminf_{r\to\infty} T_1(r) T_2(r) \geq \liminf_{r\to\infty} T_1(r)/\limsup T_2(r) = \infty.$$

Hence $\lim_{r\to\infty} T_1(r)/T_2(r) = \infty$. Those proved the lemma.

Finally we obtain the asymptotic behaviour of ρ/r, w/m *as follows.*

THEOREM 4. *Under that $m/r=c$ is constant, if*

(a) $c>\dfrac{\lambda+\mu}{\lambda}$, then $\rho/r\downarrow 0$, $w/m\downarrow 1-\dfrac{\lambda+\mu}{\lambda c}$

(b) $c<\dfrac{\lambda+\mu}{\lambda}$, then $\rho/r\downarrow 1-\dfrac{\lambda c}{\lambda+\mu}$, $w/m\downarrow 0$,

(c) $c=\dfrac{\lambda+\mu}{\lambda}$, then $\rho/r=w/m\downarrow 0$

as $m, r\to\infty$ respectively.

(proof) First we solve the steady-state equations (4), (5), (6) explicitely. They are

$$p_n=p_0\binom{m}{n}\left(\frac{\lambda}{\mu}\right)^n \quad (1\leqq n<r), \qquad p_n=p_0\frac{n!}{r^{n-r}r!}\binom{m}{n}\left(\frac{\lambda}{\mu}\right)^n \quad (r\leq n\leq m),$$

where p_0 is determined by $\sum_{n=0}^{m}p_n=1$. Using the notation in the proof of lemma 3, the sum p_n from 0 to r–1 is

$$\sum_{n=0}^{r-1}p_n=\frac{T_1(r)}{T_1(r)+T_2(r)+1}\,.$$

To prove (a), we observe that the condition is equivalent to $a>1$. Note that $a=(c-1)\lambda/\mu$ and from lemma 2,

$$\sum_{n=0}^{r-1}p_n\leq\frac{T_1(r)}{T_2(r)}\downarrow 0\,.$$

Consequently $\lim_{r\to\infty}\sum_{n=0}^{r-1}p_n=0$, $\lim_{r\to\infty}\sum_{n=r}^{cr}p_n=1$. Also $p_r=\dfrac{1}{1+T_1(r)+T_2(r)}\leq\dfrac{1}{T_2(r)}\downarrow 0$ as $r\uparrow\infty$. Hence the equations (9), (10) of lemma 1 yield that

$$\lim_{r\to\infty}\frac{\rho}{r}=0\,,\ \lim_{r\to\infty}\frac{w}{m}=1-\frac{\lambda+\mu}{\lambda c}\,.$$

(b) is proved similer to (a).

For (c), it means that $a=1$. Recall that, if $a=1$,

$$\frac{\rho}{r}=\frac{w}{m}=(1-c^{-1})\,p_r$$

by lemma 1 and $\lim_{r\to\infty}T_1(r)=\lim_{r\to\infty}T_2(r)=\infty$ by the proof 3°, 4° in lemma 3. Since $\lim_{r\to\infty}p_r=0$, it hold that $\rho/r=w/m\downarrow 0$ as $r\uparrow\infty$, which are desired results.

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