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ON CONFORMAL TRANSFORMATIONS OF WAGNER SPACES

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§ 0. Introduction.

The purpose of the present paper is to treat conformal transformations of generalized Berwald spaces (esp. Wagner spaces) and to show the following three theorems.

Theorem A. *A generalized Berwald space (esp. a Wagner space) remains to be a generalized Berwald space (esp. a Wagner space) by any conformal transformation.*

Theorem B. *The condition that a Finsler space be conformal to a Berwald space is that the space becomes a Wagner space with respect to a gradient* $a_i(x)$.

Theorem C. *The condition that a Finsler space be conformal to a locally Minkowskian space is that the space becomes a Wagner space with respect to a gradient* $a_i(x)$ *and its h-curvature tensor (in the sense of the Wagner space) vanishes. (In the above statement "h-curvature" may be replaced by "(v)h-torsion".)*

Recently, M. Hashiguchi (one of the authors) [7] treated the conformal theory of Finsler metrics and obtained the respective conditions that a Finsler space be conformal to a Berwald space and to a locally Minkowskian space. These conditions were, however, given in terms of very complicated systems of differential equations, for which appropriate geometrical meanings have been wanted. Theorems B and C give an answer about it by showing that the spaces in question construct a special class among Wagner spaces.

A *Wagner space* is the generalized Berwald space defined by V. Wagner [12], whereas a *generalized Berwald space* was defined by M. Hashiguchi [6] in a broader sense than Wagner's. On the other hand, Y. Ichijyō (the other author) [9, 10] obtained the notion of a $\{V, H\}$*-manifold* from the study about Finsler spaces modeled on a Minkowski space and showed that such a manifold is just a generalized Berwald space in the standard sense of M. Hashiguchi's. The generalized Berwald spaces contain various interesting examples [8,9] and are thought to be important Finsler spaces.

In §1 we shall first treat conformal transformations of a Berwald space. The consideration suggests us a typical transformation of a generalized Cartan connection,

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based on which Theorems A and B are proved (§2). Such a transformation is a generalization of the so-called *one-sided projective transformation* of a linear connection [5, 13] to the Finsler case, and is characterized as a transformation of a Finsler connection preserving *vectors parallel in direction*. Theorem C follows from the fact that such a transformation preserves the h-curvature tensor (resp. the $(v)h$-torsion tensor) under some conditions (§3). We shall provide §4 to state the relations between our theorems and the corresponding ones in [7] and to improve one result about two-dimensional Landsberg spaces.

This paper is a continuation of [7], and we shall usually retain the terminology and notation of [7] without comment, which is essentially based on the recent standard book [11] by M. Matsumoto.

The authors wish to express their sincere gratitude to Prof. Dr. M. Matsumoto for the invaluable suggestions and encouragement, who has organized the symposium of Finsler geometry every year and stimulated their joint studies.

§1. Berwald spaces and Wagner spaces.

1.1. An affinely connected Finsler space defined by L. Berwald [2, 3] is also called a *Berwald space*, which is the space whose connection coefficients (in the sense of L. Berwald [1]) depend on position alone. Such a space is also the one whose connection coefficients (in the sense of E. Cartan [4]) depend on position alone. V. Wagner [12] generalized the notion of the Cartan connection and called a Finsler space a generalized Berwald space if it is possible to introduce a generalized Cartan connection in such a way that the connection coefficients depend on position alone. A generalized Berwald space and a generalized Cartan connection introduced by V. Wagner have been called a Wagner space and a Wagner connection respectively in our papers [6,8].

1.2. Let $L(x, y)$ be a Finsler metric function, whose Finsler metric tensor is given by $g_{ij}=(L^2/2)_{(i)(j)}$. A Finsler connection is generally denoted by the coefficients $(F^i_{jk}, N^i_k, C^i_{jk})$. As shown in [6], a Wagner connection is characterized by the following four axioms.

(C1) It is *metrical*, i.e.,

$$(C1h)\quad g_{ij|k}=0\,,\qquad (C1v)\quad g_{ij}|_k=0\,.$$

(C2) The *deflection tensor* D vanishes identically, i.e.,

$$D^i_k=y^jF^i_{jk}-N^i_k=0\,.$$

(C3:s) It is *semi-symmetric*, i.e.,

$$T^i_{jk}=F^i_{jk}-F^i_{kj}=\delta^i_j s_k-\delta^i_k s_j$$

for some convariant vector field s_j.

(C4) The *$(v)v$-torsion tensor* S^1 vanishes identically, i.e.,

$$S^i_{jk}=C^i_{jk}-C^i_{kj}=0\,.$$

We shall call this the *Wagner connection with respect to* s_j. From Theorem 5 of [6] we have

Proposition 1.1. *Given a covariant vector field* s_j *on a Finsler space, there exists a unique Wagner connection with respect to* s_j *which we denote by* $(F^i_{jk}(s), N^i_k(s), C^i_{jk})$. *The coefficients are given by*

$$F^i_{jk}(s) = \Gamma^{*i}_{jk} + D^{ir}_{jk} s_r, \tag{1.1}$$

$$N^i_k(s) = G^i_k + D^{ir}_k s_r, \tag{1.2}$$

$$C^i_{jk} = 1/2\, g^{ir} g_{jk(r)}, \tag{1.3}$$

where $(\Gamma^{*i}_{jk}, G^i_k, C^i_{jk})$ *are the coefficients of the Cartan connection, and* D^{ir}_{jk}, D^{ir}_k *are expressed as*

$$D^{ir}_{jk} = U^{ir}_{jk} + \delta^i_j \delta^r_k, \tag{1.4}$$

$$D^{ir}_k = B^{ir}_k + y^i \delta^r_k \tag{1.5}$$

by conformal invariants U^{ir}_{jk}, B^{ir}_k *defined in* [7].

1.3. Let L be the metric function of a Berwald space, and let us consider whether the Berwald space may become a Wagner space by a conformal transformation α:

$$\bar{L} = e^{\alpha} L. \tag{1.6}$$

In the Finsler space with $\bar{L}$ a Wagner connection $(\bar{F}^i_{jk}(s), \bar{N}^i_k(s), \bar{C}^i_{jk})$ is given by

$$\bar{F}^i_{jk}(s) = \bar{\Gamma}^{*i}_{jk} + \bar{D}^{ir}_{jk} s_r, \tag{1.7}$$

$$\bar{N}^i_k(s) = \bar{G}^i_k + \bar{D}^{ir}_k s_r, \tag{1.8}$$

$$\bar{C}^i_{jk} = 1/2\, \bar{g}^{ir} \bar{g}_{jk(r)}. \tag{1.9}$$

We shall express these in terms of L. Since D^{ir}_{jk}, D^{ir}_k are conformal invariants, and it holds that

$$\bar{\Gamma}^{*i}_{jk} = \Gamma^{*i}_{jk} - U^{ir}_{jk} \alpha_r, \tag{1.10}$$

$$\bar{G}^i_k = G^i_k - B^{ir}_k \alpha_r, \tag{1.11}$$

where $\alpha_r = \partial\alpha/\partial x^r$, the above (1.7) and (1.8) become

$$\bar{F}^i_{jk}(s) = \Gamma^{*i}_{jk} - U^{ir}_{jk}\alpha_r + D^{ir}_{jk} s_r, \tag{1.12}$$

$$\bar{N}^i_k(s) = G^i_k - B^{ir}_k \alpha_r + D^{ir}_k \alpha_r \tag{1.13}$$

respectively. If we choose α_r as s_r, we have from (1.4) and (1.5)

$$\bar{F}^i_{jk}(\alpha) = \Gamma^{*i}_{jk} + \delta^i_j \alpha_k, \tag{1.14}$$

$$\bar{N}^i_k(\alpha) = G^i_k + y^i \alpha_k. \tag{1.15}$$

On the other hand, (1.9) becomes

$$\bar{C}^i_{jk} = C^i_{jk}. \tag{1.16}$$

Since Γ^{*i}_{jk} depend on position alone, $\bar{F}^i_{jk}(\alpha)$ depend on position alone. We shall call a Finsler space a *Wagner space with respect to* s_j if it is possible to introduce a Wagner connection with respect to s_j in such a way that the coefficients $F^i_{jk}(s)$ depend on position alone. In this case s_j should depend on position alone by virtue of (C3:s).

The above consideration tells us

Proposition 1.2. *By any conformal transformation* α *a Berwald space becomes a Wagner space with respect to the gradient* $\alpha_j(x)$.

§2. Conformal transformations of Wagner spaces.

2.1. *A generalized Cartan connection* is by the definition of M.Hashiguchi a Finsler connection satisfying the axioms (C1) and (C4). The axiom (C2) is imposed in the standard case, but not in this paper. By generalizing the transformation (1.14), (1.15), (1.16) of the Cartan connection, we can obtain

Proposition 2.1. *Let a generalized Cartan connection* $(F^i_{jk}, N^i_k, C^i_{jk})$ *be given in a space with a Finsler metric* L. *If for a conformal transformation* $\bar{L}=e^{\alpha}L$ *we put*

$$\bar{F}^i_{jk}=F^i_{jk}+\delta^i_j\alpha_k\,, \tag{2.1}$$

$$\bar{N}^i_k=N^i_k+y^i\alpha_k\,, \tag{2.2}$$

$$\bar{C}^i_{jk}=C^i_{jk}\,, \tag{2.3}$$

the coefficients $(\bar{F}^i_{jk}, \bar{N}^i_k, \bar{C}^i_{jk})$ *define a generalized Cartan connection in the space with the Finsler metric* $\bar{L}$.

The proof is easily obtained by checking the axioms (C1) and (C4).

2.2. Theorem A follows directly from Proposition 2.1 as follows. If F^i_{jk} depend on position alone, $\bar{F}^i_{jk}$ depend on position alone, too. A Finsler space is called a *generalized Berwald space* if it is possible to introduce a generalized Cartan connection in such a way that the coefficients F^i_{jk} depend on position alone. Thus we have proved that a generalized Berwald space remains to be a generalized Berwald space by any conformal transformation.

Especially, if $(F^i_{jk}, N^i_k, C^i_{jk})$ satisfies the axioms (C2), (C3: s) moreover, $(\bar{F}^i_{jk}, \bar{N}^i_k, \bar{C}^i_{jk})$ satisfies the axioms (C2), (C3: $s+\alpha$). Hence, as the rest of Theorem A, we have

Proposition 2.2. *By any conformal transformation* α *a Wagner space with respect to* s_j *becomes a Wagner space with respect to* $s_j+\alpha_j$.

2.3. Theorem B is proved by considering the converse of Proposition 1.2 as follows. Let us assume that a space with a Finsler metric L is a Wagner space with respect to a gradient $\alpha_j(x)$. By Proposition 2.2 a Finsler space with $\bar{L}=e^{-\alpha}L$ becomes a Wagner space with respect to the vanishing covariant vector field, which is nothing but a Berwald space. Thus the converse of Proposition 1.2 holds good, and Theorem B has been proved.

§3. One-sided projective transformations of Finsler connections.

3.1. For a moment we shall leave Finsler metrics and consider any Finsler connection $(F^i_{jk}, N^i_k, C^i_{jk})$ on a differentiable manifold M. It is noted that a Finsler connection is also given by the coefficients $(\Gamma^i_{jk}, N^i_k, C^i_{jk})$, where

$$\Gamma^i_{jk} = F^i_{jk} + C^i_{jm} N^m_k . \tag{3.1}$$

Let $C=(x^i(t))$ be a differentiable curve in M and $\tilde{C}=(x^i(t), y^i(t))$ be a differentiable curve over C in the tangent bundle of M. Tangent vectors $X(t)$ along C are called *parallel in direction along C with respect to $\tilde{C}$*, if the system of differential equations

$$\dot{X}^i + \Gamma^i_{jk}(x, y) X^j \dot{x}^k + C^i_{jk}(x, y) X^j \dot{y}^k = \lambda X^i \tag{3.2}$$

is satisfied for some λ, where a dot means d/dt.

Suggested by the transformation (2.1), (2.2), (2.3) of a generalized Cartan connection, we shall consider a transformation of a Finsler connection as follows:

$$\bar{F}^i_{jk} = F^i_{jk} + \delta^i_j s_k , \tag{3.3}$$

$$\bar{N}^i_k = N^i_k + y^i s_k , \tag{3.4}$$

$$\bar{C}^i_{jk} = C^i_{jk} , \tag{3.5}$$

where s_k is some covariant vector field. Assuming the C_2-*condition* $C^i_{jk}\, y^k=0$ for the given Finsler connection, it is easily seen that the above transformation of a Finsler connection preserves vectors parallel in direction.

Conversely, if two Finsler connections $(F^i_{jk}, N^i_k, C^i_{jk})$ and $(\bar{F}^i_{jk}, \bar{N}^i_k, \bar{C}^i_{jk})$ make any vectors parallel in direction at the same time, we have in the same way as in [5,13]

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j s_k , \tag{3.6}$$

$$\bar{C}^i_{jk} = C^i_{jk} + \delta^i_j \tilde{s}_k , \tag{3.7}$$

where $s_k=(\bar{\Gamma}^h_{hk}-\Gamma^h_{hk})/n$, $\tilde{s}_k=(\bar{C}^h_{hk}-C^h_{hk})/n$. Assuming the axiom (C4) for these Finsler connections, it is shown that (3.7) becomes (3.5) and s_k is a covariant vector field. If we assume the axiom (C2) and the C_2-condition moreover, the C_1-*condition* $y^j C^i_{jk}=0$ is also satisfied by (C4) and we have (3.4), and the C_2-condition yields (3.3). Thus we have proved

Proposition 3.1. *Let us assume that the used Finsler connections satisfy the axioms* (C2), (C4) *and the C_2-condition. Transformations of a Finsler connection given by* (3.3), (3.4), (3.5) *are the most general ones preserving any vectors parallel in direction.*

Returning to general Finsler connections, we shall call a transformation of a Finsler connection by (3.3), (3.4), (3.5) the *one-sided projective transformation with respect to s_j.*

3.2. We shall investigate how the torsion tensors and the curvature tensors change by an one-sided projective transformation. From Proposition 3.1 of [7] or directly from the definitions (2.11)~(2.18) of [7] we have

Proposition 3.2. *By an one-sided projective transformation of any Finsler connec-*

tion, the torsion tensors T^i_{jk}, R^i_{jk} *and the curvature tensor* R^i_{hjk} *are changed as follows*:

$$\bar{T}^i_{jk} = T^i_{jk} + \mathfrak{S}_{jk}\{\delta^i_j s_k\}\,, \tag{3.8}$$

$$\bar{R}^i_{jk} = R^i_{jk} + \mathfrak{S}_{jk}\{y^i\,\partial s_j/\partial x^k - (N^i_{j(m)}\,y^m - N^i_j)\,s_k\}\,, \tag{3.9}$$

$$\bar{R}^i_{hjk} = R^i_{hjk} + \mathfrak{S}_{jk}\{\delta^i_h\,\partial s_j/\partial x^k - (F^i_{hj(m)}\,y^m + C^i_{hm}(N^m_{j\ (r)}\,y^r - N^m_j))\,s_k\}, \tag{3.10}$$

and the others C^i_{jk}, P^i_{jk}, S^i_{jk}, P^i_{hjk} *and* S^i_{hjk} *remain unchanged.*

If the Finsler connection is positively homogeneous, (3.9) *and* (3.10) *become*

$$\bar{R}^i_{jk} = R^i_{jk} + y^i\,\mathfrak{S}_{jk}\{\partial s_j/\partial x^k\}\,, \tag{3.9'}$$

$$\bar{R}^i_{hjk} = R^i_{hjk} + \delta^i_h\,\mathfrak{S}_{jk}\{\partial s_j/\partial x^k\} \tag{3.10'}$$

respectively. Especially, in the case that the transformation is with respect to a gradient s_j, R^i_{jk} *and* R^i_{hjk} *remain unchanged also*:

$$\bar{R}^i_{jk} = R^i_{jk}\,, \tag{3.9''}$$

$$\bar{R}^i_{hjk} = R^i_{hjk}\,. \tag{3.10''}$$

3.3. Theorem C is obtained directly from Theorem B if we pay attention to the last case in Proposition 3.2.

A *locally Minkowskian space* is a Berwald space whose h-curvature tensor R^i_{hjk} (in the sense of the Cartan connection) vanishes. In the previous paper [7] we called such a space a Minkowski space briefly, but in order to avoid the confusion with the global one by Y. Ichijyō, we use the above terminology.

Now, let us assume that a space with a Finsler metric L is a Wagner space with respect to a gradient $a_j(x)$, which becomes a Berwald space with $\bar{L}$ by the conformal transformation $\bar{L}=e^{-\alpha}L$ as shown in 2.3. Let $\bar{R}^i_{hjk}$ and $R^i_{hjk}(a)$ be the respective h-curvature tensors of the Berwald space (in the sense of the Cartan connection) and of the original Wagner space (in the sense of the Wagner connection).

The Berwald space is a locally Minkowskian space if and only if $\bar{R}^i_{hjk}=0$, which is equivalent to $R^i_{hjk}(a)=0$ because of (3.10″). Thus Theorem C has been proved. (In a Berwald space $\bar{R}^i_{hjk}=0$ is equivalent to $\bar{R}^i_{jk}=0$. Hence, in Theorem C the h-curvature tensor $R^i_{hjk}(a)$ may be replaced by the $(v)h$-torsion tensor $R^i_{jk}(a)$ because of (3.9″).)

§4. Some remarks on the previous paper [7].

4.1. In Theorem 4.7 of [7] the condition that a Finsler space be conformal to a Berwald space was given as the existence of a solution α of the system of differential equations

$$G^i_{jkl} - B^{ir}_{jkl}\alpha_r = 0\,, \tag{4.1}$$

which expresses, in terms of the Berwald connection, the condition that the coefficients $F^i_{jk}(-a)$ of the Wagner connection with respect to $-a_j$ depend on position alone. Thus Theorem B restates Theorem 4.7 of [7].

4.2. In Theorem 4.8 of [7] the condition that a Finsler space be conformal to a locally Minkowskian space was given as the existence of a solution a of the system of the differential equations (4.1) and the following (4.2):

$$R^i_{jk}-\mathfrak{S}_{jk}\{(B^{ir}_j a_r)_{;k}+B^{ir}_{jm}B^{ms}_k a_r a_s\}=0. \tag{4.2}$$

The left-hand member of (4.2) is the $(v)h$-torsion tensor $R^i_{jk}(-a)$ of the Wagner connection with respect to $-a_j$ written in terms of the Berwald connection. Thus Theorem C restates Theorem 4.8 of [7].

4.3. Wandering from our subject, we shall remember two theorems of [7] that *if a two-dimensional Landsberg space remains to be a Landsberg space by a non-homothetic conformal transformation, the main scalar I is at most a point function* (Theorem 4.5) and *if a two-dimensional Berwald (esp. locally Minkowskian) space remains to be a Landsberg space by a non-homothetic conformal transformation, the main scalar is constant* (Theorem 4.6).

In a two-dimensional Landsberg space, however, it is known that the main scalar becomes constant if only it is at most a point function. So, the conclusion of Theorem 4.5 should be replaced by "*the main scalar I is constant*". Hence, Theorem 4.6 is contained in Theorem 4.5 and is omitted.

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