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By

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Dedicated to Professor Tatsji Kudo on his 60th birthday

0. Introduction

There are known some convenient categories for topology. We consider two of them here. In [6] R.M. Vogt established the category $\mathscr{K}$ consisting of k-spaces and continuous maps, and proved the exponential laws in $\mathscr{K}$. On the other hand, Y. Kawahara and T. Kudo [3] offered the category $\mathscr{IC}$ consisting of topological spaces and $\mathscr{I}$- maps. They proved $\mathscr{IC}$ to be cartesian closed, i.e., there is a function space functor k: $\mathscr{IC}^{op}\times\mathscr{IC}\longrightarrow\mathscr{IC}$ such that $k(-,X)$: $\mathscr{IC}\longrightarrow\mathscr{IC}$ is a right adjoint for $-\times X$: $\mathscr{IC}\longrightarrow\mathscr{IC}$.

It is remarkable that in $\mathscr{IC}$ we can treat every topological space without changing its topology.

In this note we study the category $\mathscr{IC}$ and the relation between $\mathscr{IC}$ and $\mathscr{K}$. Our purpose is to find another function space functor k': $\mathscr{IC}^{op}\times\mathscr{IC}\longrightarrow\mathscr{IC}$ which has the similar properties to that of k. Furthermore it enables us to get the cartesian closedness and exponential laws of $\mathscr{K}$ in the sence of R.M. Vogt.

1. Categories ($\mathscr{IC}$, $\mathscr{IG}$) and functors (k, κ)

Let $\mathscr{A}$ be the category of topological spaces and set maps, let $\mathscr{C}$ be of topological spaces and continuous maps. Let $\mathscr{I}$ be a full subcategory of $\mathscr{C}$ containing at least one non-empty space. In addition, $\mathscr{I}$ is required in Sec. 3 and 4 to have two properties (Axiom (**a**) and (**b**)). A map f: $X\longrightarrow Y$ in $\mathscr{A}$ is called $\mathscr{I}$-map if $f\alpha\in\mathscr{C}$ for any $\alpha\in\mathscr{C}(A, X)$ with $A\in\mathscr{I}$. By $\mathscr{IC}$ we denote the category of topological spaces and $\mathscr{I}$-maps. It is trivial that $f\in\mathscr{C}$ means $f\in\mathscr{IC}$. A space $X\in\mathscr{A}$ is called $\mathscr{I}$-generated if $\mathscr{IC}(X, Y)=\mathscr{C}(X, Y)$ for all $Y\in\mathscr{A}$. By $\mathscr{IG}$ we denote the full subcategory of $\mathscr{C}$ with objects $\mathscr{I}$-generated spaces. (Note that $\mathscr{IG}=\mathscr{K}$.) The inclusion relations of these categories are as follows,

$$\begin{array}{ccccc} & & \mathscr{IC} & \overset{j}{\subset} & \mathscr{A} \\ & & \mathit{full}\ \cup & & \cup\ i \\ \mathscr{I} & \underset{\mathit{full}}{\subset} & \mathscr{IG} & \underset{\mathit{full}}{\subset} & \mathscr{C} \end{array}$$

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We can easily verify that the finite products in $\mathscr{C}$, in $\mathscr{J}\mathscr{C}$ and in $\mathscr{A}$ coincide (because i, j and ji preserve finite limits).

Let k: $\mathscr{C}\longrightarrow\mathscr{J}\mathscr{G}$ be the functor in the sence of R.M. Vogt [6], i.e., for any $X\in\mathscr{C}$ $k(X)$ is the set X with the induced topology determined by the family $\{A\xrightarrow{\alpha}X\,|\,\alpha\in\mathscr{C}, A\in\mathscr{J}\}$ and for any $f\in\mathscr{C}(X, Y)$ $k(f)(\in\mathscr{J}\mathscr{G}(k(X), k(Y)))$ is equal to f as set maps. It is an easy consequence of the definition that $k(X)=X$ for all $X\in\mathscr{J}\mathscr{G}$ and the identity function id_X: $k(X)\longrightarrow X$ lies in $\mathscr{C}$ for all $X\in\mathscr{C}$.

The following proposition is due to R.M. Vogt.

Proposition 1. *For any* $B\in\mathscr{J}$ and $X\in\mathscr{C}$ *there is a natural isomorphism* $\mathscr{C}(B, X)\cong\mathscr{J}\mathscr{G}(B, k(X))$; $f\longmapsto f'$ where $f'=f$ as set maps.

Proposition 2. *For any* X, $Y\in\mathscr{A}$ there is a natural isomorphism, κ: $\mathscr{J}\mathscr{C}(X, Y)\longrightarrow\mathscr{J}\mathscr{G}(k(X), k(Y))$; $\kappa(f)=f$ as set maps for any $f\in\mathscr{J}\mathscr{C}(X, Y)$.

Proof. We need only prove that $\kappa(f)$ lies in $\mathscr{J}\mathscr{G}$. It suffices to show that $\kappa(f)\alpha'\in\mathscr{C}$ for any $\alpha'\in\mathscr{C}(B, k(X))$ with $B\in\mathscr{J}$. Consider the following commutative diagram,

$$\begin{array}{ccccc} & \nearrow^{\alpha'} & k(X) & \xrightarrow{\kappa(f)} & k(Y) \\ B & & \downarrow id_X & f & \downarrow id_Y \\ & \searrow_{\alpha} & X & \longrightarrow & Y \end{array}$$

By Proposition 1 $\alpha\in\mathscr{C}$. Therefore $f\alpha\in\mathscr{C}$. Using Proposition 1 again, we get that $\kappa(f)\alpha'\in\mathscr{C}$.

The last result enables us to extend the functor k: $\mathscr{C}\longrightarrow\mathscr{J}\mathscr{G}$ over $\mathscr{J}\mathscr{C}$, i.e., there is a functor κ: $\mathscr{J}\mathscr{C}\longrightarrow\mathscr{J}\mathscr{G}$ and $k=\kappa i$. On the other hand we have the fact as follows (due to R.M. Vogt).

Proposition 3. *$\mathscr{J}\mathscr{G}$ is complete and cocomplete.*

To combine Proposition 3 with Proposition 2 yields the following.

Proposition 4. *$\mathscr{J}\mathscr{C}$ is complete and cocomplete.*

The next properties follow immediately from [6] (1. 2), [3] (example 1) and Proposition 2.

Proposition 5. (i) *X is isomorphic to Y in $\mathscr{J}\mathscr{C}$ if and only if $\kappa(X)$ is isomorphic to $\kappa(Y)$ in $\mathscr{J}\mathscr{G}$.*
(ii) *$\kappa(X)$ is isomorphic to X in $\mathscr{J}\mathscr{C}$. More precisely id: $\kappa(X)\longrightarrow X$ lies in $\mathscr{C}$ and id^{-1} lies in $\mathscr{J}\mathscr{C}$.*
(iii) *$\kappa(X)=X$ for any $X\in\mathscr{J}\mathscr{G}$.*
(iv) *Let X be isomorphic to Y in $\mathscr{J}\mathscr{C}$, then the homotopy groups and singular(co) homology groups of them are isomorphic for suitable $\mathscr{J}$.*

2. Function space $k'(X, Y)$

In this section we shall define a function space functor k': $\mathscr{J}\mathscr{C}^{op}\times\mathscr{J}\mathscr{C}\longrightarrow\mathscr{J}\mathscr{C}$ and two natural transformations related to k'. They play the important roles in this note.

Let k': $\mathcal{IC}^{OP}\times\mathcal{IC}\longrightarrow\mathcal{IC}$ be a functor defined by

$$k'(X, Y)=F(\kappa(X), \kappa(Y)) \qquad X, Y\in\mathcal{IC}$$

$$k'(f,g)=F(\kappa(f), \kappa(g)) \qquad (f,g)\in\mathcal{IC}^{OP}(X, X')\times\mathcal{IC}(Y, Y')$$

where $F(-,-)$ means the ordinal function space functor with compact open topology.

Let L, M, N, P: $\mathcal{IC}^{OP}\times\mathcal{IC}^{OP}\times\mathcal{IC}\longrightarrow Set$ be the functors defined by the equations,

$$L(X, Y, Z)=\mathcal{IC}(X\times Y, Z)$$

$$M(X, Y, Z)=\mathcal{A}(X, k'(Y, Z))$$

$$N(X, Y, Z)=\mathcal{IC}(X, k'(Y, Z))$$

$$P(X, Y, Z)=\mathcal{A}(X\times Y, Z).$$

By φ: $L\longrightarrow M$ we denote a natural transformation as follows, $\varphi_{X,Y,Z}(f)=\bar{f}$; $\bar{f}(x)(y)=f(x,y)$ with $f\in L(X, Y, Z)$ $x\in X$ and $y\in Y$. By ψ: $N\longrightarrow P$ we denote a natural transformation as follows, $\psi_{X,Y,Z}(h)=\tilde{h}$; $\tilde{h}(x, y)=h(x)(y)$ with $h\in N(X, Y, Z)$, $x\in X$ and $y\in Y$.

It is easy to see that φ and ψ are well defined and natural.

Remark. The naturalities of φ and ψ yield the formulas (**B**. 1)$\sim$(**B**. 6) of [3]. They are used frequently to prove many properties.

(**B**.1) $\bar{f}a=\overline{f(a\times Y)}$ (**B**.2) $\bar{f}k'(b, Z)=\overline{f(X\times b)}$ (**B**.3) $k'(Y, c)\bar{f}=\overline{cf}$

(**B**.4) $\widetilde{ha}=\tilde{h}(a\times Y)$ (**B**.5) $\widetilde{k'(b, Z)h}=\tilde{h}(X\times b)$ (**B**.6) $\widetilde{k'(Y, c)h}=c\tilde{h}$

where $f\in L(X, Y, Z)$, $h\in N(X, Y, Z)$, $a\in\mathcal{IC}(X', X)$, $b\in\mathcal{IC}(Y', Y)$ and $c\in\mathcal{IC}(Z, Z')$.

3. ($\mathcal{I}$-) admissible and ($\mathcal{I}$-) proper

Definition 6. (i) *A space Y in $\mathcal{A}$ is said to be $\mathcal{I}$-admissible if*
$\psi_{X,Y,Z}(\mathcal{IC}(X, k'(Y, Z))\subset\mathcal{IC}(X\times Y, Z)$ *for all* $X, Z\in\mathcal{A}$.
(ii) *A space Y in $\mathcal{A}$ is said to be admissible if*
$\psi_{X,Y,Z}(\mathcal{C}(X, k'(Y, Z))\subset\mathcal{C}(X\times Y, Z)$ *for all* $X, Z\in\mathcal{A}$.

Let $\Psi=\{\Psi_{X,Y,Z}$: $\mathcal{C}(X, F(Y, Z))\rightarrow\mathcal{A}(X\times Y, Z)\}_{X,Y,Z\in\mathcal{A}}$ be a natural transformation defined by $\Psi_{X,Y,Z}(h)=\check{h}$, where $\check{h}(x, y)=h(x)(y)$ with $x\in X, y\in Y$ and $h\in\mathcal{C}(X, F(Y, Z))$.
(iii) *A space Y in $\mathcal{A}$ is said to be $\mathcal{C}$-admissible if*
$\Psi_{X,Y,Z}(\mathcal{C}(X, F(Y, Z))\subset\mathcal{C}(X\times Y, Z)$ *for all* $X, Z\in\mathcal{A}$.

Proposition 7. (i) *A space Y is $\mathcal{I}$-amdissilbe if and only if $\varepsilon_{Y,Z}\in\mathcal{IC}(k'(Y, Z)\times Y, Z)$ for all $Z\in\mathcal{A}$, where $\varepsilon_{Y,Z}=\widetilde{k'(Y, Z)}$.*
(ii) *A space Y is admissible if and only if $\varepsilon_{Y,Z}\in\mathcal{C}(k'(Y, Z)\times Y, Z)$ for all $Z\in A$.*
(iii) *A space Y is $\mathcal{C}$-admissible if and only if $e_{Y,Z}\in\mathcal{C}(F(Y, Z)\times Y, Z)$ for all $Z\in\mathcal{A}$, where $e_{Y,Z}=F(Y, Z)^{\vee}$.*

Proof. (i) $\varepsilon_{Y,Z}$ has the following (universal) property: For any $h\in\mathcal{IC}(X, k'(Y, Z))$ $\tilde{h}=\varepsilon_{Y,Z}(h\times Y)$. Hence, if $\varepsilon_{Y,Z}$ lies in $\mathcal{IC}$, then $\tilde{h}$ lies also in $\mathcal{IC}$. "Only if" part is trivial from the definition. In similar fashion (ii) and (iii) are easily verified.

This leads to the next.

Proposition 8. *If Y is admissible, then Y is $\mathscr{I}$-admissible.*

Proposition 9. *If Y is in $\mathscr{I}\mathscr{C}$ and $\mathscr{C}$-admissible, then Y is admissible.*

Proof. By Proposition 7, it suffices to show that $\varepsilon_{Y,Z}$ lies in $\mathscr{C}$ for any $Z\in\mathscr{A}$. Since $\mathscr{C}(k'(Y,Z), k'(Y,Z))=\mathscr{C}(k'(Y,Z), F(Y,\kappa(Z)))$, $k'(Y,Z)^{\vee}$ lies in $\mathscr{C}(k'(Y,Z)\times Y, \kappa(Z))$. On the other hand, we have $\varepsilon_{Y,Z}=id_Z k'(Y,Z)^{\vee}$. It completes the proof.

Axiom (a) *Any A in $\mathscr{I}$ is admissible.*

Proposition 10. *If $\mathscr{I}$ satisfies* (a), *then any Y in $\mathscr{A}$ is $\mathscr{I}$-admissible.*

Proof. By Proposition 7 we need only prove that the composite $A\xrightarrow{\langle\alpha,\beta\rangle}k'(Y,Z)\times Y\xrightarrow{\varepsilon_{Y,Z}}Z$ lies in $\mathscr{C}$ for all $\langle\alpha,\beta\rangle\in\mathscr{C}(A,k'(Y,Z)\times Y)$ with $A\in\mathscr{I}$, where $\langle\alpha,\beta\rangle$ is the unique map determined by $\alpha\in\mathscr{C}(A,k'(Y,Z))$ and $\beta\in\mathscr{C}(A,Y)$. By (B.4) $\sim$ (B.6) we have, $\varepsilon_{Y,Z}\langle\alpha,\beta\rangle=\varepsilon_{Y,Z}(k'(Y,Z)\times\beta)\langle\alpha,A\rangle=\widetilde{k'(\beta,Z)}\langle\alpha,A\rangle=\varepsilon_{A,Z}(k'(\beta,Z)\times A)\langle\alpha,A\rangle$. Since $\varepsilon_{A,Z}\in\mathscr{C}$ (Axiom (a)), we get the desired consequence.

The last result means that if $\mathscr{A}$ satisfies (a) then ψ is a natural transformation from N to L.

Definition 11. (i) *A space Y in $\mathscr{A}$ is said to be $\mathscr{I}$-proper if*
$\varphi_{X,Y,Z}(\mathscr{I}\mathscr{C}(X\times Y,Z))\subset\mathscr{I}\mathscr{C}(X,k'(Y,Z))$ *for all* $X,Z\in\mathscr{A}$.
(ii) *A space Y in $\mathscr{A}$ is said to be proper if*
$\varphi_{X,Y,Z}(\mathscr{C}(X\times Y,Z))\subset\mathscr{C}(X,k'(Y,Z))$ *for all* $X,Z\in\mathscr{A}$.

Let $\Phi=\{\Phi_{X,Y,Z}\colon\mathscr{C}(X\times Y,Z)\to\mathscr{A}(X,F(Y,Z))\}_{X,Y,Z\in\mathscr{A}}$ be a natural transformation defined by $\Phi_{X,Y,Z}(f)=\hat{f}$, where $\hat{f}(x)(y)=f(x,y)$ with $x\in X, y\in Y$ and $f\in\mathscr{C}(X\times Y,Z)$.
(iii) *A space Y in $\mathscr{A}$ is said to be $\mathscr{C}$-proper if*
$\Phi_{X,Y,Z}(\mathscr{C}(X\times Y,Z))\subset\mathscr{C}(X,F(Y,Z))$ *for all* $X,Z\in\mathscr{A}$.
It is a well known fact that any Y in $\mathscr{A}$ is $\mathscr{C}$-proper.

Proposition 12. (i) *A space Y is $\mathscr{I}$-proper if and only if* $\eta_{X,Y}\in\mathscr{I}\mathscr{C}(X,k'(Y,X\times Y))$ *for all* $X\in\mathscr{A}$, *where* $\eta_{X,Y}=\overline{X\times Y}$.
(ii) *A space Y is proper if and only if* $\eta_{X,Y}\in\mathscr{C}(X,k'(Y,X\times Y))$ *for all* $X\in\mathscr{A}$.

Proof. (i) $\eta_{X,Y}$ has the following (universal) property: For any $f\in\mathscr{I}\mathscr{C}(X\times Y,Z)$ $\bar{f}=\eta_{X,Y}k'(Y,f)$. Hence, if $\eta_{X,Y}$ lies in $\mathscr{I}\mathscr{C}$, then $\bar{f}$ lies also in $\mathscr{I}\mathscr{C}$. "Only if" part follows immediately from the definition. The proof of (ii) is similar to that of (i).

Axiom (b). *If A, $B\in\mathscr{I}$, then $A\times B\in\mathscr{I}\mathscr{C}$.*

Proposition 13. *If $\mathscr{I}$ satisfies* (a) *and* (b), *then $K\times A\in\mathscr{I}\mathscr{C}$ for any $A\in\mathscr{I}$ and $K\in\mathscr{I}\mathscr{C}$.*

Proof. We have to prove $f\in\mathscr{C}$ for any $f\in\mathscr{I}\mathscr{C}(K\times A,Z)$. If $\bar{f}\in\mathscr{C}(K,k'(A,Z))$, then $\tilde{\bar{f}}=f\in\mathscr{C}$ by (a). Hence we need only verify that $\bar{f}\in\mathscr{I}\mathscr{C}(K,k'(A,Z))=\mathscr{C}(K,k'(A,Z))$.

Suppose that $B\in\mathcal{J}$ and $\alpha\in\mathcal{C}(B, K)$, then $\bar{f}\alpha=\overline{f(\alpha\times A)}$ by (**B**. 1). On the other hand $f(\alpha\times A)\in\mathcal{C}$ by (**b**). We have the following commutative diagram,

$$\begin{array}{rccc}
 & \mathcal{J}\mathcal{C}(B\times A, Z) & \xrightarrow{\ \varphi\ } & \mathcal{A}(B, k'(A, Z)) \\
\mathcal{J}\mathcal{C}(B\times A, id_Z^{-1}) & \downarrow & & \| \\
 & \mathcal{J}\mathcal{C}(B\times A,\kappa(Z)) & \xrightarrow{\ \varphi\ } & \mathcal{A}(B, k'(A, \kappa(Z))) \\
 & \| & & \cup \\
 & \mathcal{C}(B\times A,\kappa(Z)) & \xrightarrow{\ \Phi\ } & \mathcal{C}(B, F(A, \kappa(Z))) .
\end{array}$$

Therefore $\bar{f}\alpha=\overline{f(\alpha\times A)}=\widehat{id_Z^{-1}f(\alpha\times A)}\in\mathcal{C}$.

PROPOSITION 14. *If $\mathcal{J}$ satisfies* (**a**) *and* (**b**), *then any Y in $\mathcal{A}$ is $\mathcal{J}$-proper.*

PROOF. By Proposition 12 we need only prove that $\eta_{X,Y}\in\mathcal{J}\mathcal{C}$ for any $X\in\mathcal{A}$. Let $A\in\mathcal{J}$ and $\alpha\in\mathcal{C}(A,\kappa(X))$. It follows from Proposition 13 that $\mathcal{J}\mathcal{C}(A\times\kappa(X), \kappa(Z))=\mathcal{C}(A\times\kappa(X), \kappa(Z))=\mathcal{C}(A\times\kappa(X), \kappa(Z))$. Therefore we get the commutative diagram,

$$\begin{array}{ccc}
\mathcal{J}\mathcal{C}(A\times\kappa(Y), \kappa(X\times Y)) & \xrightarrow{\ \varphi\ } & \mathcal{A}(A, k'(Y, X\times Y)) \\
\| & & \cup \\
\mathcal{C}(A\times\kappa(Y), \kappa(X\times Y)) & \xrightarrow[\ \Phi\ ]{} & \mathcal{C}(A, k'(Y, X\times Y)) .
\end{array}$$

Let $f(\in\mathcal{J}\mathcal{C})$ be the composite $\kappa(X)\times\kappa(Y)\xrightarrow{id_X\times id_Y}X\times Y\xrightarrow{id_{X\times Y}^{-1}}\kappa(X\times Y)$. Using the diagram, we have $\overline{f(\alpha\times\kappa(Y))}=\widehat{f(\alpha\times\kappa(Y))}$. By (**B**.1) $\overline{f(\alpha\times\kappa(Y))}=\bar{f}\alpha$. Hence $\bar{f}\in\mathcal{J}\mathcal{C}(\kappa(X), k'(Y, X\times Y))$. Since $\eta_{X,Y}=\bar{f}id_Z^{-1}$, we get the desired consequence.

The last result means that if $\mathcal{J}$ satisfies (**a**) and (**b**), then φ is a natural transformation from L to N.

It is obvious that $\psi\varphi=L$ and $\varphi\psi=N$ under the conditions (**a**) and (**b**).

4. Exponential laws

Throughout this section we require that $\mathcal{J}$ satisfies (**a**) and (**b**). The results of section 3 are summarized below.

THEOREM 1. *$\mathcal{J}\mathcal{C}$ is cartesian closed.*

From this, we get the exponential laws.

THEOREM 2. *There are natural isomorphisms in $\mathcal{J}\mathcal{C}$,*

(i) $k'(X\times Y, Z)\cong k'(X, k'(Y, Z))$

(ii) $k'(X, Y\times Z)\cong k'(X, Y)\times k'(X, Z)$.

PROOF. (i) Let λ: $\mathcal{J}\mathcal{C}(W, k'(X\times Y, Z))\longrightarrow\mathcal{J}\mathcal{C}(W, k'(X, k'(Y, Z)))$ be a natural isomorphism. It is a well known fact that any natural transformation λ is $\mathcal{J}\mathcal{C}(W,l)$

for some $l \in \mathscr{S}\mathscr{C}(k'(X \times Y, Z), k'(X, k'(Y, Z)))$, and l is isomorphic if and only if λ is. Combining this fact with the naturality of λ, we can easily verify that l is a natural isomorphism. We may take λ to be the following composite of natural isomorphisms $\mathscr{S}\mathscr{C}(W, k'(X \times Y, Z)) \xrightarrow{\varphi} \mathscr{S}\mathscr{C}(W \times (X \times Y), Z) \xrightarrow{\mu} \mathscr{S}\mathscr{C}((W \times X) \times Y, Z) \xrightarrow{\varphi} \mathscr{S}\mathscr{C}(W \times X, k'(Y,Z)) \xrightarrow{\varphi} \mathscr{S}\mathscr{C}(W, k'(X, k'(Y, Z)))$, where $\mu = \mathfrak{S}\mathfrak{C}(\alpha, Z)$ determined by the natural isomorphism $\alpha: (W \times X) \times Y \longrightarrow W \times (X \times Y)$.

(ii) We have only to verify the existence of a natural isomorphism $\beta: \mathscr{S}\mathscr{C}(W, k'(X, Y \times Z)) \longrightarrow \mathscr{S}\mathscr{C}(W, k'(X, Y) \times k'(X, Z))$. There are natural isomorphisms $\mathscr{S}\mathscr{C}(W, k'(X, Y \times Z)) \xrightarrow{\varphi} \mathscr{S}\mathscr{C}(W \times X, Y \times Z) \xrightarrow{(*)} \mathscr{S}\mathscr{C}(W \times X, Y) \times \mathscr{S}\mathscr{C}(W \times X, Z) \xrightarrow{\varphi \times \varphi} \mathscr{S}\mathscr{C}(W, k'(X, Y)) \times \mathscr{S}\mathscr{C}(W, k'(X, Z)) \xrightarrow{(*)'} \mathscr{S}\mathscr{C}(W, k'(X, Y) \times k'(X, Z))$, where (*) and (*)$'$ are determined by the products $Y \times Z$ and $k'(X, Y) \times k'(X, Z)$ respectively. It completes the proof. (cf. [3] Theorem 1 and 2)

Using Theorem 2, we can easily verify the following.

THEOREM 3. *$\mathscr{S}\mathscr{G}$ is cartesian closed.*

PROOF. The composite $\mathscr{S}\mathscr{G}(X \otimes Y, Z) \xrightarrow[\text{(Proposition 2)}]{\kappa^{-1}} \mathscr{S}\mathscr{C}(X \times Y, Z) \xrightarrow{\varphi} \mathscr{S}\mathscr{C}(X, k'(Y, Z)) \xrightarrow[\text{(Proposition 2)}]{\kappa} \mathscr{S}\mathscr{G}(X, \mathscr{K}_t(Y, Z))$ is a natural isomorphism. Hence we get the desired consequence.

By the similar fashion to the proof of Theorem 2, we get the next.

THEOREM 4. *There are natural isomorphisms in $\mathscr{S}\mathscr{G}$,*

$$\text{(i)} \quad \mathscr{K}_t(X \otimes Y, Z) \cong \mathscr{K}_t(X, \mathscr{K}_t(Y, Z))$$

$$\text{(ii)} \quad \mathscr{K}_t(X, Y \otimes Z) \cong \mathscr{K}_t(X, Y) \otimes \mathscr{K}_t(X, Z),$$

where $-\otimes-$ denotes the product in $\mathscr{S}\mathscr{G}$ and $\mathscr{K}_t(-, -) = \kappa(F(-, -))$.

Remark. (i) Combining Proposition 5.(i), Theorem 2 and the trivial observation that $\kappa(U \times V) = \kappa(U) \otimes \kappa(V)$ for any $U, V \in \mathscr{C}$, we can verify Theorem 3 and Theorem 4.

(ii) In this section it was shown that Theorem 1 yields Theorem 3. Conversely it is easy to verify that if $\mathscr{S}\mathscr{G}$ is cartesian closed, then $\mathscr{S}\mathscr{C}$ is also cartesian closed.

5.

Here we shall consider the relations among the function spaces. If X and Y are in $\mathscr{S}\mathscr{G}$, then $k'(X, Y) = F(X, Y)$ and $k'(X, Y) \cong \mathscr{K}_t(X, Y)$ in $\mathscr{S}\mathscr{C}$. Moreover, id: $\mathscr{K}_t(X,Y) \longrightarrow k'(X, Y)$ is in $\mathscr{C}$ and id^{-1} is in $\mathscr{S}\mathscr{C}$.

Let k: $\mathscr{S}\mathscr{C}^{op} \times \mathscr{S}\mathscr{C} \longrightarrow \mathscr{S}\mathscr{C}$ be the functor in the sence of Y. Kawahara and T. Kudo [3]. That is, let $\mathscr{U}$ be some full subcategory of $\mathscr{C}$ with $\mathscr{S} \subset \mathscr{U} \subset \mathscr{S}\mathscr{G}$, then $k(X, Y)$ is the set $\mathscr{S}\mathscr{C}(X, Y)$ with the induced topology determined by the family of set maps $\{\mathscr{S}\mathscr{C}(X, Y) \xrightarrow{n^*} F(U, Y) \mid U \in \mathscr{U}, n \in \mathscr{C}(U, X)\}$. On the other hand, according to Proposition 2 we may take as $k'(X, Y)$ the set $\mathscr{S}\mathscr{C}(X, Y)$ with the induced topology determined by the bijection κ: $\mathscr{S}\mathscr{C}(X, Y) \longrightarrow F(\kappa(X), \kappa(Y))$. Then id: $k'(X, Y) \longrightarrow k(X, Y)$ is in $\mathscr{C}$ and id^{-1} is in $\mathscr{S}\mathscr{C}$.

(Since $k'(X,-)$ and $k(X,-)$ are right adjoints for $-\times X$: $\mathscr{S}\mathscr{C}\longrightarrow\mathscr{S}\mathscr{C}$, we can directly obtain that $k'(X, Y)$ is naturally isomorphic to $k(X, Y)$ in $\mathscr{S}\mathscr{C}$.) Let P be a one point space, then $k'(P, X)\cong X$ in $\mathscr{S}\mathscr{C}$, really $k'(P, X)=\kappa(X)$.

Finally, there are some examples of which hold the Axioms (**a**) and (**b**) (cf. [6]).

Let $\mathscr{L}\mathscr{C}$ be the full subcategory of $\mathscr{C}$ consisting of all locally compact spaces. Combining Proposition 9 with the well known fact that any Y in $\mathscr{L}\mathscr{C}$ is $\mathscr{C}$-admissible, we get that any full subctaegory of $\mathscr{L}\mathscr{C}$ holds (**a**).

The followings hold (**b**) too.

(i) The full subcategory consisting of a one point space only;

(ii) the full subcategory consisting of all compact Hausdorff spaces;

(iii) the full subcategory consisting of all locally compact Hausdorff spaces;

(iv) $\mathscr{L}\mathscr{C}$ itself.

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References

[1] J. Dugundji, *Topology*, Allyn and Bacon, Inc., Boston, 1965.

[2] Y. Kawahara, *A construction of cartesian closed categories*, Tech. Rep. Yamaguchi Univ., **1**(5) (1976) 579–583.

[3] Y. Kawahara and T. Kudo, *A construction of closed categoies related to k-spaces*, Mem. Fac. Sci., Kyushu Univ., Ser. A, Math., **30** (1976) 113–121.

[4] S. MacLane, *Categories for the working mathematician*, Springer-Verlag, Berlin, 1971.

[5] N.E. Steenrod, *A convenient category of topological spaces*, Mich. Math. J., **14** (1967) 133–150.

[6] R.M. Vogt, *Convenient categories of topological spaces for homotopy theory*, Arch. Math., **22** (1971) 545–555.