RANDERS SPACES WITH RECTILINEAR GEODESICS Dedicated to Professor Dr. Makoto Matsumoto on the occasion of his sixtieth birthday

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By

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Abstract

In the present paper we treat the Randers change of Finsler metrics to determine the Randers spaces with rectilinear geodesics.

§ 0. Introduction.

In his paper [17] G. Randers modified the Riemannian metric under the necessity of using a metric of asymmetrical properties and introduced the metric

$$ds = (a_{ij}(x)\, dx^i dx^j)^{1/2} + b_i(x)\, dx^i\,, \tag{0.1}$$

where a_{ij} is a Riemannian metric tensor, and b_i is a covariant vector. In spite of its simplicity in form, the Finsler space with this metric enjoys interesting properties full of suggestion, and was named the *Randers space* by R. S. Ingarden [11], and has been studied by many authors, from various standpoints in physical and mathematical aspects, e.g., [4], [6], [7], [9], [10], [11], [12], [13], [14], [18], [20], [21] etc. (See the concerned article in [16].)

The Randers space is thought to be the simplest possible asymmetrical modification of the Riemannian space (§1). So, it seems to be important to investigate what geometrical properties remain under the modification by some b_i. As an example of such a property, we consider the property that the space be *with rectilinear geodesics* (§2). Generally we treat a change of Finsler metrics called the *Randers change,* which has been defined by M. Matsumoto [14] as

$$ds \to d\bar{s} = ds + b_i(x)\, dx^i\,, \tag{0.2}$$

and obtain the condition that it hold the above property (§3). As the special case that ds be Riemannian, we can determine the Randers spaces with rectilinear geodesics (§4).

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To construct generally the geometry of the space with rectilinear geodesics belongs to the fourth problem of D. Hilbert, who gave an interesting example [8], and since the studies of P. Funk [5] and L. Berwald [1, 2, 3] the problem has also been an important but difficult one in the Finsler geometry. (See [15], [19].) In the Riemannian spaces such a space is nothing but a space of constant curvature. In the Finsler geometry, however, it seems that the condition that the space be of constant curvature is too strong. So, our research also aims at the problem to find a reasonable Finsler space corresponding to a Riemannian space of constant curvature.

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§ 1. Randers spaces.

Let (M, L) be an n-dimensional Finsler space, that is, an n-dimensional differentiable manifold M with a fundamental function $L(x, y)$ $(y^i=\dot{x}^i)$. At each point x of M we consider the indicatrix I_x defined as the hypersurface $I_x=\{y \in M_x \mid L(x, y)=1\}$ of the tangent space M_x at x. In case of a Riemannian space with $L(x, y)=(a_{ij}(x)y^iy^j)^{1/2}$, each indicatrix I_x is the quadratic hypersurface $a_{ij}(x)y^iy^j=1$ with respect to the coordinates y^i of M_x with the center $y=0$. Hence, a Finsler space, whose indicatrix I_x is a quadratic hypersurface with respect to y^i at each point x of M, is thought to be the simplest possible modification of a Riemannian space. If its center y is not zero, the Finsler metric has an asymmetrical property $L(x, -y) \neq L(x, y)$. Such Finsler spaces are given by

Proposition 1.1. *Let (M, L) be a Finsler space. If at each point x of M the indicatrix I_x is a quadratic hypersurface with respect to the coordinates y^i of M_x, the fundamental function $L(x, y)$ has the form*

$$L(x,y) = \alpha(x,y)+\beta(x,y)\,, \tag{1.1}$$

$$L(x,y) = -\alpha(x,y)+\beta(x,y)\,, \tag{1.2}$$

or

$$L(x,y) = \alpha(x,y)^2/\beta(x,y)\,, \tag{1.3}$$

where $\alpha(x,y) = (a_{ij}(x)y^iy^j)^{1/2}$, $\beta(x,y) = b_i(x)\,y^i$.

Proof. Let the indicatrix I_x be given by

$$p_{ij}(x)\,y^iy^j+2q_i(x)\,y^i+r(x) = 0\,.$$

Since $l^i=y^i/L(x, y)$ has the unit length, it satisfies

$$p_{ij}(x)\,l^il^j+2q_i(x)\,l^i+r(x) = 0\,,$$

that is,

$$r(x)\,L(x,y)^2+2q_i(x)\,y^iL(x,y)+p_{ij}(x)\,y^iy^j = 0\,,$$

from which we have

$$L(x,y) = (-q_i(x)\,y^i \pm (\,(q_i(x)\,q_j(x) - r(x)\,p_{ij}(x)\,)\,y^i y^j)^{1/2})/r(x)\,,$$

or

$$L(x,y) = -\,p_{ij}(x)\,y^i y^j/2q_i(x)\,y^i\,,$$

according as $r(x)\neq 0$ or $r(x)=0$. These are reduced to the form (1.1), (1.2) or (1.3).

Let us assume in the above proposition that $L(x, y)$ is positive valued, i.e., $L(x, y)>0$ for any $y\neq 0$. Then, $L(x, y)$ should have the form (1.1), and the Finsler space is just a Randers space. Thus we can understand that G. Randers attained the metric (0.1) as the simplest possible asymmetrical modification of the Riemannian metric.

A *Randers space* (M, L) is by definition a Finsler space with a fundamental function $L(x, y)=(a_{ij}(x)y^iy^j)^{1/2}+b_i(x)y^i$, where a_{ij} is a Riemannian metric tensor, and b_i is a covariant vector. (a_{ij}) is positive definite, but it does not need to assume that L is positive valued. We have easily

Proposition 1.2. *In a Randers space* $(M, \alpha+\beta)$ $(\alpha=(a_{ij}y^iy^j)^{1/2}$, $\beta=b_iy^i)$, *the fundamental function* $\alpha+\beta$ *is positive valued if and only if* $(a_{ij}-b_ib_j)$ *is positive definite at each* $x\in M$.

§ 2. Finsler spaces with rectilinear geodesics.

A geodesic in a Finsler space (M, L) is given by

$$d^2x^i/ds^2+2G^i(x, dx/ds) = 0\,, \tag{2.1}$$

if we use the arc-length s as the parameter. $G^i(x, y)$ is (2)p-homogeneous in y^i, and is expressed as $G^i=\gamma^i_{jk}y^jy^k/2$, by putting

$$\gamma^i_{jk} = g^{ih}(\partial g_{jh}/\partial x^k+\partial g_{kh}/\partial x^j-\partial g_{jk}/\partial x^h)/2\,,$$

where $g_{ij} = \partial^2(L^2/2)/\partial y^i\partial y^j$ is the fundamental tensor, and $(g^{ij})=(g_{ij})^{-1}$.

A Finsler space (M, L) is called *with rectilinear geodesics*, if the manifold M is covered by coordinate neighbourhoods in which the geodesics can be represented by $(n-1)$ linear equations of the coordinates, or equivalently by

$$x^i(s) = R(s)\,m^i+n^i \qquad (m^i, n^i\in R)\,. \tag{2.2}$$

Now, let h_{ij} be the angular metric tensor: $h_{ij}=g_{ij}-l_il_j$, where $l_i=\partial L/\partial y^i$. Then, we have

Proposition 2.1. *A Finsler space* (M, L) *is with rectilinear geodesics if and only if the manifold* M *is covered by coordinate neighbourhoods in which it holds*

$$G^i = pl^i \tag{2.3}$$

for some (2)p*-homogeneous local function* $p(x, y)$, *or equivalently*

$$h_{ij}G^j = 0\,. \tag{2.4}$$

Proof. Let x_0 be a fixed point of M and y_0 be a fixed tangent vector at x_0. If (M, L) is with rectilinear geodesics, there exists a coordinate neighbourhood around x_0, in which the geodesic passing through x_0 and tangent to $y_0/L(x_0, y_0)$ is given by

$$x^i(s) = R(s)\, y_0^i + x_0^i \qquad (R(0) = 0\,,\ R'(0) = L(x_0, y_0)^{-1})\,.$$

Then, we have $d^2x^i/ds^2+(-R''/R')(dx^i/ds)=0$. Hence, putting $p(x_0, y_0)=-R''(0)/2R'(0)^3$, we have $G^i(x_0, y_0)=p(x_0, y_0)l_0^i$. Since x_0, y_0 can be arbitrarily chosen, it follows (2.3).

Conversely, if M is covered by coordinate neighbourhoods in which it holds (2.3), the equations (2.1) of a geodesic become

$$d^2x^i/ds^2+2p(x, dx/ds)\,(dx^i/ds) = 0\,,$$

which are, assuming $dx^i/ds>0$ without loss of generality, reduced to $d\log(dx^i/ds)/ds=-2p$, from which we have (2.2) by putting $R(s)=\int \exp(-2\int p ds)ds$.

On the other hand, since p in (2.3) becomes $p=l_jG^j$, it is evident that (2.3) is equivalent to (2.4).

Especially, let (M, L) be a locally Minkowski space. Since M is covered by coordinate neighbourhoods in which $G^i=0$, we have

Proposition 2.2. *A locally Minkowski space is with rectilinear geodesics.*

§ 3. Randers changes of Finsler metrics.

If we see a Randers metric $\alpha+\beta$ ($\alpha=(a_{ij}y^iy^j)^{1/2}$, $\beta=b_iy^i$) as a modification of a Riemannian metric α by β, we can generally consider the following change of Finsler metrics by $\beta=b_iy^i$:

$$L \rightarrow \bar{L} = L+\beta\,. \tag{3.1}$$

This change $L\rightarrow\bar{L}$ was first studied by M. Matsumoto [14]. We shall call this the *Randers change* by β, and consider the properties that remain under this change.

Since $\partial^2\bar{L}/\partial y^i\partial y^j=\partial^2L/\partial y^i\partial y^j$, that is, $\bar{L}^{-1}\bar{h}_{ij}=L^{-1}h_{ij}$, we have first

Proposition 3.1. *The tensor $L^{-1}h_{ij}$ is invariant by any Randers change.*

Next, we shall get the transformation formula of G^i by a Randers change. For the purpose we shall express the equations of a geodesic for the metric $\bar{L}=L+\beta$ in terms of the arc-length s for the metric L. The Euler-Lagrange equations

$$d(\partial\bar{L}/\partial y^i)/ds-\partial\bar{L}/\partial x^i = 0 \tag{3.2}$$

become

$$d(\partial L/\partial y^i)/ds-\partial L/\partial x^i+2b_{[ij]}(dx^j/ds) = 0\,, \tag{3.3}$$

where we put $2b_{[ij]} = \partial b_i/\partial x^j-\partial b_j/\partial x^i$.

Interrupted here the discussion we note

Theorem 3.1. *A Randers change $L\rightarrow\bar{L}=L+\beta$ by $\beta=b_iy^i$ is projective, that is, any geodesic remains to be a geodesic by the change, if and only if b_i is gradient*: $b_{[ij]}=0$.

Returning to (3.3) and calculating from $L=(g_{ij}y^iy^j)^{1/2}$, (3.3) is written

$$d^2x^i/ds^2+2G^i(x, dx/ds)+2g^{ij}b_{[jk]}(dx^k/ds)=0\,. \tag{3.4}$$

If we change the parameter s in (3.4) to the arc-length $\bar{s}$ for the metric $\bar{L}$, we have

$$\begin{aligned} d^2x^i/d\bar{s}^2+2G^i(x, dx/d\bar{s})+2g^{ij}b_{[jk]}\,(dx^k/d\bar{s})\,(ds/d\bar{s}) \\ +(dx^i/d\bar{s})\,(b_{j|k}(dx^j/d\bar{s})\,(dx^k/d\bar{s})-2g^{hj}b_hb_{[jk]}(dx^k/d\bar{s})\,(ds/d\bar{s})\,)=0\,, \end{aligned} \tag{3.5}$$

where $b_{j|k}$ is the h-covariant derivative of b_j with respect to the Cartan connection of the Finsler space with L. In fact, from $\bar{s}=\int\bar{L}(x, dx/ds)ds$ we have $d\bar{s}/ds=1+b_j(dx^j/ds)$, and

$$d^2\bar{s}/ds^2=(\partial b_j/\partial x^k)\,(dx^j/ds)\,(dx^k/ds)+b_j(d^2x^j/ds^2)\,,$$

the latter of which becomes from (3.4)

$$d^2\bar{s}/ds^2=b_{j|k}(dx^j/ds)\,(dx^k/ds)-2g^{hj}b_hb_{[jk]}(dx^k/ds)\,. \tag{3.6}$$

Since we have $dx^i/ds=(dx^i/d\bar{s})(d\bar{s}/ds)$, and

$$d^2x^i/ds^2=(d^2x^i/d\bar{s}^2)\,(d\bar{s}/ds)^2+(dx^i/d\bar{s})\,(d^2\bar{s}/ds^2)\,, \tag{3.7}$$

we can derive (3.5) by substituting into (3.7) from (3.4), (3.6).

The equation (3.5) shows that

$$\begin{aligned} 2\bar{G}^i(x, dx/d\bar{s}) &= 2G^i(x,dx/d\bar{s})+2g^{ij}b_{[jk]}(dx^k/d\bar{s})\,(ds/d\bar{s}) \\ &+(dx^i/d\bar{s})\,(b_{j|k}(dx^j/d\bar{s})\,(dx^k/d\bar{s})-2g^{hj}b_hb_{[jk]}(dx^k/d\bar{s})\,(ds/d\bar{s})\,)\,. \end{aligned}$$

Since for arbitrary x^i, y^i we can choose the geodesic satisfying $dx^i/d\bar{s}=y^i/\bar{L}(x, y)$, $ds/d\bar{s}=L(x, y)/\bar{L}(x, y)$ at x, we have

$$\bar{G}^i=G^i+Lg^{ij}b_{[jk]}y^k+L\bar{L}^{-1}(b_{j|k}y^jy^k/2-Lg^{hj}b_hb_{[jk]}y^k)l^i\,. \tag{3.8}$$

The formula (3.8) has been already obtained by M. Matsumoto [14]. We have tried to derive this from the equations of geodesics.

Paying attention to Proposition 3.1 we have from (3.8)

Proposition 3.2. *By a Randers change* (3.1) *the quantity* $h_{ij}G^j$ *is transformed as*

$$\bar{L}^{-1}\bar{h}_{ij}\bar{G}^j=L^{-1}h_{ij}G^j+b_{[ij]}y^j\,. \tag{3.9}$$

Thus, from Proposition 2.1 we have

Theorem 3.2. *A Randers change* $L\to\bar{L}=L+\beta$ *by* $\beta=b_iy^i$ *holds the property that the Finsler space be with rectilinear geodesics, if* b_i *is gradient.*

Theorem 3.3. *A Finsler space* (M, L) *becomes a Finsler space* $(M, L+\beta)$ *with rectilinear geodesics by a Randers change by* $\beta=b_iy^i$, *if and only if the manifold* M *is covered by coordinate neighbourhoods in which it holds*

$$h_{ij}G^j+Lb_{[ij]}y^j=0\,. \tag{3.10}$$

§ 4. Randers spaces with rectilinear geodesics.

Now, we are in position to treat a Randers space $(M, \alpha+\beta)$, where $\alpha=(a_{ij}y^iy^j)^{1/2}$, $\beta=b_iy^i$. Putting $L=\alpha$ in Theorem 3.3, the condition that the Randers space be with rectilinear geodesics is that M is covered by coordinate neighbourhoods in which it holds

$$h_{ij}G^j+\alpha b_{[ij]}y^j=0\,. \tag{4.1}$$

In this case, we have $h_{ij}=a_{ij}-(\partial\alpha/\partial y^i)(\partial\alpha/\partial y^j)$, and $G^j=\{{}^{\,j}_{rs}\}y^ry^s/2$, where $\{{}^{\,j}_{rs}\}$ is the Christoffel symbol for the Riemannian metric tensor a_{ij}. Hence, $h_{ij}G^j$ in (4.1) is expressed as

$$h_{ij}G^j=(a_{ij}a_{kl}-a_{ik}a_{jl})\,\{{}^{\,j}_{rs}\}\,y^ky^ly^ry^s/2\alpha^2\,. \tag{4.2}$$

Since α is an irrational function of y^i, (4.1) is equivalent to

$$h_{ij}G^j=0\,,\qquad b_{[ij]}=0\,. \tag{4.3}$$

Thus we have

Theorem 4.1. *A Rander space $(M, \alpha+\beta)$ $(\alpha=(a_{ij}y^iy^j)^{1/2}$, $\beta=b_iy^i)$ is with rectilinear geodesics if and only if the corresponding Riemannian space (M, α) is with rectilinear geodesics and b_i is gradient. Then, any geodesic of (M, α) remains to be a geodesic of $(M, \alpha+\beta)$.*

The latter assertion of the above theorem follows from Theorem 3.1. Also, Theorem 3.1 makes Theorem 3.2 trivial, because we have no information about the converse of Theorem 3.2. It is shown, however, from Theorem 4.1 that the converse holds if we confine a Randers change within Randers metrics.

Theorem 4.2. *A Randers change $L\to\bar{L}=L+\beta$ by $\beta=b_iy^i$ changes a Randers metric L to a Randers metric $\bar{L}$. And, it holds the property that the Randers space be with rectilinear geodesics, if and only if b_i is gradient. Then, this change is projective.*

On the other hand, from Proposition 2.1 the condition $h_{ij}G^j=0$ is equivalent to $G^i=qy^i$, that is, in a Riemannian case

$$\{{}^{\,i}_{rs}\}\,y^ry^s/2=qy^i\,, \tag{4.4}$$

where q is some (1)p-homogeneous function. Differentiating the both sides of (4.4) by y^j and y^k successively, we have

$$\{{}^{\,i}_{jk}\}=(\partial^2q/\partial y^j\partial y^k)\,y^i+(\partial q/\partial y^j)\,\delta_k{}^i+(\partial q/\partial y^k)\,\delta_j{}^i\,. \tag{4.5}$$

The contraction of (4.5) with respect to i and j yields $\partial q/\partial y^j=\{{}^{\,r}_{rj}\}/(n+1)$, from which it follows $\partial^2q/\partial y^j\partial y^k=0$. Hence, $\{{}^{\,i}_{jk}\}$ has the form

$$\{{}^{\,i}_{jk}\}=(\{{}^{\,r}_{rj}\}/(n+1))\,\delta^i_k+(\{{}^{\,r}_{rk}\}/(n+1))\,\delta^i_j\,, \tag{4.6}$$

that is, the Riemannian space (M, α) is projectively flat. It is well known that a Riemannian space is projectively flat if and only if it is of constant curvature. So, Theorem 4.1 is restated as

Theorem 4.3. *A Randers space* $(M, \alpha+\beta)$ $(\alpha=(a_{ij}y^iy^j)^{1/2}, \beta=b_iy^i)$ *is with rectilinear geodesics if and only if the corresponding Riemannian space* (M, α) *is of constant curvature and* b_i *is gradient. Then, the geodesics in* $(M, \alpha+\beta)$ *coincide with the geodesics in* (M, α).

Finsler spaces with rectilinear geodesics contain Riemannian spaces of constant curvature and locally Minkowski spaces, and are thought to be the Finsler spaces corresponding to Riemannian spaces of constant curvature. Of course, we have an example of a non-Minkowski Randers space with rectilinear geodesics. S. Kikuchi [12] has shown that a Randers space $(M, \alpha+\beta)$ $(\beta=b_iy^i)$ is locally Minkowski if and only if the corresponding Riemannian space (M, α) is flat and b_i is parallel with respect to the Riemannian connection. So, in order to get an example of a non-Minkowski Randers space with rectilinear geodesics it suffices to modify the euclidean metric α by a non-constant gradient vector b_i.

Example. Let R^n be the n-dimensional euclidean space. The Randers space $(R^n-\{0\}, L)$ with the following fundamental function L is with rectilinear geodesics and not locally Minkowski:

$$L=(\delta_{ij}y^iy^j)^{1/2}+b_iy^i\,, \tag{4.7}$$

where $b_i=\partial b/\partial x^i$ for $b=(\delta_{ij}x^ix^j)^{1/2}$.

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