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ON FINSLER METRICS ASSOCIATED WITH A LAGRANGIAN

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Abstract

This article is a revised note of the lecture [1] presented by the authors to "The XXIst National Symposium on Finsler Geometry" held at Yokosuka during October 15–18, 1986. We discuss some aspects of the treatment as a Finsler metric of a non-homogeneous generalized metric.

Introduction

In the previous Symposium on Finsler Geometry held at Kagoshima in the summer of 1985, Professor R. Miron emphasized the importance of a generalized metric which is not necessarily assumed to be positively homogeneous. In fact, a lot of results in Finsler geometry remain valid without the assumption of homogeneity. (cf. R. Miron [14], Gh. Atanasiu-M. Hashiguchi-R. Miron [2], which were also presented to the above Symposium.)

"Since the term ⟨Finsler geometry⟩ sounds as if the geometry is antiquated, it does not yet attract a general attention in spite of our modern approaches."said he, "On the other hand, non-homogeneous geometrical objects like Lagrangians are important from the standpoint of applications. So the term ⟨Finsler geometry⟩ should be replaced by ⟨Lagrange geometry⟩ which was first termed by J. Kern [7]." Then he proposed the following definitions.

Let M be an n-dimensional differentiable manifold, and $T(M)$ its tangent bundle. A coordinate system $x=(x^i)$ in M induces a canonical coordinate system $(x, y)=(x^i, y^i)$ in $T(M)$. We put $\partial_i=\partial/\partial x^i$, $\dot{\partial}_i=\partial/\partial y^i$, and $T^0(M)=\{(x, y)\in T(M) \mid y\neq 0\}$.

A Finsler tensor field $g_{ij}(x, y)$ of type (0, 2) in M is called a *generalized Lagrange metric* if it is symmetric and non-degenerate: $g_{ij}=g_{ji}$, $\det(g_{ij})\neq 0$. Especially, a generalized Lagrange metric $g_{ij}(x, y)$ is called a *Lagrange metric* when g_{ij} is given by $g_{ij}=\dot{\partial}_i\dot{\partial}_j L$ for some Finsler function $L(x, y)$ in M.

A generalized Lagrange metric $g_{ij}(x, y)$ is called a *generalized Finsler metric* if it is positively homogeneous of degree 0: $g_{ij}(x, \lambda y)=g(x, y)$ for $\lambda>0$. Especially, a Lagrange metric $g_{ij}(x, y)$ is called a *Finsler metric* when g_{ij} is given by $g_{ij}=(\dot{\partial}_i\dot{\partial}_j L^2)/2$ for some Finsler function $L(x, y)$ in M which is positively homogeneous of degree 1: $L(x, \lambda y)$

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$=\lambda L(x, y)$ for $\lambda>0$.

Let R be the real line, considered as a one-dimensional differentiable manifold, with a fixed global coordinate t. A differentiable function $L(t, x, y)$ defined on a domain D of $R\times T^0(M)$ is called a *Lagrangian*. It is not necessarily to be $\partial L/\partial t\neq 0$. A Lagrangian L is called *regular* if $\det(\dot{\partial}_i\dot{\partial}_j L)\neq 0$. Since a Lagrangian L is usually assumed to be regular, the above terminology proposed by Prof. Miron seems to be reasonable.

On the occasion of his visit to the University of Bari, in the autumn of 1985, M. Hashiguchi, one of the authors, introduced the above terminology in his lecture [5]. Then O. Amici and B. Casciaro, the other authors, gave a remark that the geometry based on Lagrangians has a long tradition since C. Carathéodory [4], with various interesting applications, e. g., B. Segre [17], but we need careful consideration in order to make the teminology establish fully, because every regular Lagrangian L, even non-homogeneous, can be treated as a fundamental function L^* of a Finsler space of dimension $n+1$ (cf. C. Lanczos [12, pp. 280-290]).

The present lecture is a report of our discussion about the above remark. In the first section the result remarked above is shown (Theorem 1.1), and in the following section an example of a special Finsler space is given from this standpoint (Example 2.1). In the last section two kinds of regularity and other types of Finslerization are discussed for a Lagrangian (Theorem 3.1, Theorem 3.3). The terminology and notations follow those in [1], with slight modifications.

1. A Finsler metric associated with a regular Lagrangian

For an n-dimensional differentiable manifold M we consider the product manifold $M^*=R\times M$. A coordinate system $x=(x^i)$ in M induces respective coordinate systems $x^*=(x^\alpha)=(t, x^i)$ and $(x^*, y^*)=(x^\alpha, y^\alpha)=(t, x^i, \dot{t}, y^i)$ in M^* and $T(M^*)$, where and in the following we assume that Latin indices take values 1, 2, $\cdots$, n and Greek 0, 1, 2, $\cdots$, n. We put for $y^0\neq 0$

$$z^i=y^i/y^0. \tag{1.1}$$

Now, with a Lagrangian $L(t, x, y)$ defined on a domain D of $R\times T^0(M)$ we associate a differentiable function $L^*(x^*, y^*)$ given by

$$L^*(x^*, y^*)=L(x^0, x, z)y^0, \tag{1.2}$$

which is defined on the domain $\{(x^*, y^*)\in T(M^*)\,|\, y^0\neq 0,\ (x^0, x, z)\in D\}$. It should be noted that L^* is homogeneous of degree 1 with respect to y^*:

$$L^*(x^*, \lambda y^*)=\lambda L^*(x^*, y^*) \text{ for } \lambda\neq 0. \tag{1.3}$$

We shall show that L^* gives a Finsler metric function in M^* if L is regular. Putting

$$l_i=\dot{\partial}_i L,\quad h_{ij}=L(\dot{\partial}_i\dot{\partial}_j L), \tag{1.4}$$

and

(1.5) $$g^*_{\alpha\beta}=(\dot{\partial}_\alpha\dot{\partial}_\beta L^{*2})/2,$$

we have by direct calculations

(1.6) $$g^*_{00}(x^*,\ y^*)=L^2-2Ll_iz^i+(l_il_j+h_{ij})z^iz^j,$$

(1.7) $$g^*_{i0}(x^*,\ y^*)=Ll_i-(l_il_j+h_{ij})z^j,$$

(1.8) $$g^*_{ij}(x^*,\ y^*)=l_il_j+h_{ij},$$

and

(1.9) $$\det(g^*_{\alpha\beta}(x^*,\ y^*))=L^2\det(h_{ij}),$$

where L, l_i, h_{ij} in the right-hand sides mean the respective values at $(x^0,\ x,\ z)$.

We can see from (1.9) that the matrix $(g^*_{\alpha\beta})$ is regular at $(x^*,\ y^*)$ if and only if $L\det(h_{ij})$ does not vanish at the corresponding point $(x^0,\ x,\ z)$. In the following we shall consider L at the points such that $L\neq 0$. Then, since the condition $\det(h_{ij})\neq 0$ is equivalent to the regularity condition $\det(\dot{\partial}_i\dot{\partial}_jL)\neq 0$, we have

Theorem 1.1. *Let $L(t,\ x,\ y)$ be a Lagrangian defined on a domain D of $R\times T^0(M)$. Then the function $L^*(x^*,\ y^*)$ defined by* (1.2) *on the domain $D^*=\{(x^*,\ y^*)\in T(M^*)\mid y^0\neq 0,\ (x^0,\ x,\ z)\in D,\ L^*(x^*,\ y^*)>0\}$ is a Finsler metric function in M^* if and only if L is regular.*

The function L^* given in Theorem 1.1 for a regular Lagrangian L is called the Finsler metric (function) *associated with* L.

A simple example of a regular Lagrangian is given in a simple dynamical system as the difference between the kinetic energy $T=\alpha^2/2$ and the potential one U:

(1.10) $$L(x,\ y)=T(x,\ y)-U(x),$$

where $\alpha(x,\ y)=(a_{ij}(x)y^iy^j)^{1/2}$ is a Riemannian metric. In this case we have $h_{ij}=La_{ij}$. Since (h_{ij}) is regular, we have a Finsler metric of Kropina type (cf. C. Shibata [18]):

(1.11) $$L^*(x^*,\ y^*)=(a_{ij}(x)y^iy^j-2U(x)(y^0)^2)/2y^0.$$

For a regular Lagrangian $L=\alpha^2$ we have a Finsler metric

(1.12) $$L^*(x^*,\ y^*)=(a_{ij}(x)y^iy^j)/y^0.$$

If we consider this Finser metric in the section given by $y^0=\beta$, where $\beta(x,\ y)$ is a non-vanishing 1-form, we have a Kropina metric α^2/β [10, 11]. If b_i is a gradient vector $\partial_if(x)$, the Kropina space $(M,\ \alpha^2/\beta)$ is considered as a subspace $x^0=f(x)$ of the Finsler space $(M^*,\ L^*)$.

In general, for a regular Lagrangian $L(t,\ x,\ y)$ we put

(1.13) $$(\tilde{h}^{ij})=(h_{ij})^{-1},\quad \tilde{l}^i=\tilde{h}^{ij}l_j,\quad \lambda=1+l_i\tilde{l}^i.$$

Then the inverse matrix $(g^{*\alpha\beta})$ of $(g^*_{\alpha\beta})$ is given by

(1.14) $$g^{*00}(x^*,\ y^*)=\lambda/L^2,$$

(1.15) $$g^{*i0}(x^*,\ y^*)=\lambda z^i/L^2-\tilde{l}^i/L,$$

(1.16) $$g^{*ij}(x^*, y^*)=\lambda z^i z^j/L^2-(\tilde{l}^i z^j+\tilde{l}^j z^i)/L+\tilde{h}^{ij},$$

where λ, L, $\tilde{l}^i$, $\tilde{h}^{ij}$ in the right-hand sides mean the respective values at (x^0, x, z). These formulas will be useful for the study of Finsler metrics of Kropina type.

2. Projectivity of the Finslerization of a Lagrange space

Let $L(t, x, y)$ be a regular Lagrangian, and $L^*(x^*, y^*)$ the Finsler metric associated with L. We shall call (M, L) a *Lagrange space*, and the change from (M, L) to the Finsler space (M^*, L^*) the *Finslerization* of the Lagrange space (M, L), and investigate an invariant by this change.

For a differentiable curve C: $x^i=x^i(t)$ in (M, L) we can define a curve C^* in (M^*, L^*) called the *lift* of C by $x^0=t$, $x^i=x^i(t)$. Then the length s^* of C^* defined by

(2.1) $$s^*=\int L^*(x^*(t), \dot{x}^*(t))dt$$

coincides with the integral

(2.2) $$s=\int L(t, x(t), \dot{x}(t))dt,$$

because of $\dot{x}^0=1$. We shall call the integral s given by (2.2) the *length* of C. In general, the lift C^* of C depends on the choice of a parameter of C, and the length of C does also so.

Conversely, any differentiable curve C^*: $x^0=f(\tau)$, $x^i=x^i(\tau)$ in (M^*, L^*) may be considered as the lift of some curve in (M, L), provided $f'(\tau)\neq 0$, because we can take $x^0=t$ as a parameter of C^*. It should be noted that the length of C^* is independent on the choice of a parameter of C^*, since L^* is homogeneous of degree 1 in the sense of (1.3). In the following, taking x^0 as a parmeter of a curve in M^*, we shall express it as $x^0=t$, $x^i=x^i(t)$.

The relation between extremals of s and s^* is given by

(2.3) $$d(\dot{\partial}_0 L^*)/dt-\partial_0 L^*=-(d(\dot{\partial}_k L)/dt-\partial_k L)\dot{x}^k(t),$$

(2.4) $$d(\dot{\partial}_k L^*)/dt-\partial_k L^*=d(\dot{\partial}_k L)/dt-\partial_k L,$$

where $\partial_\alpha L^*$, $\dot{\partial}_\alpha L^*$in the left-hand sides mean the functions of $x^*(t)$, $\dot{x}^*(t)$, and $\partial_k L$, $\dot{\partial}_k L$ in the right-hand sides mean the functions of t, $x(t)$, $\dot{x}(t)$. Thus, since the Euler equations for (2.2) are equivalent to the ones for (2.1), if we call an extremal of (2.2) a *geodesic* of a Lagrange space (M, L), we have the following well-known theorem (cf. [16]):

Theorem 2.1. *A curve C of a Lagrange space (M, L) is a geodesic if and only if the lift C^* is a geodesic of the associated Finsler space (M^*, L^*).*

Therem 2.1 shows that a Lagrange space may be treated geometrically as a Finsler space on some aspects. Paying attention to the projectivity of the Finslerization shown in

Theorem 2.1, we shall consider the following special Finsler space.

A Finsler space is called *with rectilinear geodesics*, if the underlying manifold is covered by coordinate neighborhoods such that the geodesics can be represented by $(n-1)$ linear equations of the coordinates x^i, or equivalently by the equations of the form $x^i = a^i t + b^i$ $(a^i, b^i \in R)$. For a Lagrange space we also use the same terminology. Then from Theorem 2.1 we have

Theorem 2.2. *Let (M, L) be a Lagrange space with rectilinear geodesics. Then the associated Finsler space (M^*, L^*) is also with rectilinear geodesics.*

As an application of the above theorem we can derive a well-known example of a Finsler space with rectilinear geodesics due to L. Berwald [3]. In a two-dimensional euclidean space R^2, let $k(x^0, x^1)$ be a solution of the differential equation

$$\partial_0 k - k\partial_1 k = 0. \tag{2.5}$$

Then the function

$$L^*(x^0, x^1, y^0, y^1) = (k(x^0, x^1)y^0 + y^1)^2/y^0 \tag{2.6}$$

is a Finsler metric in R^2, and the Finsler space (R^2, L^*) is with rectilinear geodesics.

In fact, L^* is associated with a Lagrangian

$$L(t, x^1, y^1) = (k(t, x^1) + y^1)^2, \tag{2.7}$$

in which $h_{11} = 2L \neq 0$. Since L is regular, L^* is a Finsler metric in R^2. Now, let C: $x^1 = f(t)$ be a geodesic of a one-dimensional Lagrange space (R, L). By the condition (2.5) the Euler equation $d(\dot\partial_1 L)/dt - \partial_1 L = 0$ becomes $d^2x^1/dt^2 = 0$, from which we have $f(t) = at + b$ $(a, b \in R)$. Since (R, L) is with rectilinear geodesics, (R^2, L^*) is also so.

In the similar way we can get examples of Finsler spaces with rectilinear geodesics. For example, since a Riemannian space (M, α) with rectilinear geodesics is nothing but a space of constant curvature, and an extremal of $\int \alpha^2$ dt becomes an extremal of $\int \alpha dt$, we have by the Finslerization of a Lagrange space (M, α^2)

Example 2.1. Let (M, α) be a Riemannian space of constant curvature. The Finsler space (M^*, L^*) endowed with the Finsler metric L^* given by (1.12) is with rectilinear geodesics.

3. *F*-regular Lagrangians

A Finsler metric $L(x, y)$ is not regular as a Lagrangian. We hope to find a concept generalizing both of a regular Lagrangian and a Finsler metric. Suggested by the definition of a Finsler metric, for a Lagrangian $L(t, x, y)$ we put

$$g_{ij} = (\dot\partial_i \dot\partial_j L^2)/2, \tag{3.1}$$

and call L to be *F-regular* if $\det(g_{ij}) \neq 0$, that is, L^2 is regular. An F-regular Lagrangian

$L(t, x, y)$ is a Finsler metric, in the case that L is independent on t and is positively homogeneous of degree I with respect to y.

With an F-regular Lagrangian $L(t, x, y)$ we associate a function $L^*(x^*, y^*)$ defined by (1.2), and define $g^*_{\alpha\beta}$ by (1.5). If $L(x, y)$ is a Finsler metric, we have an interesting result $L^*(x^*, y^*)=L(x, y)$, but since L^* is independent on y^0, the matrix $(g^*_{\alpha\beta})$ is not regular, that is, L^* is not a Finsler metric. In order to get the condition that L^* becomes a Finsler metric, we use the following lemma (cf. [13, Proposition 30.1]).

Lemma 3.1. *Let* (s_{ij}) *be a regular symmetric matrix of degree* n, *and* $a \neq 0$, a_i, $b_i \in R$. *Putting*

$$t_{ij}=as_{ij}-a_ib_i,\ t=a-s^{ij}a_ib_j, \tag{3.2}$$

where $(s^{ij})=(s_{ij})^{-1}$, *we have*

$$\det(t_{ij})=a^{n-1}t\det(s_{ij}). \tag{3.3}$$

The matrix (t_{ij}) *is regular if and only if* $t \neq 0$, *and then the inverse matrix* (t^{ij}) *is given by*

$$t^{ij}=(ts^{ij}-a^ib^j)/at, \tag{3.4}$$

where $a^i=s^{ij}a_j$, $b^i=s^{ij}b_j$.

Proof. (3.3) is derived as follows:

$$\begin{aligned}\det(t_{ij})&=\det(as_{ij}-a_ib_j)\\ &=\det\begin{pmatrix} as_{ij}-a_ib_j & 0 \\ b_j & 1\end{pmatrix}=\det\begin{pmatrix} as_{ij} & a_i \\ b_j & 1\end{pmatrix}\\ &=-a^{n-1}S^{ij}a_ib_j+\det(as_{ij})=a^{n-1}t\det(s_{ij}),\end{aligned}$$

where the third equality follows from the additions of the last row multiplied by a_i to the i-th row $(i=1, \cdots, n)$, the fourth by the expansion with respect to the last row, and the last from $s^{ij}=S^{ij}/\det(s_{ij})$, where S^{ij} is the cofactor of s_{ij} in (s_{ij}). The last statement follows directly by verifying $t_{ij}t^{jk}=\delta^k_i$ from (3.2) and (3.4). Q. E. D.

Now, for an F-regular Lagrangian, if we put

$$(g^{ij})=(g_{ij})^{-1},\ l^i=g^{ij}l_j,\ p=l-l_il^i, \tag{3.5}$$

since $h_{ij}=g_{ij}-l_il_j$, we have from Lemma 3.1

$$\det(h_{ij})=p\det(g_{ij}), \tag{3.6}$$

and so from (1.9)

$$\det(g^*_{\alpha\beta})(x^*, y^*)=(L^2p\det(g_{ij}))(x^0, x, z). \tag{3.7}$$

Thus we have proved the following theorems.

Theorem 3.1. *Let* $L(t, x, y)$ *be a Lagrangian. If* L *is regular, then* L *is* F*-regular. Conversely, when* L *is* F*-regular,* L *is regular if and only if* $p \neq 0$.

Theorem 3.2. *Let $L(t, x, y)$ be an F-regular Lagrangian defined on a domain D of $R\times T^0(M)$. Then the function $L^*(x^*, y^*)$ defined by* (1.2) *on the domain $D^*=\{(x^*, y^*)\in T(M^*)|\ y^0\neq 0,\ (x^0, x, z)\in D,\ L^*(x^*, y^*)>0\}$ is a Finsler metric in M^* if and only if $p\neq 0$.*

For a Finsler metric we have $p=0$. Denoting by n the dimension of the underlying manifold M, we can see that the set G_n of F-regular Lagrangians has two important disjoint subsets, that is, the one H_n is the set of regular Lagrangians and the other F_n is the set of Finsler metrics. By the Finslerization (1.2), H_n goes into F_{n+1}, but the rest goes out of G_{n+1}

There exists an F-regular Lagrangian belonging out of $H_n\cup F_n$.

Example 3.1. Let $\alpha(x, y)=(a_{ij}(x)y^iy^j)^{1/2}$ be a Riemannian metric. Then the following function is an F-regular Lagrangian which is neither regular nor Finslerian:

$$L(x, y)=\alpha(x, y)+1. \tag{3.8}$$

Proof. The matrix (g_{ij}) is given by

$$g_{ij}=(La_{ij}-\alpha_i\alpha_j)/\alpha, \tag{3.9}$$

where $\alpha_i=\dot{\partial}_i\alpha$. Putting $(a^{ij})=(a_{ij})^{-1}$, $\alpha^i=a^{ij}\alpha_j$, we have $t=L-\alpha_i\alpha^i=\alpha$. Since $t\neq 0$, we can see from Lemma 3.1 that (g_{ij}) is regular, and (g^{ij}) is given by

$$g^{ij}=\alpha(a^{ij}+\alpha^i\alpha^j/\alpha)/L. \tag{3.10}$$

Thus we have $l_i=\alpha_i$, $l^i=\alpha^i$, from which we have $p=0$. Q. E. D.

The function $L^*(x^*, y^*)$ associated with L in Example 3.1 is given by

$$L^*(x^*, y^*)=\alpha(x, y)+y^0, \tag{3.11}$$

provided $y^0>0$. Since $p=0$, it is not a Finsler metric. However, if we consider this L^* in the section given by $y^0=\beta$ $(=b_i(x)y^i)$, we have a Randers metric $\alpha+\beta$ [15] (cf. R. S. Ingarden [6]).

In general, for an F-regular Lagrangian $L(t, x, y)$ we put

$$z_i=g_{ij}z^j,\quad q_i=l_i-z_i/L,\quad q^i=l^i-z^i/L. \tag{3.12}$$

Then $g^*_{\alpha\beta}$ given by (1.6), (1.7), (1.8) are rewritten as

$$g^*_{00}(x^*, y^*)=L^2(p+q_iq^i), \tag{3.13}$$

$$g^*_{i0}(x^*, y^*)=Lq_i, \tag{3.14}$$

$$g^*_{ij}(x^*, y^*)=g_{ij}, \tag{3.15}$$

and especially, in the case that L is regular, corresponding to (1.14), (1.15), (1.16) the inverse matrix $(g^{*\alpha\beta})$ of $(g^*_{\alpha\beta})$ is given by

$$g^{*00}(x^*, y^*)=1/pL^2, \tag{3.16}$$

$$g^{*i0}(x^*, y^*)=-q^i/pL, \tag{3.17}$$

(3.18) $$g^{*ij}(x^*, y^*)=g^{ij}+q^i q^j/p,$$

where p, L, q_i, q^i, g_{ij}, g^{ij} in the right-hand sides mean the respecitve values at (x^0, x, z).

It is noted that (3.7) is also derived from (3.13), (3.14), (3.15), and then (3.6) is derived from (1.9), (3.7).

Lastly, we consider other methods which convert any Lagrangians to Finsler metrics. For example, we have

Theorem 3.3. *Let $L(t, x, y)$ be an F-regular Lagrangian defined on a domain D of $R\times T^0(M)$, and* $f(L)$ be a differentiable function of L, *satisfying $f'(L)\neq 0$. Then the function $L_f(x^*, y^*)$ defined by*

(3.19) $$L_f(x^*, y^*)=f(L(x^0, x, z))y^0,$$

on the domain $D^*=\{(x^*, y^*)\in T(M^*)\,|\, y^0\neq 0,\ (x^0, x, z)\in D,\ L_f(x^*, y^*)>0\}$ is a Finsler metric in M^* if and only if

(3.20) $$p=1-(1-Lf''/f')l_i l^i\neq 0.$$

For example, for $f(L)=L^2$ we have a Finsler metric

(3.21) $$L_f(x^*, y^*)=L^2(x^0, x, z)y^0.$$

If $L(x, y)$ is a Finsler metric, then (3.21) becomes

(3.22) $$L_f(x^*, y^*)=L^2(x, y)/y^0,$$

provided $y^0>0$, which is a generalization of (1.12).

We hope to consider changes of a Lagrangian $L(x, y)$ to Finsler metrics in M (not in M^*) such that L is invariant if it is a Finsler metric. For example, let $f(x, y)$ be a non-vanishing differentiable function which is positively homogeneous of degree 1, and we put

(3.23) $$L'_f(x, y)=L(x, y/f(x, y))f(x, y).$$

Does this L'_f become a Finsler metric for some f, e. g., α, β ?

The Finslerization of a Lagrange space can be applied to the Kostant-Souriau gauge theory (cf. [8.9]) by obtaining interesting Finsler structures as we will show in a next paper.

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