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ON AN INEQUALITY OF RAY-CHAUDHURI AND WILSON FOR t-DESIGNS WITH GROUP ACTION

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Abstract

We shall extend an inequality of Ray-Chaudhuri and Wilson for t-designs with group action.

1. Introduction and Summary

Throughout this paper X denotes a finite set of v elements called points and $\binom{X}{s}$ denotes the set of all subsets of X containing s points ; members of this set are called s-subsets of X.

Let $\mathfrak{B}$ be a subset of $\binom{X}{k}$ (whose elements called blocks). A t-(v, k, λ) design (or simply a t-design) is a pair $(X, \mathfrak{B})$ satisfying the following requirement :

any t-subset of X is contained in exactly λ blocks.

The cardinality of $\mathfrak{B}$ will be called b. Note that the number of blocks which contain all of i points is

$$b_i = \lambda \binom{v-i}{t-i} \Big/ \binom{k-i}{t-i}.$$

It is well known (cf, Wilson [6]) that for $i + j \leq t$, the number of blocks of a t-(v, k, λ) design $(X, \mathfrak{B})$ which contain i given points but are disjoint with any of a set of j otherpoints is

$$b_i^j = \lambda \binom{v-i-j}{k-i} \Big/ \binom{v-t}{k-t}.$$

Notice that $b_i = b_i^0$ and $b = b_0 = b_0^0$.

For $i = 0, 1, 2, \cdots$, the higher incidence matrix N_i of a t-design $(X, \mathfrak{B})$ is

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the $\binom{v}{i} \times b$ whose rows are indexed by the i-subsets of X and whose columns are indexed by blocks, with the entry in row S and column β being 1 if $S \subset \beta$ and 0 otherwise. An automorphism group G of t-design $(X, \mathfrak{B})$ is a group satisfying the following : (1) G acts on X, (2) $\beta g \in \mathfrak{B}$ for all $g \in G, \beta \in \mathfrak{B}$, (3) if $x \in \beta$, then $xg \in \beta g$. Here we note that if G acts on X, then G acts on $\binom{X}{s}$ for any s. Suppose that a finite group G acts on X and that P is a normal subgroup of G. Let Ω^P denote the set of points in Ω fixed by P. Then, Ω^P is G-invariant. Ω/G denotes the set of orbits of G on Ω. Noda [4] and independently Kreher [3] proved the following

Proposition 1. *Suppose that $(X, \mathfrak{B})$ is a $2s$-(v, k, λ) design which admits an automorphism group G. If $v \geq k + s$, then the following holds :*

$$|\mathfrak{B}/G| > |\binom{X}{s}/G|.$$

This result is an extension of the following proposition which is proved by Ray-Chaudhuri and Wilson [5] .

Proposition 2. *Suppose that $(X, \mathfrak{B})$ is a $2s$-(v, k, λ) design with $v \geq k + s$. Then*

$$|\mathfrak{B}| \geq |\binom{X}{s}|.$$

But their proof suffices for the version stated below.

Proposition 3. *Suppose that $(X, \mathfrak{B})$ is a $2s$-(v, k, λ) design with $v \geq k + s$. Let p be a prime number which does not divide $b_s{}^i$ for $0 \leq i \leq s$. Then we have over p-element field F_p,*

$$rank\ N_s = \binom{v}{s}.$$

The purpose of this paper is to generalize Proposition 1 as follows.

Theorem. *Suppose that $(X, \mathfrak{B})$ is a $2s$-(v, k, λ) design which admits an automorphism group G. Let p be a prime number which does not divide $b_s{}^i$ for $0 \leq i \leq s$ and let P be a normal p-subgroup of G. If $v \geq k + s$, then the following holds :*

$$|\mathfrak{B}^P/G| \geq |\binom{X}{s}^P/G|.$$

The above inequality is also an extension of Yoshida's inequality for 2-design [7].

2. Lemmas and Propositions

To prove Theorem we need some lemmas and propositions. In this section K denotes a p-element field F_p.

Lemma 1. *If a matrix A with entries in K is non-singular, then the inverse A^{-1} is expressible as a polynomial of the matrix A.*

Proof. We omit a proof.

Lemma 2. *Let N be a $m \times n$ matrix of rank m with entries in K. Then rank N = rank NN^t.*

Proof. We use the fact that *rank* NN^t = *rank* N^tN. So we must show that *rank* N = *rank* N^tN. It is clear that there exist two non-singular $n \times n$ matrices U, V such that

$$UN^tNV = \begin{pmatrix} a_1 & & & & & \\ & \ddots & & & 0 & \\ & & a_i & & & \\ & & & 0 & & \\ & 0 & & & \ddots & \\ & & & & & 0 \end{pmatrix} \tag{1}$$

, where for every j $a_j \neq 0$.

Let $v_1, v_2, \ldots, v_n$ be the row vectors of UN^t. Then by (1) we see that $v_1, v_2, \ldots, v_i$ are linearly independent. Since *rank* $UN^t = m$, we can find $m-i$ linearly independent vectors $v'_{i+1}, v'_{i+2}, \ldots, v'_m$ in $\{v_{i+1}, \ldots, v_n\}$. Let W be the matrix whose row vectors are $v_1, \ldots, v_i, v'_{i+1}, \ldots, v'_m$. Note that W is non-singular. By (1) we see that

$$WNV = \begin{pmatrix} a_1 & & & & & \\ & \ddots & & & 0 & \\ & & a_i & & & \\ & & & 0 & & \\ & 0 & & & \ddots & \\ & & & & & 0 \end{pmatrix}.$$

From this it is clear that *rank* WNV = *rank* $N = i$. Hence $i = m$.

In order to state Higman's result we shall follow the first section of Higman [2]. Let R be a commutative ring with identity, and X, Y, Z, be finite non-empty sets. We define $M_R(X, Y)$ to be the totality of maps $A : X \times Y \to R$ and we call A an X by Y matrix over R. If $A \in M_R(X, Y)$ and $B \in M_R(Y, Z)$, then $AB \in M_R(X, Z)$ is defined by

$$AB(x, z) = \Sigma_{y \in Y} A(x, y) B(y, z) \quad (x \in X, z \in Z).$$

Then $M_R(X, X)$ is a R-algebra.

If $\mathscr{P}, \mathscr{Q}$ are partition of X, Y, respectively, then we say that $A \in M_R(X, Y)$ has property $(\mathscr{P}, \mathscr{Q})$ if for all $S \in \mathscr{P}$, $T \in \mathscr{Q}$,

$\Sigma_{t \in T} A(s, t)$ is independent of $s \in S$.

If $A \in M_R(X, Y)$ has property $(\mathscr{P}, \mathscr{Q})$, and $S \in \mathscr{P}$, $T \in \mathscr{Q}$, we set $\delta(A)(S, T) = \Sigma_{t \in T} A(s, t)$, for some $s \in S$. Higman [2] proved the following :

Proposition 4. *If $A \in M_R(X, Y)$ has property $(\mathscr{P}, \mathscr{Q})$ and $B \in M_R(Y, Z)$ has property $(\mathscr{Q}, \mathscr{U})$, then $AB \in M_R(X, Z)$ has property $(\mathscr{P}, \mathscr{U})$ and $\delta(AB) = \delta(A)\delta(B)$.*

Corollary. $\mathfrak{A} = \{A \in M_R(X, X) \mid A$ *has property* $(\mathscr{P}, \mathscr{P})\}$ *is a subalgebra of $M_R(X, X)$, and the map δ is an algebra homomorphism of $\mathfrak{A}$ onto a subalgebra $\underline{\mathfrak{A}}$ of $M_R(\mathscr{P}, \mathscr{P})$.*

3. Proof of Theorem

Our proof is similar to that of Theorem [1]. Now we shall prove Theorem. Since P is a normal subgroup of G, $\binom{X}{s}^P$ and $\mathfrak{B}^P$ are G-invariant. Also $\binom{X}{s} - \binom{X}{s}^P$ and $\mathfrak{B} - \mathfrak{B}^P$ are G-invariant. Hence we see that

$$\binom{X}{s}^P = S_1^G \cup S_2^G \cup \cdots \cup S_m^G, \quad \mathfrak{B}^P = \beta_1^G \cup \beta_2^G \cup \cdots \cup \beta_l^G,$$

$$\binom{X}{s} - \binom{X}{s}^P = S_{m+1}^G \cup S_{m+2}^G \cup \cdots \cup S_{m+m'}^G \qquad (S_{m+i} \notin \binom{X}{s}^P),$$

$$\mathfrak{B} - \mathfrak{B}^P = \beta_{l+1}^G \cup \beta_{l+2}^G \cup \cdots \cup \beta_{l+l'}^G \quad (\beta_{l+j} \notin \mathfrak{B}^P),$$

where S_i^G and β_j^G are the G-orbits of S_i and β_j, respectively. Clearly S_i^G $(m+1 \leq i \leq m+m')$ is an union of P-orbits and so is β_j^G $(l+1 \leq j \leq l+l')$. Now we note the following trivial lemma.

Lemma 3. *$p \mid |S^P|$ for any $S \in \binom{X}{s} - \binom{X}{s}^P$ and $p \mid |\beta^P|$ for any $\beta \in \mathfrak{B} - \mathfrak{B}^P$.*

Hence we see that $p \mid |S_i^G|$ $(m+1 \leq i \leq m+m')$.
Also, we get $p \mid |\beta_j^G|$ $(l+1 \leq j \leq l+l')$.

Let N_S be the higher incidence matrix of the t-design $(X, \mathfrak{B})$. The following Lemma 4 is important for our proof.

Lemma 4. *The number of 1's in every row of the submatrix $N_s \mid_{S_i^G \times \beta_j^G}$ $(1 \leq i \leq m, l+1 \leq j \leq l+l')$ is a multiple of p, where $N_s \mid_{S_i^G \times \beta_j^G}$ is the* *restriction of mapping* N_s *on* $S_i^G \times \beta_j^G$.

Proof. See the proof of Lemma 10 [1].

Similarly the following holds :

Lemma 5. *The number of* 1'*s in every row of the submatrix* $N_s^t \mid_{\beta_j^G \times S_i^G}$ $(1 \leq j \leq 1,\ m+1 \leq i \leq m+m')$ *is a multiple of* p.

Let $\mathscr{P} = \{S_1^G, S_2^G, \cdots, S_{m+m'}^G\}$ and $\mathscr{Q} = \{\beta_1^G, \beta_2^G, \cdots, \beta_{l+l'}^G\}$. $\mathscr{P}$ and $\mathscr{Q}$ are partitions of $\binom{X}{s}$ and $\mathfrak{B}$, respectively. It is easy to prove the following :

Lemma 6. *The higher incidence matrix* N_s *of the* t-*design* $(X, \mathfrak{B})$ *has property* $(\mathscr{P}, \mathscr{Q})$. *Also* N_s^t *has property* $(\mathscr{Q}, \mathscr{P})$.

By the above lemma we may apply δ in Proposition 4 to N_s and N_s^t.

From now on we consider integral matrices as ones with entries in the p-element field F_p. Let $\mathscr{P}_1 = \{S_1^G, \cdots, S_m^G\}$, $\mathscr{P}_2 = \{S_{m+1}^G, \cdots, S_{m+m'}^G\}$, $\mathscr{Q}_1 = \{\beta_1^G, \cdots, \beta_l^G\}$ and $\mathscr{Q}_2 = \{\beta_{l+1}^G \cdots, \beta_{l+l'}^G\}$. From Lemma 4, $\delta(N_s)$ has the form

$$\delta(N_s) = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix} \tag{2}$$

, where A_{11} is a $\mathscr{P}_1$ by $\mathscr{Q}_1$ matrix, and A_{22} is a $\mathscr{P}_2$ by $\mathscr{Q}_2$ matrix. From Lemma 5, $\delta(N_s^t)$ has the from

$$\delta(N_s^t) = \begin{pmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{pmatrix} \tag{3}$$

, where B_{11} is a $\mathscr{Q}_1$ by $\mathscr{P}_1$ matrix, and B_{22} is a $\mathscr{Q}_2$ by $\mathscr{P}_2$ matrix. By applying Proposition 4, we obtain that $N_sN_s^t$ has property $(\mathscr{P}, \mathscr{P})$

$$\text{and } \delta(N_s)\,\delta(N_s^t) = \delta(N_sN_s^t). \tag{4}$$

Put $M = N_sN_s^t$. From Proposition 3 and Lemma 2, it follows that M is non-singular. By Lemma 1 $M^{-1} = f(M)$, where $f(x)$ is a polynomial. By Corollary M^{-1} has $(\mathscr{P}, \mathscr{P})$ property. By applying Proposition 4 to $I = MM^{-1}$, we obtain that $\delta(I) = \delta(M)\,\delta(M^{-1})$. (The notation "$I$" denotes the identity matrix.) It is clear that $\delta(I)$ = the identity matrix of size $m + m'$. Thus $\delta(M)$ is non-singular. From (2),(3) and (4) it follows that $A_{11}\ B_{11}$ is non-singular. Then *rank* $A_{11}\ B_{11} = m$. A_{11} must have rank at least m. Since A_{11} has size $m \times l$. we have

$m \leq l$, which proves Theorem.

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