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AN ELEMENTARY AXIOM SYSTEM OF NONSTANDARD SET THEORY

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Abstract

We provide an axiom system of nonstandard set theory which is easy for beginners in nonstandard analysis to use. To avoid the introduction of new operation and predicate symbols, the theory has a large number of fundamental symbols and axioms.

Introduction

In most axiomatic systems of nonstandard set theory, every standard infinite set has nostandard elements as in Nelson [1], Hrbacek [2], and the author [3]. For example, the standard real number field R intrinsically contains infinitesimals. This viewpoint yields a simple description of nonstandard analysis. However, many working mathematicians prefer to adopt the usual model theoretic viewpoint, from which R is the real number field and *R is the hyperreal number field containing infinitesimals. In [3], the author has proposed a nonstandard set theory UNST which has adopted this viewpoint. In [4], Kinoshita has provided a more elementary system than UNST. The purpose of the present paper is to provide a nonstandard set theory WUNST which is more elementary than UNST and is easy for beginners in nonstandard analysis to use. The number of symbols and axioms in an axiomatic system is usually made as small as possible. In nonstandard set theory, however, a predicate or operation symbol may have different meanings in standard, internal, and external universes. Whenever we introduce a new predicate or operation symbol, we need to check them. To save labor, WUNST has a large number of fundamental symbols and axioms, some of

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which could be expressed by using other ones. The theory WUNST has neither the axioms of replacement nor regularity because we seldom use them in ordinary mathematics.

The theory WUNST is obtained by weakening axioms of UNST in [3]. But it is strong enough for most applications ; for example, Loeb measure theory.

1. Definition of WUNST

We define the nonstandard set theory WUNST. Every set which WUNST deals with is called an external set. The non-logical symbols of WUNST are the following (i)-(iv) :

(i) Variables : A, B, C, ⋯, a, b, c, ⋯.
These are variables ranging over external sets.

(ii) Constants : U, I, $*$, 0, N.
These symbolize the specific external sets.

(iii) Operation symbols : $\{a\}$, $\{a, b\}$, (a, b), $A \times B$, $\bigcup A$, $A \cup B$, $P(A)$, $^*P(A)$, $\bigcap A$, $A \cap B$, $A - B$, $f(a)$, $f[A]$, $f^{-1}[A]$.

(iv) Predicate symbols : $A \in B$, $A \subset B$, $f : A \to B$, $f : A \to B (1:1)$, $f : A \to B$ (onto), $f : A \to B$ (1:1, onto).

Remark. In the above list, operation and predicate symbols are the parts obtained by erasing variables A, B, a, b, and f. For example, P( ) is the 1-placed operation symbol. We often write P for P( ).

Atomic formulas in WUNST are built up from the equality symbol = and the symbols in (i)-(iv). Formulas in WUNST are built up from atomic formulas by means of connectives $[\phi] \wedge [\psi]$ (and) ; $[\phi] \vee [\psi]$ (or) ; $\neg [\phi]$ (not) ; $[\phi] \to [\psi]$ (implies) ; $[\phi] \leftrightarrow [\psi]$ (iff) ; and quantifiers $\forall \alpha [\phi]$ (for all α) ; $\exists \alpha [\phi]$ (there exists α), where α are variables.

Every set A such that $A \in U$ ($A \in I$, resp.) is called a usual (internal, resp.) set. Intuitively, U can be identified with the universe of discourse of usual mathematics. A formula containing none of the four symbols U, I, $*$, and $^*P$ is called a Z-formula. Let ϕ be a Z-formula. Then $^U\phi$ denotes the formula obtained by replacing each occurrence $\forall \alpha$ ($\exists \alpha$, resp.) by $\forall \alpha \in U$ ($\exists \alpha \in U$, resp.), where α is a variable. Moreover, $^I\phi$ denotes the formula obtained by replacing each occurrence $\forall \alpha$ ($\exists \alpha$, N, P, resp.) by $\forall \alpha \in I$ ($\exists \alpha \in I$, $*(N)$, $^*P$ resp.), where α is a variable. The axioms of WUNST are the following (A. 1) -(A. 9) :

(A. 1) $[A \in B \wedge B \in U] \to A \in U$.

(A. 2) $[A \subset B \wedge B \in U] \to A \in U$.

(A. 3) $[A \in B \wedge B \in I] \to A \in I$.

(A. 4) This consists of the following (1)-(23) :

(1) (Axiom of Extensionality) $A=B \leftrightarrow \forall x\ [x \in A \leftrightarrow x \in B]$.
(2) $A \subset B \leftrightarrow \forall x\ [x \in A \rightarrow x \in B]$.
(3) $x \in \{a\} \leftrightarrow x=a$.
(4) (Axiom of Pairing) $x \in \{a, b\} \leftrightarrow [x = a \lor x = b]$.
(5) (Axiom of Union) $x \in \bigcup A \leftrightarrow \exists y\ [x \in y \land y \in A]$.
(6) $x \in A \cup B \leftrightarrow [x \in A \lor x \in B]$.
(7) (Axiom of Power Set) $x \in P(A) \leftrightarrow x \subset A$.
(8) (Axiom of Empty Set) $\forall x\ [x \notin 0]$.
(9) (Axiom of Natural Numbers) $N \in U \land [0 \in N \land \forall x \in N\ [x \cup \{x\} \in N]] \land \forall A\ [[0 \in A \land \forall x \in A\ [x \cup \{x\} \in A]] \rightarrow N \subset A]$.
(10) (Axiom Schema of Comprehension) Let $\phi(x)$ be a formula with a free variable x and possibly other free variables. Then

$$\forall A\ \exists B\ \forall x\ [x \in B \leftrightarrow [x \in A \land \phi(x)]].$$

(11) $[[\exists !\, y\ [(x, y) \in f] \land [(x, z) \in f]] \rightarrow f(x)=z] \land [\neg\ [\exists !\, y\ [(x, y) \in f]] \rightarrow f(x)=0]$,
where the symbol ! is used to stand for "there exists a unique".
(12) (Axiom of Choice) $\forall A\ \exists f\ \forall x \in A\ [x \neq 0 \rightarrow f(x) \in x]$.
(13) $(a, b)=(c, d) \leftrightarrow [a = c \land b = d]$.
(14) $x \in A \times B \leftrightarrow \exists a \in A\ \exists b \in B\ [x= (a, b)]$.
(15) $x \in A-B \leftrightarrow [x \in A \land x \notin B]$.
(16) $x \in \bigcap A \leftrightarrow [A \neq 0 \land \forall y \in A\ [x \in y]]$.
(17) $x \in A \cap B \leftrightarrow [x \in A \land x \in B]$.
(18) $f: A \rightarrow B \leftrightarrow [f \subset A \times B \land \forall x \in A\ \exists !\, y\ [(x, y) \in f]]$.
(19) $f: A \rightarrow B\ (1:1) \leftrightarrow [f: A \rightarrow B \land \forall x, y \in A\ [x \neq y \rightarrow f(x) \neq f(y)]]$.
(20) $f: A \rightarrow B\ (\text{onto}) \leftrightarrow [f: A \rightarrow B \land \forall y \in B\ \exists x \in A\ [f(x)=y]]$.
(21) $f: A \rightarrow B\ (1:1, \text{onto}) \leftrightarrow [f: A \rightarrow B\ (1:1) \land f: A \rightarrow B\ (\text{onto})]$.
(22) $y \in f\,[A] \leftrightarrow \exists x \in A\ [(x, y) \in f]$.
(23) $x \in f^{-1}\,[B] \leftrightarrow \exists y \in B\ [(x, y) \in f]$.

(A. 5) Let Γ be a n-placed operation symbol in (iii) (n= 1, 2) which is not *P. Then

$$\forall x \in U\ [\Gamma(x) \in U] \text{ for } n=1\,;$$
$$\forall x, y \in U\ [\Gamma(x, y) \in U] \text{ for } n=2.$$

(A. 6) $*: U \rightarrow I$. $*$ is a mapping of U into I.
If $A \in U$, then we write *A for $*(A)$.

(A. 7) (Transfer Principle) Let $\phi(x_1, \cdots, x_n)$ be a Z-formula all of whose free variables are among $x_1, \cdots, x_n$. Then

$$\forall x_1, \cdots, x_n \in U\ [{}^U\phi(x_1, \cdots, x_n) \leftrightarrow {}^I\phi({}^*x_1, \cdots, {}^*x_n)].$$

Let F(A) be the formula

$$\exists n \exists f\ [n \in N \wedge f: n \to A\ (1:1,\ \text{onto})].$$

Then ${}^{I}F(A)$ is

$$\exists n \in I\ \exists f \in I\ [n \in {}^{*}N \wedge f: n \to A\ (1:1,\ \text{onto})].$$

The formulas $F(A)$ and ${}^{I}F(A)$ are read "A is finite" and "A is $*$-finite", respectively.

(A. 8) (Axiom Schema of Enlarging) Let ϕ $(a, b, x_1, \cdots, x_n)$ be a Z-formula all of whose free variables are among $a, b, x_1, \cdots, x_n$. Then

$$\forall x_1, \cdots, x_n \in U \begin{bmatrix} \forall D \in U\ [F(D) \to \exists b \in U\ \forall a \in D\ {}^{U}\phi(a, b, x_1, \cdots, x_n) \\ \to \exists c \in I\ \forall a \in U\ {}^{I}\phi({}^{*}a, c, {}^{*}x_1, \cdots, {}^{*}x_n) \end{bmatrix}.$$

(A. 9) (Weak Extension Principle)

$$\forall A \in U\ \forall B \in I\ \forall f$$
$$[[f: A \to B] \to \exists g \in I\ [g: {}^{*}A \to B \wedge \forall x \in A\ [g({}^{*}x) = f(x)]]].$$

2. Conservation theorem

The following theorem asserts that every "usual" statement proved in WUNST holds true as a theorem of conventional mathematics. This shows that WUNST can be used for study of conventional mathematics.

Conservation theorem for WUNST. *If ϕ is a Z-formula and ${}^{U}\phi$ is the theorem of WUNST, then ϕ is the theorem of ZFC (Zermelo-Fraenkel set theory with the axiom of choice).*

Proof. This follows from the conservation theorem for UNST in [3].

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