Several examples of system of fundamental sequences to show a connection among the built-up systems

Rep. Fac. Sci., Kagoshima Univ., (Math., Phys. & Chem.),
No. 22, p. 1–6, 1989.

Several examples of system of fundamental sequences to show a connection among the built-up systems

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(Received September 4, 1989)

Abstract

We mentioned the including relations among the built-up systems which are (n)-built-up and (n)-diagonal-built-up and etc. in our previous paper [3]. In this paper, we show that each inclusion of those is proper.

1. Notions and notations

Let $\varDelta$ be an countable ordinal or the first uncountable ordinal. We will use Greek letters $\alpha, \beta, \gamma, \ldots$, for ordinal numbers in $\varDelta$. When α is limit ordinal in $\varDelta$ and $\alpha_i < \varDelta$ for all $i < \omega$, the sequence $\langle \alpha_i \rangle_{i<\omega}$ is a *fundamental sequence for* α if $\alpha_i < \alpha_{i+1} < \alpha$ for all $i < \omega$ and $\lim_{i<\omega} \alpha_i = \alpha$. We write $\alpha[i]$ for α_i. Assume that a fundamental sequence for each limit ordinal in $\varDelta$ is given. We call $\mathbf{P}: \varDelta \to \varDelta^\omega$ a *system of fundamental sequences for* $\varDelta$ if

$$\mathbf{P}(\beta) = \begin{cases} \lambda x.\beta_x & \text{if } \beta \text{ is limit ordinal, where } \langle \beta_i \rangle_{i<\omega} \text{ is the fundamental sequence for } \beta, \\ \lambda x.\gamma & \text{if } \beta = \gamma + 1, \\ \lambda x.0 & \text{if } \beta = 0. \end{cases}$$

We shall write $\alpha[x]_{\mathbf{P}}$ or simply $\alpha[x]$ for $(\mathbf{P}(\alpha))(x)$ whenever $\alpha < \varDelta$ and $x < \omega$. We define notation $\alpha \underset{n}{\rightarrow} \beta$ recursively, i) if $\alpha[n] = \beta$ then we write $\alpha \underset{n}{\rightarrow} \beta$, ii) if $\alpha[n] \underset{n}{\rightarrow} \beta$ then we write $\alpha \underset{n}{\rightarrow} \beta$. We also use $\alpha \underset{n}{\Rightarrow} \beta$ if either $\alpha \underset{n}{\rightarrow} \beta$ or $\alpha = \beta$.

Definition 1.1. (1) $\mathbf{P}$ *is* (n)*-built-up if* $\alpha[x+1] \underset{n}{\rightarrow} \alpha[x]$ *for every limit ordinal* $\alpha < \varDelta$ *and* $x < \omega$.

(2) $\mathbf{P}$ *is* (n)*-diagonal-built-up if* $\alpha[x+1] \underset{x+n}{\rightarrow} \alpha[x]$ *for every limit ordinal* $\alpha < \varDelta$ *and* $x < \omega$.

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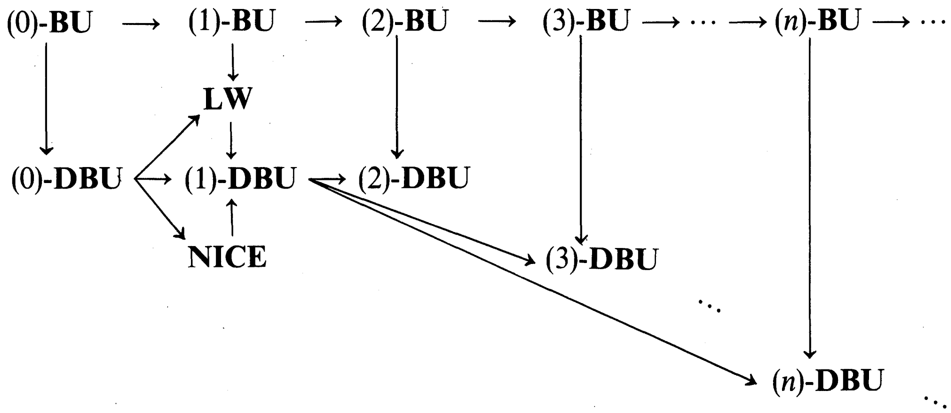
(3) **P** *is LW* (*used by Löb-Wainer in* [5]) *if* $\alpha[1] \underset{1}{\rightarrow} \alpha[0]$ *and* $\alpha[x+1] \underset{x}{\rightarrow} \alpha[x]$ *for every limit ordinal* $\alpha < \Delta$ *and* $0 < x < \omega$.

(4) **P** *is nice if* $\alpha[x+1] \underset{x+1}{\Longrightarrow} \alpha[x] + 1$ *for every limit ordinal* $\alpha < \Delta$ *and* $x < \omega$.

Let (*n*)-**BU**, (*n*)-**DBU**, **NICE**, **LW** be the class of all (*n*)-built-up, (*n*)-diagonal-built-up, nice and LW systems of fundamental sequences for Δ, respectively. In [3], we showed that the following relations holds;



where $S \rightarrow S'$ means that S' contains S. We asserted in [3] that each arrow means that S' contains S properly, but we did not prove properness. We will prove this in the next section.

2. Proof of Properness

By means of an example which is not S system but S', we show properness of inclusion $S \rightarrow S'$.

Lemma 2.1. *The system of fundamental sequences for* $\omega \cdot 2 + 1$ *determined by the following is not* (*k*)*-built-up but* (*k* + 1)*-built-up*;

$$\omega[x] = x,$$

$$\omega \cdot 2[x] = \begin{cases} x & \text{for } x \leq k+1, \\ \omega + x & \text{for } x > k+1. \end{cases}$$

Proof. Only ω and $\omega \cdot 2$ are limit ordinals in $\omega \cdot 2 + 1$. We can easily check that $\omega[x+1] \underset{k+1}{\rightarrow} \omega[x]$ for $x < \omega$, then we show that $\omega \cdot 2[x+1] \underset{k+1}{\rightarrow} \omega \cdot 2[x]$ for all $x < \omega$. We distinguish three cases. Case 1) $x < k+1$. $\omega \cdot 2[x+1] = x+1$ and $\omega \cdot 2[x] = x$, and $(x+1)[k+1] = x$, then $\omega \cdot 2[x+1] \underset{k+1}{\rightarrow} \omega \cdot 2[x]$. Case 2) $x = k+1$. $\omega \cdot 2[x+1] = \omega + x + 1 = \omega + k + 2$ and $\omega \cdot 2[x] = x = k+1$. $\omega + k + 2$ becomes $k+1$ by applying "$[k+1]$" $k+3$ times. Namely, $\omega \cdot 2[x+1] \underset{k+1}{\rightarrow} \omega \cdot 2[x]$. Case 3) $x > k$

$+ 1$. $\omega\cdot 2[x+1] = \omega + x + 1$ and $\omega\cdot 2[x] = \omega + x$, and $(\omega + x + 1)[k+1] = \omega + x$, then $\omega\cdot 2[x+1] \xrightarrow[k+1]{} \omega\cdot 2[x]$. Hence, this system in $(k+1)$-built-up. It dose not hold that $\omega\cdot 2[k+2] \xrightarrow[k]{} \omega\cdot 2[k+1]$ because $\omega\cdot 2[k+2] = \omega + k + 2 \xrightarrow[k]{} \omega > \omega\cdot 2[k+1]$ and $\omega[k] = k < \omega\cdot 2[k+1]$. Namely, this system is not (k)-built-up.

Lemma 2.2. *The system of fundamental sequences for* $\omega^2 + 1$ *determined by the following is not* (0)*-diagonal-built-up but nice*;

$$(\omega\cdot(n+1))[x] = \omega\cdot n + 2\cdot x,$$

$$\omega^2[x] = \begin{cases} 1 & \textit{for } x = 0, \\ \omega\cdot x + x & \textit{for } x > 0. \end{cases}$$

Proof. Remark that each limit ordinal in $\omega^2 + 1$ has the form $\omega\cdot(n+1)$ for $n < \omega$ or ω^2. $(\omega\cdot(n+1))[x+1] = \omega\cdot n + 2\cdot x + 2$ and $(\omega\cdot n + 2\cdot x + 2)[x+1] = \omega\cdot n + 2\cdot x + 1$ $= \omega\cdot(n+1)[x] + 1$, then $(\omega\cdot(n+1))[x+1] \xrightarrow[x+1]{} \omega\cdot(n+1)[x] + 1$. Assume that $x > 0$. $\omega^2[x+1] = \omega\cdot(x+1) + x + 1$ and $\omega\cdot(x+1) + x + 1 \xrightarrow[x+1]{} \omega\cdot(x+1) \xrightarrow[x+1]{} \omega\cdot x + 2\cdot x + 2$ $\xrightarrow[x+1]{} \omega\cdot x + 2\cdot x + 1 = \omega^2[x] + 1$, then $\omega^2[x+1] \xrightarrow[x+1]{} \omega^2[x] + 1$ for $x > 0$. $\omega^2[1] =$ $\omega + 1 \xrightarrow[1]{} \omega \xrightarrow[1]{} 2 = 1 + 1 = \omega^2[0] + 1$, namely, $\omega^2[1] \xrightarrow[1]{} \omega^2[0] + 1$. Hence, this system is nice. Since $\omega^2[1] \xrightarrow[0]{} \omega > \omega^2[0]$ and $\omega[0] = 0 < 1 = \omega^2[0]$, it dose not hold that $\omega^2[1] \xrightarrow[0]{} \omega^2[0]$. Hence, this system is not (0)-diagonal-built-up.

Lemma 2.3. *The system of fundamental sequences for* $\omega^2 + 1$ *determined by the following is neither* (0)*-diagonal-built-up nor nice but LW*;

$$(\omega\cdot(n+1))[x] = \omega\cdot n + x,$$

$$\omega^2[x] = \begin{cases} 1 & \textit{for } x = 0, \\ \omega\cdot x + x & \textit{for } x > 0. \end{cases}$$

Proof. Firstly, we show that this system is LW. $(\omega\cdot(n+1))[x+1] = \omega\cdot n + x + 1$ and $(\omega\cdot n + x + 1)[x] = \omega\cdot n + x = (\omega\cdot(n+1))[x]$, then $(\omega\cdot(n+1))[x+1] \xrightarrow[x]{} (\omega\cdot(n+1))$ $[x]$. $(\omega\cdot(n+1))[1] \xrightarrow[1]{} (\omega\cdot(n+1))[0]$ also holds. Let $x > 0$. $\omega^2[x+1] = \omega\cdot(x+1) +$ $x + 1$ and $(\omega\cdot(x+1) + x + 1) \xrightarrow[x]{} \omega\cdot(x+1)$ and $(\omega\cdot(x+1))[x] = \omega^2[x]$, then $\omega^2[x+1]$ $\xrightarrow[x]{} \omega^2[x]$. Since $\omega^2[1] = \omega + 1$ and $(\omega + 1) \xrightarrow[1]{} \omega$ and $\omega[1] = 1 = \omega^2[0]$, then $\omega^2[1] \xrightarrow[1]{} \omega^2[0]$. Hence, this system is LW. Secondly, we show that this system is not (0)-diagonal-built-up. $\omega^2[1] = \omega + 1$ and $(\omega + 1)[0] = \omega > \omega^2[0]$ and $\omega[0] = 0$ $< \omega^2[0]$, then it dose not hold that $\omega^2[1] \xrightarrow[0]{} \omega^2[0]$. Namely, this system is not (0)-diagonal-built-up. Lastly, we show that this system is not nice. $\omega^2[1] \xrightarrow[1]{} \omega > \omega^2[0]$

$+ 1$ and $\omega[1] = 1 < \omega^2[0] + 1$, then it dose not hold that $\omega^2[1] \xrightarrow[1]{} \omega^2[0] + 1$. Hence, this system is not nice.

Lemma 2.4. *The system of fundamental sequences for $\omega^2 + 1$ determined by the following is not LW, not nice and not (j)-diagonal-built-up for $j < k$, but is (k)-diagonal-built-up*;

$$(\omega\cdot(n + 1))[x] = \omega\cdot n + x,$$

$$\omega^2[x] = \omega\cdot(x + k) + x + k.$$

Proof. We first show that this system is (k)-diagonal-built-up. $(\omega\cdot(n + 1))[x + 1] = \omega\cdot n + x + 1$ and $(\omega\cdot n + x + 1)[x + k] = \omega\cdot n + x = (\omega\cdot(n + 1))[x]$, then $(\omega\cdot(n + 1))[x + 1] \xrightarrow[x+k]{} (\omega\cdot(n + 1))[x]$. $\omega^2[x + 1] = \omega\cdot(x + 1 + k) + x + 1 + k$ and $\omega\cdot(x + 1 + k) + x + 1 + k \xrightarrow[x+k]{} \omega\cdot(x + 1 + k)$ and $(\omega\cdot(x + 1 + k))[x + k] = \omega\cdot(x + k) + x + k = \omega^2[x]$, then $\omega^2[x + 1] \xrightarrow[x+k]{} \omega^2[x]$. Hence, this system is (k)-diagonal-built-up. Next, we show that this system is not LW. $\omega^2[x + 1] \xrightarrow[x]{} \omega\cdot(x + 2) > \omega^2[x]$ and $(\omega\cdot(x + 2))[x] = \omega\cdot(x + 1) + x < \omega^2[x]$, then it dose not hold that $\omega^2[x + 1] \xrightarrow[x]{} \omega^2[x]$. Hence, this system is not LW. We next show that this system is not nice. $\omega^2[x + 1] \xrightarrow[x+1]{} \omega\cdot(x + 2) > \omega^2[x] + 1$ and $(\omega\cdot(x + 2))[x + 1] = \omega\cdot(x + 1) + (x + 1) < \omega^2[x] + 1$, then it dose not hold that $\omega^2[x + 1] \xrightarrow[x+1]{} \omega^2[x] + 1$. Hence, this system is not nice. Lastly, we show that this system is not (j)-diagonal-built-up for $j < k$. $\omega^2[x + 1] = \omega\cdot(x + 1 + k) + x + 1 + k$ and $\omega\cdot(x + 1 + k) + x + 1 + k \xrightarrow[x+j]{} \omega\cdot(x + 1 + k) > \omega^2[x]$ and $(\omega\cdot(x + 1 + k))[x + j] = \omega\cdot(x + k) + x + j < \omega^2[x]$, then it dose not hold that $\omega^2[x + 1] \xrightarrow[x+j]{} \omega^2[x]$. Hence, this system is not (j)-diagonal-built-up.

Lemma 2.5. *The system of fundamental sequences for $\omega^2 + 1$ determined by the following is not LW but nice*;

$$(\omega\cdot(n + 1))[x] = \omega\cdot n + 2\cdot x,$$

$$\omega^2[x] = \omega\cdot x + 2\cdot x + 1.$$

Proof. $(\omega\cdot(n + 1))[x + 1] = \omega\cdot n + 2\cdot x + 2$ and $(\omega\cdot n + 2\cdot x + 2)[x + 1] = \omega\cdot n + 2\cdot x + 1 = (\omega\cdot(n + 1))[x] + 1$, then $(\omega\cdot(n + 1))[x + 1] \xrightarrow[x+1]{} (\omega\cdot(n + 1))[x] + 1$. $\omega^2[x + 1] = \omega\cdot(x + 1) + 2\cdot x + 3$ and $\omega\cdot(x + 1) + 2\cdot x + 3 \xrightarrow[x+1]{} \omega\cdot(x + 1)$ and $(\omega\cdot(x + 1))[x + 1] = \omega\cdot x + 2\cdot x + 2 = \omega^2[x] + 1$, then $\omega^2[x + 1] \xrightarrow[x+1]{} \omega^2[x] + 1$. Hence, this system is nice. Since $\omega^2[x + 1] \xrightarrow[x]{} \omega\cdot(x + 1) > \omega^2[x]$ and $(\omega\cdot(x + 1))[x] = \omega\cdot x + 2\cdot x < \omega^2[x]$, it dose not hold that $\omega^2[x + 1] \xrightarrow[x]{} \omega^2[x]$. Namely, this system is not LW.

ω_n is defined recursively; i) $\omega_0 = 1$, ii) $\omega_{n+1} = \omega^{(\omega_n)}$. The example in the next lemma is (k)-diagonal-built-up, where $k > 1$, but dose not have the following property;

for all limit ordinal α *and* β *in* Δ *and*
for all $x < \omega$,

if $\alpha \underset{x}{\rightarrow} \beta$ *then* $\alpha \underset{x+1}{\rightarrow} \beta$.

So the condition "$k = 0$ *or* 1" can not eliminate from the lemma 2.3.(2) in [3] "*If* **P** *is* (k)-*diagonal-built-up for some* $k = 0,\ 1,\ \alpha \underset{m}{\rightarrow} \beta$ *and* $m < s$, *then* $\alpha \underset{s}{\rightarrow} \beta$.".

Lemma 2.6. *The system of fundamental sequences for* $\varepsilon_0 + \omega + 1$ *determined by the following is not* $(k+1)$-*diagonal-built-up but* (k)-*diagonal-built-up for* $k > 1$;

$$(\omega^\alpha + \beta)[x] = \begin{cases} \omega^\alpha + \beta[x] & \text{if } \beta \neq 0, \\ \omega^{\alpha[x]} & \text{if } \beta = 0 \text{ and } \alpha \text{ is limit}, \\ \omega^\gamma \cdot x & \text{if } \beta = 0 \text{ and } \alpha = \gamma + 1, \end{cases}$$

$$\varepsilon_0[x] = \begin{cases} \omega^{x+k} & \text{if } x \leq k, \\ \omega_{x+1-k} & \text{if } x > k, \end{cases}$$

$$(\varepsilon_0 + \omega)[x] = \begin{cases} \omega^{2\cdot k} & \text{if } x = 0, \\ \varepsilon_0 & \text{if } x = 1, \\ \varepsilon_0 + x & \text{if } x > 1. \end{cases}$$

Proof. Since each limit ordinal α in ε_0, $\alpha[x+1] \underset{1}{\rightarrow} \alpha[x]$ holds (cf. [1] [2] [3] [4]), $\alpha[x+1] \underset{x+k}{\rightarrow} \alpha[x]$ also holds (cf. [1] [2] [3]). Assume that $x < k$. $\varepsilon_0[x+1] = \omega^{x+1+k}$ and $\varepsilon_0[x] = \omega^{x+k} \cdot (\omega^{x+1+k})[x+k] = \omega^{x+k} \cdot (x+k)$ and $\omega^{x+k} \cdot (x+k) \underset{x+k}{\rightarrow} \omega^{x+k} \cdot (x+k-1) \underset{x+k}{\rightarrow} \omega^{x+k} \cdot (x+k-2) \underset{x+k}{\rightarrow} \cdots \underset{x+k}{\rightarrow} \omega^{x+k} = \varepsilon_0[x]$. Assume that $x = k$. $\varepsilon_0[x+1] = \omega_{x+1+1-k} = \omega^\omega$ and $(\omega^\omega)[x+k] = \omega^{x+k} = \varepsilon_0[x]$. Assume that $x > k$. $\varepsilon_0[x+1] = \omega_{x+1+1-k} \underset{x+k}{\rightarrow} \omega_{x+1-k} = \varepsilon_0[x]$ (cf. [4]). Hence, $\varepsilon_0[x+1] \underset{x+k}{\rightarrow} \varepsilon_0[x]$ for all x. For $x \geq 1$, $(\varepsilon_0 + \omega)[x+1] \underset{x+k}{\rightarrow} (\varepsilon_0 + \omega)[x]$ is easy. $(\varepsilon_0 + \omega)[1] = \varepsilon_0$ and $\varepsilon_0[k] = \omega^{k+k} = \omega^{2\cdot k} = (\varepsilon_0 + \omega)[0]$. Hence, $(\varepsilon_0 + \omega)[x+1] \underset{x+k}{\rightarrow} (\varepsilon_0 + \omega)[x]$ for all x. After all, for each limit ordinal α in $\varepsilon_0 + \omega + 1$, $\alpha[x+1] \underset{x+k}{\rightarrow} \alpha[x]$. This system is (k)-diagonal-built-up. Whereas, it dose not hold that $(\varepsilon_0 + \omega)[1] \underset{k+1}{\rightarrow} (\varepsilon_0 + \omega)[0]$. Indeed, $(\varepsilon_0 + \omega)[1] = \varepsilon_0$ and $\varepsilon_0[k+1] = \omega_{k+1+1-k} = \omega^\omega > (\varepsilon_0 + \omega)[0]$ and $\omega^\omega[k+1] = \omega^{k+1} < \omega^{2\cdot k} = (\varepsilon_0 + \omega)[0]$. Hence, this system is not $(k+1)$-diagonal-built-up.

Theorem. *Each arrow in the figure illustrated the previous section means properly containing.*

Proof. Recall that (0)-diagonal-built-up, and hence LW, nice and (n)-diagonal-built-up for each $n < \omega$, for the first uncountable ordinal exists, but (n)-built-up system for the ordinal does not exist for all $n < \omega$([3]). Considering this fact, this theorem is immediately consequence from Lemmata 2.1–2.6.

We can see by Lemmata 2.1–2.6 that any arrow can not be added in that figure. In this sence, that figure is complete.

3. Other remarks

We can construct a (n)-built-up system $\mathbf{Q}$ for Δ from a given ($n+1$)-built-up system $\mathbf{P}$ for Δ. Arrow-notation in $\mathbf{P}$ and $\mathbf{Q}$ will be written $\xrightarrow{\mathbf{P}}$ and $\xrightarrow{Q}$, respectively. We define $\mathbf{Q}$ as follows:

$$(\mathbf{Q}(\alpha))(x) = (\mathbf{P}(\alpha))(x+1) \quad \text{for all } \alpha < \Delta.$$

Let α be an arbitrary limit ordinal in Δ. We will show that $\alpha[x+1]_Q \xrightarrow[n]{Q} \alpha[x]_Q$. $\alpha[x+1]_Q = \alpha[x+2]_\mathbf{P}$ and $\alpha[x]_Q = \alpha[x+1]_\mathbf{P}$. Since $\mathbf{P}$ is ($n+1$)-built-up, $\alpha[x+2]_\mathbf{P} \xrightarrow[n+1]{\mathbf{P}} \alpha[x+1]_\mathbf{P}$ holds, namely, $\alpha[x+2]_\mathbf{P}[n+1]_\mathbf{P} \cdots [n+1]_\mathbf{P} = \alpha[x+1]_\mathbf{P}$. That is $\alpha[x+1]_Q [n]_Q \cdots [n]_Q = \alpha[x]_Q$, namely, $\alpha[x+1]_Q \xrightarrow[n]{Q} \alpha[x]_Q$. Hence, Q is (n)-built-up. In the same way as above, we can construct (n)-diagonal-built-up system for Δ from a given ($n+1$)-diagonal-built-up system for Δ.

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