A Note on Automorphisms of Finsler Bundles

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A Note on Automorphisms of Finsler Bundles*

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Abstracts

In the present paper, we shall consider the action of an automorphism of Finsler bundles to the affine space of Finsler connections, which is an analogy of the action of gauge transformation of principal fibre bundles to the affine space of all connections.

Key Words: Finsler bundle, Finsler connections, automorphisms.

1. Introduction

As is well-known, the set of all connections of a principal bundle has the structure of affine space with a vector space, and gauge transformations of the bundle, which induce identity maps on the base space, act on this affine space. In the case where the given bundle is a Finsler bundle, the second author has investigated some topics as the analogy of such a theory, and obtained some results (cf. Otsuji [6]).

On the other hand, Matsumoto [4] studied automorphisms of a Finsler bundle, which are called *left translations* of the bundle in his paper, and investigated the conditions that an automorphism preserves a Finsler connection invariant. In other words, he obtained the conditions that a gauge transformation of a Finsler bundle is contained in the isotropy group of a Finsler connection.

In this short report, suggested by Matsumoto's paper [4], we also study the gauge transformations of Finsler bundles and Finsler vector bundles, and investigate their actions to the affine space of Finsler connections. Furthermore, we give another simple proof of the

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main theorem in [4]. It must be noted that our present study is a special case of general theory, since Finsler bundles or Finsler vector bundles are defined as special fiber bundles over the tangent bundle of the base manifold.

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2. Finsler bundles and Finsler vector bundles

Let M be an n-dimensional differentiable manifold, and $\pi_T : TM \to M$ its tangent bundle. A principal bundle FM over TM is called a *Finsler bundle* if it is the pull-back $\pi_T{}^* P$ of a principal G-bundle $\pi_P : P \to M$ over M by the projection π_T:

$$FM = \{(y, z) \in TM \times P; \pi_T(y) = \pi_P(z)\}.$$

We denote by π_1 and π_2 the natural projection from FM to TM and P respectively. Then, $\pi_1 : FM \to TM$ is also a G-bundle over TM.

First we state the bundle structure of FM. We take a local trivialization $\{U_\lambda, \Phi_\lambda\}$ of the G-bundle $\pi_P : P \to M$. Putting $\tilde{U}_\lambda = \pi_T{}^{-1}(U_\lambda)$, $\{\tilde{U}_\lambda\}$ gives a covering of TM. Then, on each $\tilde{U}_\lambda \times G$, we define a diffeomorphism $\tilde{\Phi}_\lambda : \tilde{U}_\lambda \times G \to \pi_1{}^{-1}(\tilde{U}_\lambda)$ by

$$\tilde{\Phi}_\lambda(y, g) = (y, \Phi_\lambda(x, g)), \quad x = \pi_T(y).$$

The covering $\{\tilde{U}_\lambda\}$ with local diffeomorphisms $\{\tilde{\Phi}_\lambda\}$ gives a local trivialization of FM. If we put $\Phi_\lambda{}^{-1} = (\pi_P, \phi_\lambda)$ for a differential mapping $\phi_\lambda : \pi_P{}^{-1}(U_\lambda) \to G$, the inverse $\tilde{\Phi}_\lambda{}^{-1} : \pi_1{}^{-1}(\tilde{U}_\lambda) \to \tilde{U}_\lambda \times G$ is given by

$$\tilde{\Phi}_\lambda{}^{-1} = (\pi_1, \tilde{\phi}_\lambda), \quad \tilde{\phi}_\lambda = \pi_2{}^* \phi_\lambda.$$

The transition functions $\tilde{f}_{\lambda\mu} : \tilde{U}_\lambda \cap \tilde{U}_\mu \to G$ of FM given by $(\tilde{\Phi}_\mu \circ \tilde{\Phi}_\lambda{}^{-1})(y, g) = (y, \tilde{f}_{\mu\lambda}(y) g)$ are expressed as follows:

$$\tilde{f}_{\mu\lambda}(y) = \tilde{\phi}_\mu(u) \tilde{\phi}_\lambda(u)^{-1} = (\pi_T{}^* f_{\mu\lambda})(y)$$

for the transition functions $f_{\mu\lambda}$ of P at $y \in \tilde{U}_\lambda \cap \tilde{U}_\mu$ for $^\forall u \in \pi_2{}^{-1}(y)$. Then we have

Proposition 2.1. *Let FM be a principal G-bundle over TM. If its transition functions are given by* $\pi_T^* f_{\lambda\mu}$ *for the ones* $f_{\lambda\mu}$ *of a principal G-bundle P over M, then FM is equivalent to the Finsler bundle* $\pi_T^* P$*, that is,* $FM=\pi_T^* P$.

Next we consider Finsler vector bundles which associate with a Finsler bundle FM. Suppose that the structure group G of FM acts on a vector space F of finite dimension k. Let $\rho: G \to GL(F)$ be the given linear representation of G. Then an action of G on the product manifold $FM\times F$ is defined by $(u, v)g=(ug, \rho(g^{-1})v)$. By this action, we can introduce an equivalent relation "$\approx$" on $FM\times F$ as follows.

Let (u_1, v_1) and (u_2, v_2) be elements of $FM\times F$. Then we define $(u_1, v_1)\approx(u_2, v_2)$ if and only if there exists an element $g\in G$ satisfying $(u_1, v_1)=(u_2, v_2)g$. We denote by $FM\times_\rho F$ the quotient space $FM\times F/\approx$ which admits the structure of a vector bundle over TM of rank (= fibre dimension) k. Such a vector bundle associated with FM is called a *Finsler vector bundle.*

Let $\pi_F: \boldsymbol{F}=FM\times_\rho F \to TM$ be a Finsler vector bundle. We denote by $[u, v]$ the equivalent class of $(u, v)\in FM\times F$. Then we define a local diffeomorphism $\tilde{\Psi}_\lambda: \pi_F^{-1}(\tilde{U}_\lambda) \to \tilde{U}_\lambda\times F$ by

$$\tilde{\Psi}_\lambda([u, v])=(\pi_1(u), \rho(\tilde{\phi}_\lambda(u))v)$$

for the given local diffeomorphism $\tilde{\Phi}_\lambda^{-1}=(\pi_1, \tilde{\phi}_\lambda): \pi_1^{-1}(\tilde{U}_\lambda) \to \tilde{U}_\lambda\times G$ of FM. Then $\{\tilde{U}_\lambda, \tilde{\Psi}_\lambda\}$ gives a local trivializtion of $\boldsymbol{F}$. Then we have

Proposition 2.2. *Let* $\boldsymbol{F}$ *be a vector bundle over TM with a standard fibre F. Suppose that the transition functions of* $\boldsymbol{F}$ *are given by* $\rho\circ(\pi_T^* f_{\lambda\mu})$ *for a linear representation* $\rho: G\to GL(F)$ *and the transition functions* $f_{\lambda\mu}$ *of a principal G-bundle P over M. Then* $\boldsymbol{F}$ *is isomorphic to the vector bundle* $FM\times_\rho F$ *associated with the Finsler bundle* $FM=\pi_T^* P$*, that is,* $\boldsymbol{F}=FM\times_\rho F$.

3. Automorphisms preserving a Finsler connection invariant

In this section, we shall consider automorphisms of FM, and give another proof of the main theorem in Matsumoto [4]. We denote by $Aut(FM)$ the set of all automorphisms which induce identity maps on the tangent bundle of the base manifold M:

$$Aut(FM)=\{f: FM \to FM;\ f(ug)=f(u)g,\ \forall u\in FM,\ \forall g\in G\}.$$

We may call the group $Aut(FM)$ the *gauge group* of FM and each element of $Aut(FM)$ a *gauge transformation* of FM. We call in this report, however, each element of $Aut(FM)$ an *automorphism* of FM merely. For an arbitrary $f\in Aut(FM)$, we define a mapping $g_\lambda: U_\lambda \to G$ by

$$g_\lambda(y)=\tilde{\phi}_\lambda(f(u))\tilde{\phi}_\lambda(u)^{-1}$$

for $^\forall u\in\pi^{-1}(y)$, where we see easily that the right-hand-side is independent on the choice of $u\in\pi^{-1}(y)$. If $\tilde{U}_\lambda\cap\tilde{U}_\mu\neq\phi$, then we see easily the relation

$$g_\mu(y)=Ad_{f_{\mu\lambda}(y)}(g_\lambda(y))$$

for $^\forall y\in\tilde{U}_\lambda\cap\tilde{U}_\mu$, where $Ad_{f_{\mu\lambda}(y)}: G\to G$ means the inner automorphism of the structure group G by $\tilde{f}_{\lambda\mu}(y)\in G$. Hence the family $g=\{g_\lambda\}$ is a section of the *automorphism bundle* $G_{FM}=FM\times_{Ad}G$ of FM. $Aut(FM)$ is isomorphic to $\Gamma(G_{FM})$ as a group, where $\Gamma(G_{FM})$ is the group of all sections of the bundle G_{FM}. In fact, if a section $g=\{g_\lambda\}$ of G_{FM} is given, then an automorphism $f\in Aut(FM)$ corresponds as follows:

$$f: \tilde{\Phi}_\lambda^{-1}(y, \tilde{\phi}_\lambda(u))\to\tilde{\Phi}_\lambda^{-1}(y, L_{g_\lambda(y)}\tilde{\phi}_\lambda(u))$$

where the notation $L_{g_\lambda(y)}$ means the left translation of G by $g_\lambda(y)\in G$.

In the following, we denote by $\{\tilde{U}_\lambda, (x^i, y^i)\}$ the canonical covering on TM induced from a covering by the system of coordinate neighborhoods $\{U_\lambda, (x^i)\}$ on M. We also denote by $(x^i, y^i, z^\alpha_\beta)$ the local coordinate on a neighborhood $\pi_1^{-1}(\tilde{U}_\lambda): \tilde{\Phi}^{-1}(y, z)=(x^i, y^i, z^\alpha_\beta)$. If we put $g_\lambda(y)=(g^\alpha_{\lambda\beta}(y))$, the local expression of $f\in Aut(FM)$ is given as follows:

$$f: u=(x^i, y^i, z^\alpha_\beta)\to f(u)=(x^i, y^i, g^\delta_{\lambda\beta}z^\alpha_\delta). \tag{3.1}$$

A connection of FM is called a *Finsler connection*, and is given by a family $\omega=\{\omega_\lambda\}$ of local 1-forms ω_λ on each $\tilde{U}_\lambda$ satisfying the cocycle condition:

$$\omega_\mu=\tilde{f}_{\lambda\mu}^{-1}d\tilde{f}_{\lambda\mu}+\tilde{f}_{\lambda\mu}^{-1}\omega_\lambda\tilde{f}_{\lambda\mu}. \tag{3.2}$$

Then we have also a global 1-form $\tilde{\omega}$ on FM as follows:

$$\tilde{\omega}=z^{-1}dz+z^{-1}(\pi^*\omega)z. \tag{3.3}$$

If an automorphism $f\in Aut(FM)$ or $g\in\Gamma(G_{FM})$ is given, because of (3.1) and (3.3), the action of $g\in\Gamma(G_{FM})$ to the global form $\tilde{\omega}$ is given as follows:

$$\begin{aligned}g^*\tilde{\omega}&=(gz)^{-1}d(gz)+(gz)^{-1}(\pi^*\omega)(gz)\\&=z^{-1}dz+z^{-1}(\pi^*(g^{-1}dg+g^{-1}\omega g))z.\end{aligned}$$

Hence we get the law of the action by $g\in\Gamma(G_{FM})$ to a connection $\omega=\{\omega_\lambda\}$ as follows:

(3.4) $$g^*\omega=g^{-1}dg+g^{-1}\omega g,$$

which is well-known formula in gauge theory. From this formula, we can easily prove the main theorem in [4]. Since $g_\lambda(y)=(g^i_{\lambda j}(y))\in G$, we may identify $g\in\Gamma(G_{FM})$ as a section of $End(\boldsymbol{F})$, where $\boldsymbol{F}=FM\times_\rho F$ is an associated bundle with FM. Then the formula (3.4) is written as follows:

$$g^*\omega=g^{-1}dg+g^{-1}\omega g=g^{-1}(dg+\omega g-g\omega)+\omega=g^{-1}\nabla g+\omega,$$

where $\nabla:\Gamma(\boldsymbol{F})\to\Gamma(\boldsymbol{F}\otimes TTM^*)$ means the covariant derivation with respect to $\omega=\{\omega_\lambda\}$. Hence we have

Theorem 3.1. *A Finsler connection $\omega=\{\omega_\lambda\}$ on FM is invariant by the action of $g\in\Gamma(G_{FM})$ if and only if the condition $\nabla g=0$ is satisfied, where we identify $g\in\Gamma(G_{FM})$ with a section of $End(\boldsymbol{F})$.*

Remark 3.1. If a *non-linear connection* is given on TM, then the covariant derivation ∇ splits as $\nabla=\nabla^h+\nabla^v$ (cf. Aikou [1] and Example below). Then the condition $\nabla g=0$ in the theorem is written as $\nabla^h g=\nabla^v g=0$, which is just the condition given in Matsumoto [4].

Next we consider automorphisms of vector bundles which associate with a Finsler bundle FM. If an automorphism $g=\{g_\lambda\}\in\Gamma(G_{FM})$ is given, we get a natural automorphism of an arbitrary vector bundle $\boldsymbol{F}=FM\times_\rho F$ as follows.

For a given $g\in\Gamma(G_{FM})$, an automorphism $f\in Aut(\boldsymbol{F})$ is induced on an arbitrary vector bundle $\boldsymbol{F}=FM\times_\rho F$ associated with FM as follows:

(3.5) $$f:\tilde{\Psi}_\lambda^{-1}(y,v)\to\tilde{\Psi}_\lambda^{-1}(y,\rho(g_\lambda(y))v).$$

Example. Let $\sigma:\boldsymbol{Q}\to\sigma(\boldsymbol{Q}):=\boldsymbol{H}\subset TTM$ be a non-linear connection on TM, that is, σ is a splitting of the exact sequence

$$\boldsymbol{0}\to\boldsymbol{V}\to TTM\underset{\sigma}{\rightleftarrows}\boldsymbol{Q}\to\boldsymbol{0},$$

where $\boldsymbol{V}=Ker.\,d\pi$ and the quotient bundle $\boldsymbol{Q}=TTM/\boldsymbol{V}$. If we define a vector bundle $\boldsymbol{H}$ by $\boldsymbol{H}=\sigma(\boldsymbol{Q})$, the tangent bundle TTM of TM splits as $TTM=\boldsymbol{H}\oplus\boldsymbol{V}$. $\boldsymbol{H}$ and $\boldsymbol{V}$ are vector bundles over TM of rank n, and called the *horizontal* and *vertical vector bundle* respectively.

For the canonical covering by the system of coordinate neighborhoods $\{\tilde{U}_\lambda,(x^i,y^i)\}$ on TM, the following vector fields $\{X_i\}$, $\{Y_i\}$ $(1\leq i\leq n)$ define local frame fields on $\tilde{U}_\lambda$ of $\boldsymbol{H}$ and

V respectively:

$$X_i=\sigma\Big(\frac{\partial}{\partial x^i}\Big)=\frac{\partial}{\partial x^i}-N_i^m\frac{\partial}{\partial y^m},\quad Y_i=\frac{\partial}{\partial y^i},$$

where $N_j^i(x, y)$ $(1\leq i, j\leq n)$ are some well-defined functions on each $\tilde{U}_\lambda$, which are called the *coefficients* of the non-linear connection σ. From these expressions and Proposition 2.2, we see easily that $\boldsymbol{H}$ and $\boldsymbol{V}$ are Finsler vector bundles associated with $\pi_T{}^*LM$, where LM is the frame bundle of M. For example, the vector bundle $\boldsymbol{H}$ can be expressed as $\boldsymbol{H}=\pi_T{}^*LM\times_\rho V$, where the representation $\rho: GL(n, \boldsymbol{R})\rightarrow GL(V)$ is the natural one. Hence, by (3.5), an automorphism $f\in Aut(\boldsymbol{F})$ acts on $\boldsymbol{H}$ as follows:

$$f(X_i)=g_{\lambda i}^m X_m$$

for the corresponding $g\in\Gamma(G_{FM})$. In the case of $\boldsymbol{V}$, we also get the similar expressions for $f\in Aut(\boldsymbol{V})$.

Remark 3.2. We denote by g_{FM} the *associate bundle* $FM\times_{Ad}g$ of a Finsler bundle FM for the Lie algebra g of the structure group G, and by $\Omega^1(g_{FM})$ the vector space of g_{FM}-valued 1-form on TM. Then we see that the set of all connections of FM admits the structure of affine space with the vector space $\Omega^1(g_{FM})$ (for details and general theory, see Mogi-Itoh [5]). The second author has treated the action (3.4) on the affine space, and consider the so-called *Yang-Mills functional* in the case where the structure group G of FM is compact, and obtained the condition that a Finsler connection is a critical point of the functional (cf. Otsuji [6]). Since Finsler bundles are special principal bundles over the tangent bundle TM of a manifold M, the almost results in [6] were obtained by the same methods in Riemannian geometry. For example, any critical point of the Yang-Mills functional is invariant under the action of any gauge transformation $g\in\Gamma(G_{FM})$, and if a Finsler connection is a critical point of the functional, then its curvature form is a harmonic form. It must be noted that TM is not compact even if the base manifold M is compact. Hence, even if the curvature form is harmonic, the connection is not necessary a critical point of the functional.

4. An almost symplectic structure on tangent bundles

In this section, we are concerned with a Finsler manifold (M, L) and an almost symplectic structure naturally defined on its tangent bundle (cf. Ichijyō [3]). Furthermore, as an example of automorphism of a Finsler vector bundle, we treat an automorphism which preserves the almost symplectic structure invariant (cf. Chapter III in Hermann [2]).

If a Finsler metric $L(x, y)$ is given, then a Riemannian metric G on its tangent bundle TM is defined by

$$G = g_{ij}\, dx^i \otimes dx^i + g_{ij}\, \theta^i \otimes \theta^j,$$

where we put $g_{ij}(x, y) = (\partial^2 L/\partial y^i \partial y^j)/2$ and $\theta^i = dy^i + N^i_m\, dx^m$ for the coefficients $N^i_j(x, y)$ which are derived naturally from $g_{ij}(x, y)$ as follows:

$$N^i_j = \frac{1}{2} g^{ij}\left(\frac{\partial g_{lk}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\right) y^k.$$

Then the following 2-form Ω is well-defined on TM:

$$\Omega = 2 g_{ij}\, dx^i \wedge \theta^j,$$

which defines an almost symplectic structure on the Riemannian manifold (TM, G). This form is written as $\Omega = 2d(y_m\, dx^m)$, where we put $y_m = g_{mr}\, y^r$. If we put $\tau = y_m\, dx^m$, this form is a globally defined 1-form on TM and satisfies $\Omega = 2d\tau$. It is obvious that $\tau \in \Gamma(\boldsymbol{H}^*)$, where $\boldsymbol{H}^*$ is the dual bundle of the horizontal subbundle $\boldsymbol{H}$.

In the following, we consider the condition that an automorphism $f \in Aut\,(\boldsymbol{H}^*)$ corresponding to $g = \{g_\lambda\} \in \Gamma(G_{FM})$ preserves the form Ω, that is, we consider the following situation:

$$\Omega = 2d\tau = 2d(f(\tau)).$$

If we put $f(\tau) = y_m'\, dx^m$, we have $d(y_m' - y_m)\, dx^m = 0$. By direct calculations, we have

$$X_i\, B_j = X_j\, B_i, \quad Y_j\, B_i = 0$$

for $B_i = y_i' - y_i$. Hence we have $B_i = \partial B/\partial x^i$ for a local function $B(x)$, and so we get

$$f(\tau) = \tau + \pi_T{}^*\, \xi, \tag{4.1}$$

where ξ is a closed form on the base manifold M which has local expression $\xi = dB$. Because of $y_i' = (g_\lambda{}^{-1})^m_i\, y_m$, this condition is written as follows:

$$(g_\lambda{}^{-1})^m_i\, y_m - y_i = \frac{\partial B(x)}{\partial x^i}.$$

Hence we have the similar results of Theorem 8.1 in Hermann [2] as follows:

Theorem 4.1. *Let (M, L) be a Finsler manifold, and (TM, G) its tangent bundle with the Riemannian metric G canonically defined from the Finsler metric L. An automorphism*

$g=\{g_\lambda\}\in\Gamma(G_{FM})$ preserves the almost symplectic form Ω if and only if it satisfies (4.1) *for a closed* 1-*form* ξ *on the base manifold M.*

The correspondence $g\in\Gamma(G_{FM})\rightarrow\xi$ is an isomorphism from $\Gamma(G_{FM})$ to the group of closed 1-*form on M.*

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