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On Locally Flat Generalized (α, β)-Metrics and Conformally Flat Generalized Randers Metrics

Yoshihiro ICHIJYŌ[1)] and Masao HASHIGUCHI[2)]

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Abstract

A Finsler metric L on a differentiable manifold M is called an (α, β)-metric, when L is a positively homogeneous function of degree 1 of a Riemannian metric α and a non-vanishing 1-form β on M. In the present paper, we generalize the notion of (α, β)-metric by replacing β by a singular Riemannian metric, and for such a generalized (α, β)-metric L satisfying some assumptions: e.g., a generalized Randers metric $L=\alpha+\beta$, we give a condition that L be locally flat and a condition that L be conformally flat, in the tensorial form expressed in terms of the given metrics α and β.

Key words: Finsler metric, Generalized (α, β)-metric, Generalized Randers metric, Locally flat, Conformally flat.

1. Introduction

On a differentiable manifold M we shall consider a Finsler metric $L(\alpha, \beta)$ which is a positively homogeneous function of degree 1 of a Riemannian metric α and a singular Riemannian metric β on M. Denoting a point of M and a tangent vector at that point by $x=(x^i)$ and $y=(y^i)$ respectively, we put

$$\alpha(x, y)=(a_{ij}(x)y^i y^j)^{1/2}, \quad \beta(x, y)=(b_{ij}(x)y^i y^j)^{1/2}. \tag{1.1}$$

In the case of $b_{ij}=b_i b_j$, where b_i is a non-vanishing covariant vector field on M, it is reduced to an (α, β)-metric. So we shall call such a Finsler metric a *generalized* (α, β)*-metric.*

An interesting example is given by

1) Department of Mathematical Science, Faculty of Integrated Arts and Sciences, University of Tokushima, 1-1 Minami-josanjima-cho, Tokushima 770, Japan.
2) Department of Mathematics, Faculty of Science, Kagoshima University, 1-21-35 Korimoto, Kagoshima 890, Japan.

(1.2) $$L(x, y)=(a_{ij}(x)y^i y^j)^{1/2}+\{(b_i(x)y^i)^2+(b_i(x)f^i{}_j(x)y^j)^2\}^{1/2},$$

where $\alpha(x, y)=(a_{ij}(x)y^i y^j)^{1/2}$ is a Riemannian metric on M, b_i is a non-vanishing covariant vector field on M, and $f^i{}_j$ is an almost Hermitian structure of the Riemannian manifold (M, α):

(1.3) $$f^i{}_r f^r{}_j=-\delta^i{}_j,\quad a_{ij}f^i{}_h f^j{}_k=a_{hk}.$$

This metric is called an (a, b, f)*-structure*, and gives an important example of a Rizza manifold (cf. [4]). A *Rizza manifold* (M, L, f) is by definition a Finsler manifold (M, L), endowed with an almost complex structure $f^i{}_j(x)$ on M: $f^i{}_r f^r{}_j=-\delta^i{}_j$, satisfying the condition

(1.4) $$L(x, \phi_\theta y)=L(x, y),$$

where $\phi_\theta{}^i{}_j=(\cos\theta)\delta^i{}_j+(\sin\theta)f^i{}_j$, or equivalently, the condition

(1.5) $$g_{ij}(x, y)f^i{}_r(x)y^r y^j=0,$$

where g_{ij} is the fundamental tensor field.

On a differentiable manifold M endowed with a generalized (α, β)-metric $L(\alpha, \beta)$, we have the Cartan connection $C\Gamma=(\Gamma^*{}_j{}^i{}_k,\ G^i{}_k,\ C_j{}^i{}_k)$ and the Berwald connection $B\Gamma=(G_j{}^i{}_k,\ G^i{}_k,\ 0)$ of the Finsler manifold (M, L) and further the Levi-Civita connection $\Gamma=(\{{}_j{}^i{}_k\})$ of the Riemannian manifold (M, α). With respect to Γ we denote the covariant differentiation and the curvature tensor field by ∇_k and $R_h{}^i{}_{jk}$ respectively.

A Rizza manifold (M, L, f) is called a *Kaehlerian Finsler manifold* if $f^i{}_{j|k}=0$ is satisfied, where $_{|k}$ denotes the h-covariant differentiation with respect to the Cartan connection $C\Gamma$. An (a, b, f)-structure $L(\alpha, \beta)$ on M satisfying

(1.6) $$\nabla_k b_i=0,\quad \nabla_k f^i{}_j=0$$

gives an example of a Kaehlerian Finsler manifold. In fact, the Finsler connection $F\Gamma=(\{{}_j{}^i{}_k\},\ y^j\{{}_j{}^i{}_k\},\ C_j{}^i{}_k)$ is just the Cartan connection $C\Gamma$ of (M, L), because $F\Gamma$ satisfies the axiomatic system for $C\Gamma$ due to Matsumoto [11] (cf. [3]).

Now, a Finsler manifold (M, L) or a Finsler metric L is called *locally flat* if for any point p of M there exists a local coordinate neighbourhood (U, x) containing p such that L is a locally Minkowski metric on U.

For a Randers space $(M, \alpha+\beta)$, where

(1.7) $$\alpha(x, y)=(a_{ij}(x)y^i y^j)^{1/2},\quad \beta(x, y)=b_i(x)y^i,$$

we have the following theorem due to Kikuchi [9].

Theorem 1.1. *A Randers space $(M, \alpha+\beta)$ is locally flat if and only if*

$$R_h{}^i{}_{jk}=0, \quad \nabla_k b_i=0. \tag{1.8}$$

A Finsler manifold (M, L) or a Finsler metric L is called *conformally flat* if L is locally conformal to a locally flat metric, that is, for any point p of M there exist a local coordinate neighbourhood (U, x) containing p and a function $\sigma(x)$ on U such that $e^\sigma L$ is a locally Minkowski metric on U.

In our previous paper [6], we discussed the condition that a Randers space $(M, \alpha+\beta)$ be conformally flat. We put

$$M_j=(1/b^2)\{b^r(\nabla_r b_j)-(\nabla_r b^r)b_j/(n-1)\}, \tag{1.9}$$

where $(a^{ij})=(a_{ij})^{-1}$, $b^i=a^{ir}b_r$, $b^2=a^{ij}b_i b_j$, and $n=\dim M$. Putting

$$M_j{}^i{}_k=\{{}_j{}^i{}_k\}+\delta_j{}^i M_k+\delta_k{}^i M_j-a_{jk}M^i, \tag{1.10}$$

where $M^i=a^{ir}M_r$, we have a conformally invariant linear connection $\overset{m}{\Gamma}=(M_j{}^i{}_k)$. We denote its curvature tensor field by $M_h{}^i{}_{jk}$, which is also conformally invariant. Then from Theorem 1.1 we obtained

Theorem 1.2. *A Randers space $(M, \alpha+\beta)$ is conformally flat if and only if*

$$M_h{}^i{}_{jk}=0, \quad \nabla_k M_j=\nabla_j M_k, \quad \nabla_k b_j=b_k M_j-b_r M^r a_{jk}. \tag{1.11}$$

The above results were generalized in the lectures [7], [8] to the case of generalized (α, β)-metric. The present paper is a revised note of the lectures. Putting $\lambda=\beta L_\alpha/\alpha L_\beta$, where $L_\alpha=\partial L/\partial\alpha$, $L_\beta=\partial L/\partial\beta$, and $\mu=a^{ij}b_{ij}$, $\nu=b^{ij}b_{ij}$, where $(a^{ij})=(a_{ij})^{-1}$, $b^i{}_j=a^{ir}b_{rj}$, $b^{ij}=a^{jr}b^i{}_r$, we assume the following, if necessary.

Assumption 1.1. λ is an irrational function of y^i.

Assumption 1.2. $n\nu-\mu^2\neq 0$.

2. Locally flat generalized (α, β)-metrics

We shall obtain a condition that a generalized (α, β)-metric $L(\alpha, \beta)$ on M be locally flat, under Assumption 1.1.

The h-covariant differentiation with respect to the Berwald connection $B\Gamma$ of a Finsler manifold (M, L) is denoted by $_{;k}$ and the h-curvature tensor field by $K_h{}^i{}_{jk}$. Since $B\Gamma$ satisfies $L_{;k}=0$ and $y^i{}_{;k}=0$, we have $L_{;k}=\alpha_{;k}L_\alpha+\beta_{;k}L_\beta=(L_\alpha/2\alpha)(a_{ij;k}y^iy^j)+(L_\beta/2\beta)(b_{ij;k}y^iy^j)=0$, that is,

$$\lambda(a_{ij;k}y^iy^j)+(b_{ij;k}y^iy^j)=0. \tag{2.1}$$

If (M, L) is a Berwald space, then the coefficients $G_j{}^i{}_k$ of $B\Gamma$ are functions of x^i alone. Since $a_{ij;k}y^iy^j$ and $b_{ij;k}y^iy^j$ become polynomials of y^i, from Assumption 1.1 we have $a_{ij;k}y^iy^j=0$ and $b_{ij;k}y^iy^j=0$, that is, $a_{ij;k}=0$ and $b_{ij;k}=0$, the former of which yields $G_j{}^i{}_k=\{{}_j{}^i{}_k\}$. Then we have $\nabla_k b_{ij}=0$, and also $K_h{}^i{}_{jk}=R_h{}^i{}_{jk}$. Further, if (M, L) is locally flat, then from $K_h{}^i{}_{jk}=0$ we have $R_h{}^i{}_{jk}=0$.

The converse is also true. In fact, if $\nabla_k b_{ij}=0$ is satisfied, then the linear Finsler connection $F\Gamma=(\{{}_j{}^i{}_k\}, y^j\{{}_j{}^i{}_k\}, 0)$ is just the Berwald connection $B\Gamma$ of (M, L), because $F\Gamma$ satisfies the axiomatic system for $B\Gamma$ due to Okada [12]. Hence (M, L) is a Berwald space, and we have $K_h{}^i{}_{jk}=R_h{}^i{}_{jk}$. Further, if $R_h{}^i{}_{jk}=0$ is satisfied, then we have $K_h{}^i{}_{jk}=0$, so (M, L) is locally flat. Thus we have the same result as Theorem 1.1 for a Randers space.

Theorem. 2.1. *A Finsler manifold with a generalized (α, β)-metric satisfying Assumption* 1.1 *is locally flat if and only if*

$$R_h{}^i{}_{jk}=0, \quad \nabla_k b_{ij}=0. \tag{2.2}$$

We shall give an example of a generalized (α, β)-metric satisfying Assumption 1.1. A generalized (α, β)-metric L of type $L=\alpha+\beta$ is called a *generalized Randers metric*, and a Finsler manifold $(M, \alpha+\beta)$ a *generalized Randers space*. For such a metric L we have $\lambda=\beta/\alpha$, so L satisfies Assumption 1.1 because of the regularity of α and the singularity of β. Thus we have

Theorem 2.2. *A generalized Randers space $(M, \alpha+\beta)$ is locally flat if and only if $R_h{}^i{}_{jk}=0$, $\nabla_k b_{ij}=0$.*

Remark 2.1. In the case of Kropina type $L=\alpha^2/\beta$ we have $\lambda=-2\beta^2/\alpha^2$, so Assumption 1.1 is not satisfied. On the other hand, in the case of Matsumoto type $L=\alpha^2/(\alpha-\beta)$ (cf. [1]) we have $\lambda=\beta/\alpha-2\beta^2/\alpha^2$, so Assumption 1.1 is satisfied.

From the proof of Theorem 2.1 we have

Theorem 2.3. *A Finsler manifold with a generalized (α, β)-metric satisfying Assumption* 1.1, *e.g., a generalized Randers space, is a Berwald space if and only if* $\nabla_k b_{ij}=0$.

Remark 2.2. In the above theorems we need not assume Assumption 1.1 for the converse statement.

Remark 2.3. A generalized (α, β)-metric is also called a *Finsler metric of type* (α, β_2), which is introduced in [2] from some physical consideration. A generalized Randers space is then called a *2nd-order Randers space.* In general, a *Finsler metric of type* (α, β_m) is considered by taking β as the m-th root β_m of an m-form in M.

3. A conformally invariant linear connection

In order to obtain a condition that a generalized (α, β)-metric be conformally flat, we shall first find a conformally invariant symmetric linear connection, under Assumption 1.2. We need not here assume Assumption 1.1.

Let (M, L) be an $n(\geqq 2)$-dimensional Finsler manifold with a generalized (α, β)-metric $L=L(\alpha, \beta)$. By a conformal change

$$L=L(\alpha, \beta) \rightarrow \tilde{L}=e^{\sigma} L(\alpha, \beta), \tag{3.1}$$

we have also a generalized (α, β)-metric $\tilde{L}=L(\tilde{\alpha}, \tilde{\beta})$, where $\tilde{\alpha}=e^{\sigma} \alpha$, $\tilde{\beta}=e^{\sigma} \beta$. Putting $\tilde{\alpha}(x, y) = (\tilde{a}_{ij}(x) y^i y^j)^{1/2}$, $\tilde{\beta}(x, y)=(\tilde{b}_{ij}(x) y^i y^j\}^{1/2}$, we have $\tilde{a}_{ij}=e^{2\sigma} a_{ij}$, $\tilde{b}_{ij}=e^{2\sigma} b_{ij}$.

Since the Christoffel symbols $\widetilde{\{{}_j{}^i{}_k\}}$ constructed from $\tilde{a}_{ij}$ are written as

$$\widetilde{\{{}_j{}^i{}_k\}} = \{{}_j{}^i{}_k\} + \delta_j{}^i \sigma_k + \delta_k{}^i \sigma_j - a_{jk} \sigma^i, \tag{3.2}$$

where $\sigma_k=\partial\sigma/\partial x^k$, $\sigma^i=\alpha^{ir} \sigma_r$, we have

$$\tilde{\nabla}_k \tilde{b}_{ij}=e^{2\sigma}\{\nabla_k b_{ij}-b_{kj} \sigma_i-b_{ki} \sigma_j+a_{ik}(b_{jr} \sigma^r)+a_{jk}(b_{ir} \sigma^r)\}, \tag{3.3}$$

from which we have

$$\tilde{b}^{rs} \tilde{\nabla}_r \tilde{b}_{sj}=b^{rs} \nabla_r b_{sj}-\nu\sigma_j+\mu(b_{jr} \sigma^r), \tag{3.4}$$

$$\tilde{\nabla}_r \tilde{b}^r{}_j=\nabla_r b^r{}_j-\mu\sigma_j+n(b_{jr} \sigma^r). \tag{3.5}$$

If we eliminate the term $b_{jr}\sigma^r$ from (3.4), (3.5), we have

$$n\tilde{b}^{rs}\tilde{\nabla}_r\tilde{b}_{sj}-\mu\tilde{\nabla}_r\tilde{b}^r{}_j=nb^{rs}\nabla_r b_{sj}-\mu\nabla_r b^r{}_j-(n\nu-\mu^2)\sigma_j. \tag{3.6}$$

It is noted that μ and ν are conformally invariant. If we assume Assumption 1.2, we can put

$$L_j=(n/(n\nu-\mu^2))\{b^{rs}\nabla_r b_{sj}-(\mu/n)\nabla_r b^r{}_j\}, \tag{3.7}$$

and from (3.6) we have

$$\sigma_j=L_j-\tilde{L}_j. \tag{3.8}$$

Substituting σ_j from (3.8) into (3.2), and putting

$$L_j{}^i{}_k=\{{}_j{}^i{}_k\}+\delta_j{}^i L_k+\delta_k{}^i L_j-a_{jk}L^i, \tag{3.9}$$

where $L^i=a^{ir}L_r$, we have $\tilde{L}_j{}^i{}_k=L_j{}^i{}_k$. $L_j{}^i{}_k$ define a conformally invariant symmetric linear connection on M. Thus we have shown

Theorem 3.1. *In a Finsler manifold with a generalized (α, β)-metric satisfying Assumption 1.2 there exists a conformally invariant symmetric linear connection* $\overset{l}{\Gamma}=(L_j{}^i{}_k)$.

We shall call the linear connection $\overset{l}{\Gamma}$ the *conformally invariant linear connection* of a generalized (α, β)-metric. We denote its curvature tensor field by $L_h{}^i{}_{jk}$, which is also conformally invariant.

Remark 3.1. In the case of $b_{ij}=b_i b_j$, where b_i is a non-vanishing covariant vector field on M, a generalized (α, β)-metric is reduced to a usual (α, β)-metric. Then we have $\mu=b^2$, $\nu=b^4$, where $b^2=a^{ij}b_i b_j$. Thus we have $n\nu-\mu^2=(n-1)b^4\neq 0$, so Assumption 1.2 is satisfied. Further, we have

$$L_j=M_j+(n/(n-1)b^2)(b^r b^s\nabla_r b_s)b_j, \tag{3.10}$$

where M_j is given by (1.9). M_j is also a geometrical object satisfying $\sigma_j=M_j-\tilde{M}_j$, from which in [6] we have obtained the conformally invariant linear connection $\overset{m}{\Gamma}=(M_j{}^i{}_k)$ of an (α, β)-metric, but from (3.8) we obtained another conformally invariant linear connection $\overset{l}{\Gamma}$ of an (α, β)-metric. It is noted that the additional term in (3.10) is a conformally invariant covariant vector field of an (α, β)-metric. A geometrical object L_j which obeys (3.8) is not unique (cf. Ichijyō [5], Kikuchi [10]).

4. Conformally flat generalized Randers metrics

In the same way as shown in [6], using the conformally invariant linear connection $\overset{l}{\Gamma}$ and Theorem 2.2 we can obtain a condition that a generalized Randers metric be conformally flat, under Assumption 1.2.

Let (M, L) be a generalized Randers space, where $L=\alpha+\beta$. By a conformal change (3.1) we have a generalized Randers metric $\tilde{L}=\tilde{\alpha}+\tilde{\beta}$, where $\tilde{\alpha}=e^{\sigma}\alpha$, $\tilde{\beta}=e^{\sigma}\beta$. If ($M$, $\tilde{L}$) is locally flat, from Theorem 2.2 we have $\tilde{R}_h{}^i{}_{jk}=0$ and $\tilde{\nabla}_k\tilde{b}_{ij}=0$, from the latter of which we have $\tilde{L}_j=0$. Then from (3.9) we have $\tilde{L}_j{}^i{}_k=\{\widetilde{{}_j{}^i{}_k}\}$, so we have $\tilde{L}_h{}^i{}_{jk}=\tilde{R}_h{}^i{}_{jk}=0$, that is, $L_h{}^i{}_{jk}=0$. On the other hand, from (3.8) we have $\sigma_j=L_j$, so L_j is locally gradient: $\nabla_k L_j=\nabla_j L_k$, and from (3.3) we have $\nabla_k b_{ij}=b_{kj}L_i+b_{ki}L_j-a_{ik}b_{jr}L^r-a_{jk}b_{ir}L^r$. Since the conditions locally obtained above are expressed in the tensorial form in terms of the given Finsler metric, we have globally

$$L_h{}^i{}_{jk}=0,\quad \nabla_k L_j=\nabla_j L_k,\quad \nabla_k b_{ij}=b_{kj}L_i+b_{ki}L_j-a_{ik}b_{jr}L^r-a_{jk}b_{ir}L^r. \tag{4.1}$$

Conversely, if (4.1) is satisfied, then we locally have a function $\sigma(x)$ such that $\sigma_j=L_j$. Then $\tilde{L}=e^{\sigma}L$ satisfies $\tilde{R}_h{}^i{}_{jk}=0$, $\tilde{\nabla}_k\tilde{b}_{ij}=0$, so it follows from Theorem 2.2 that $\tilde{L}$ is locally flat. Thus we have proved

Theorem 4.1. *A generalized Randers space* (M, L) *with a metric* $L=\alpha+\beta$ *satisfying Assumption* 1.2 *is conformally flat if and only if the condition* (4.1) *is satisfied.*

We can express (4.1) in terms of the linear connection $\overset{l}{\Gamma}$ itself as follows:

$$L_h{}^i{}_{jk}=0,\quad \overset{l}{\nabla}_k L_j=\overset{l}{\nabla}_j L_k,\quad \overset{l}{\nabla}_k b_{ij}=-2L_k b_{ij}, \tag{4.2}$$

where $\overset{l}{\nabla}_k$ denotes the covariant differentiation with respect to $\overset{l}{\Gamma}$.

An advantage of (4.2) is suggested by the proverb "Do your own business for yourself", but $\overset{l}{\Gamma}$ is not metrical with respect to α: $\overset{l}{\nabla}_k a_{ij}=-2L_k a_{ij}$. If we want to express (4.2) in terms of a metrical linear connection of the Riemannian manifold (M, α), by the well-known metrization method we may change $L_j{}^i{}_k$ to

(4.3) $$V_j{}^i{}_k = L_j{}^i{}_k + a^{ir}(\overset{l}{\nabla}_k a_{rj})/2,$$

which is written as

(4.4) $$V_j{}^i{}_k = L_j{}^i{}_k - \delta_j{}^i L_k,$$

that is,

(4.5) $$V_j{}^i{}_k = \{{}_j{}^i{}_k\} + \delta_k{}^i L_j - a_{jk} L^i.$$

$V_j{}^i{}_k$ define a semi-symmetric metrical linear connection $\overset{v}{\Gamma} = (V_j{}^i{}_k)$. It is shown that the condition (4.2) is equivalent to

(4.6) $$V_h{}^i{}_{jk} = 0, \ \overset{v}{\nabla}_k L_j = \overset{v}{\nabla}_j L_k, \ \overset{v}{\nabla}_k b_{ij} = 0,$$

where $\overset{v}{\nabla}_k$ and $V_h{}^i{}_{jk}$ denote the covariant differentiation and the curvature tensor field with respect to $\overset{v}{\Gamma}$.

The linear connection $\overset{v}{\Gamma}$ is not necessarily conformally invariant, but it satisfies $\overset{v}{\nabla}_k a_{ij} = 0$, $\overset{v}{\nabla}_k b_{ij} = 0$. Thus it is at a glance shown that $\overset{v}{\nabla}_k \mu = 0$, $\overset{v}{\nabla}_k \nu = 0$, that is, μ and ν are constant on each connected component of M.

From the proof of Theorem 4.1 we have

Theorem 4.2. *A generalized Randers space* (M, L) *with a metric* $L = \alpha + \beta$ *satisfying Assumption* 1.2 *is conformal to a Berwald space if and only if*

(4.7) $$\nabla_k L_j = \nabla_j L_k, \ \nabla_k b_{ij} = b_{kj} L_i + b_{ki} L_j - a_{ik} b_{jr} L^r - a_{jk} b_{ir} L^r,$$

which is equivalent to each of the following:

(4.8) $$\overset{l}{\nabla}_k L_j = \overset{l}{\nabla}_j L_k, \ \overset{l}{\nabla}_k b_{ij} = -2 L_k b_{ij},$$

(4.9) $$\overset{v}{\nabla}_k L_j = \overset{v}{\nabla}_j L_k, \ \overset{v}{\nabla}_k b_{ij} = 0.$$

Now, the discussion in this section is generally valid for a Finsler manifold with a generalized (α, β)-metric L, provided L satisfies Assumption 1.1 and Assumption 1.2. On the other hand, by Remark 2.2 we need not assume Assumption 1.1 for the converse statements of the above theorems. Thus we have generally

Theorem 4.3. *Let* (M, L) *be a Finsler manifold with a generalized* (α, β)*-metric* L *satisfying Assumption* 1.2. *If one of the equivalent conditions* (4.7), (4.8), (4.9) *is satisfied, then* (M, L) *is conformal to a Berwald space. Then* μ *and* ν *are constant on each connected component of* M. *If one of the equivalent conditions* (4.1), (4.2), (4.6) *is satisfied, then* (M, L) *is conformally flat.*

Theorem 4.4. *Let* (M, L) *be a Finsler manifold with a generalized* (α, β)*-metric* L *satisfying Assumption* 1.1 *and Assumption* 1.2. *If* (M, L) *is conformal to a Berwald space, then the conditions* (4.7), (4.8), (4.9) *are satisfied. If* (M, L) *is conformally flat, then the conditions* (4.1), (4.2), (4.6) *are satisfied.*

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