Rep. Fac. Sci. Kagoshima Univ. (Math., Phys. & Chem.),
No.29, 15-21, 1996.

Asymmetric MDS Method for the General Euclidean Model

Hiroshi YADOHISA*

(Received September 10, 1996)

Abstract

Vector representation of asymmetry for the general Euclidean model, which includes many kind of models (eg., INDSCAL, IDIOSCAL, ASYMSCAL, GEMSCAL, etc), is proposed. We consider that (dis)similarity data among the objects are given as several square or rectangular matrices. We shall propose the method for representation of asymmetry in the dissimilarity data as vectors in the objects depicted space.

Key words: multidimensional scaling (MDS), asymmetry, (dis)similarity, INDSCAL, ASYMSCAL, GEMSCAL

1 Introduction

The general Euclidean model (GEM) for three-way data was introduced by F. W. Young (see, Young, 1984) and is appropriate to many type of data. This model is about multidimensional scaling (MDS) of three-way dissimilarity data, which is consisting of several square or rectangle matrices. The GEM is defined by the following:

$$d_{ij}^k = \left[(y_i - x_j)\, V_i W_k \,(y_i - x_j)'\right]^{1/2}, \tag{1.1}$$

where d_{ij}^k is the GEM distance approximates the dissimilarity o_{ij}^k between objects and is given as ith row and jth column in the kth dissimilarity matrix, x_j is r elements vector of coordinates which show the point of the object j in an Euclidean space, y_j is also r elements vector of coordinates which specifies the point in an Euclidean space. y_i is identical to x_i if the dissimilarity matrices are square and are restricted to x_i if those matrices are rectangle. V_i is a square, symmetric, order r and positive semi-definite matrix

* Department of Mathematics, Kagoshima University, Kagoshima 890, Japan.

of weights associated with row object i. W_k is a square, symmetric, order r and positive semi-definite matrix of weights associated with dissimilarity matrix k.

Usually the asymmetry in dissimilarity data matrices is often considered as error and ignored. On the other hand, the same asymmetry has been strongly attracting many intensive workers. They feel that the lack of information contained in the skew-symmetric component often makes the results far from satisfaction as well as that asymmetry itself can be the prime target of analysis. A number of attempts, therefore, have been made to represent asymmetry in MDS. See, e.g., Coombs (1964), Constantine and Gower (1978), Gower (1977) and Weeks (1982).

The purpose of this paper is to discuss a simple and natural model for three-way (dis)similarity data with representing the asymmetry in the data. For each object under analysis, a vector value is given as a score of asymmetry as well as its location coordinates. Any prior information on the immanent structure of asymmetry is not required.

2 Method

For square dissimilarity matrices $\left(o_{ij}^k \right)$ between n objects, the GEM distance d_{ij}^k represents asymmetry by means of V_i's, which may be noticed by noting $d_{ij}^k = d_{ji}^k$ for $V_1 = \cdots = V_n = E$. Interpretation of V_i's in the results of analysis would be, however, quite difficult, even if they have only diagonal elements.

We use the restricted GEM distance

$$d_{ij}^k = \left[(y_i - x_j) W_k (y_i - x_j)' \right]^{1/2} \tag{2.1}$$

for approximating the symmetric part s_{ij}^k of o_{ij}^k, say $s_{ij}^k = (o_{ij}^k + o_{ji}^k)/2$, to give the vectors x_i of coordinates. A vector is then determined, for each object, from the location x_i and all the remaining anti-symmetric parts $a_{ij}^k = o_{ij}^k - s_{ij}^k$ $(j \neq i)$ by taking average of the stress vectors caused by the forced (neglecting asymmetry) allocation of the objects.

We propose the vector model for two cases, (i) W_k is only one, that is, we can use I instead of W_k by some transformation and (ii) W_k is semi-positive definite and reduced rank $r_k < r$. If data matrices are symmetric, these model corresponds to the methods of Classical MDS (e.g., Torgerson, 1965) and GEMSCAL (Young, 1984, 1987), respectively. Asymmetric MDS methods for the case that W_k is diagonal and full rank r has been proposed by Yadohisa and Niki (1995) as an extension of INDSCAL (Carroll and Chang, 1972).

Hereafter, we use the following notations:

$$1_n = (1 \ldots 1)', \quad I_n = \mathrm{diag}\, 1_n \text{ (identity matrix)}, \tag{2.2}$$

$$\lambda(x) = \begin{cases} 0, & \text{if } x = 0; \\ x/|x|, & \text{otherwise}; \end{cases} \quad \text{(direction vector)}. \tag{2.3}$$

3 Vector model for two-way asymmetric data

Here we describe simplest case, that is, a square asymmetric dissimilarity matrix are given (see Niki,1990; Yadohisa and Niki, 1994). In this case, the weigtt matrix W_k is only one and an identity matrix I is used insted of the W_k.

The data matrix O $[n \times n]$ can be decomposed into the symmetric component $S = \{O + O'\}/2$ and skew-symmetric one $T = \{O - O'\}/2$. From the matrix S, a set X $[n \times R] = \{x_i \mid i = 1, \ldots, n\}$ of the location coordinates of n objects can be determined by applying some suitable symmetric MDS method (e.g., Torgason, 1965). We determine the "asymmetric vectors" $A = \{a_i \mid i = 1, \ldots, n\}$ at X analyzing the matrix $\boldsymbol{T}$.

In our model, each element t_{ij} of $\boldsymbol{T}$ is compared with the projection of a_i on the directed line given by $p_i - p_j$:

$$a_i = t_{ij}\,\lambda(p_i - p_j) + e_{ij}, \tag{3.1}$$

where $\lambda(x)$ denotes the direction vector as defined above and e_{ij} is a random vector.

The least square estimate $\hat{\boldsymbol{a}}$ of $\boldsymbol{a}$ for which

$$\boldsymbol{e}^2 = \sum_{j \neq i} |e_{ij}|^2 = \sum_{j \neq i} e'_{ij} e_{ij} \tag{3.2}$$

attains its minimum is given by

$$\hat{a}_i = \frac{1}{n-1} \sum_{j \neq i} t_{ij}\,\lambda\,(p_i - p_j)\,. \tag{3.3}$$

4 Vector model for three-way asymmetric data

We propose, in this section, the asymmetric vector model to give an extension of GEMSCAL. The weight matrices W_k are semi-positive definite. GEMSCAL is one of the most general model for three-way data and represents the stimuli (object) and individuals in the same space called "joint space", and the directions of vectors, called "principal direction", indicate the directions in the space that are most salient to individuals. This model makes the assumption that an individual has several orthogonal directions in joint space. This means one individual has a particular set of orthogonal directions to be most important, another individual has some other directions to be most important. These directions are principal directions. The length of principal direction show the relative salience of the directions to the individuals, that is, the individual differences are represented by orientation and length of orthogonal principal directions in the joint space. The asymmetry in dissimilarity data represented as vectors in "joint space" and "personal space", which corresponds to "group stimuli space" and "individual (perceptual) space" in INDSCAL model, respectively (see, Young, 1984, 1987).

4.1 Model

The dissimilarity data matrix O_k is decomposed into the symmetric part S_k and the skew-symmetric part T_k as mentioned above. By applying GEMSCAL to matrix S_k, we can determine the coordinates matrix X $[n \times R]$ in "group stimuli space" and the N symmetric matrices W_k $[R \times R]$ ($k = 1, 2, \ldots, N$) of weights. Added to these, the coordinates matrices for personal spaces and "principal direction" are given as $X_k = (x_1^k, \ldots, x_n^k)' = X\,P_k(Q_k)^{1/2}$ and $P_k(Q_k)^{1/2}$, respectively, where P_k is an $r \times r_k$ column orthonormal matrix containing the eigenvectors of W_k, Q_k is $r_k \times r_k$ diagonal matrix contain the eigenvalues of W_k, this remains that W_k is decomposed as follows;

$$W_k = P_k Q_k (P_k)'. \tag{4.1}$$

We assume that there exists a joint $n \times R$ matrix $A = (a_1 \;\cdots\; a_n)'$ of which R dimensional row vectors represent asymmetry or "stress" hidden behind the configuration X of n objects. In addition, as same way of extension of INDSCAL, we assume that the stress vectors in the personal individual spaces are given by the matrices

$$A_k = (a_1^k \;\cdots\; a_n^k)' = A\,P_k(Q_k)^{1/2} \quad (k = 1, 2, \ldots, N). \tag{4.2}$$

The coordinates matrix in the k-th personal space is $XP_k(Q_k)^{1/2}$. Then, the stress caused by the forced location of the i-th and j-th objects can be expressed with

$$t_{ij}^k\,\lambda\left(x_i^k - x_j^k\right) = t_{ij}^k\,\lambda\,(x_i - x_j)\,P_k(Q_k)^{1/2}. \tag{4.3}$$

We consider the following model with vector errors ϵ_{ij}^k:

$$a_i^k = a_i P_k(Q_k)^{1/2} = t_{ij}^k\,\lambda\,(x_i - x_j)\,P_k(Q_k)^{1/2} + \epsilon_{ij}^k \tag{4.4}$$

and fit the matrix A so as to minimize

$$\phi_i = \sum_{k=1}^{N}\sum_{j \neq i} \epsilon_{ij}^k\,(P_k Q_k (P_k)')^{-1}(\epsilon_{ij}^k)' \tag{4.5}$$

$$= \sum_{k=1}^{N}\sum_{j \neq i} \epsilon_{ij}^k\,(W_k)^{-1}(\epsilon_{ij}^k)'. \tag{4.6}$$

Then we have an estimate $\hat{a}_i$ for the stress vector a_i of the i-th object in the group stimuli space:

$$\hat{a}_i = \frac{1}{N\,(n-1)} \sum_{k=1}^{N}\sum_{j \neq i} t_{ij}^k\,\lambda\,(x_i - x_j). \tag{4.7}$$

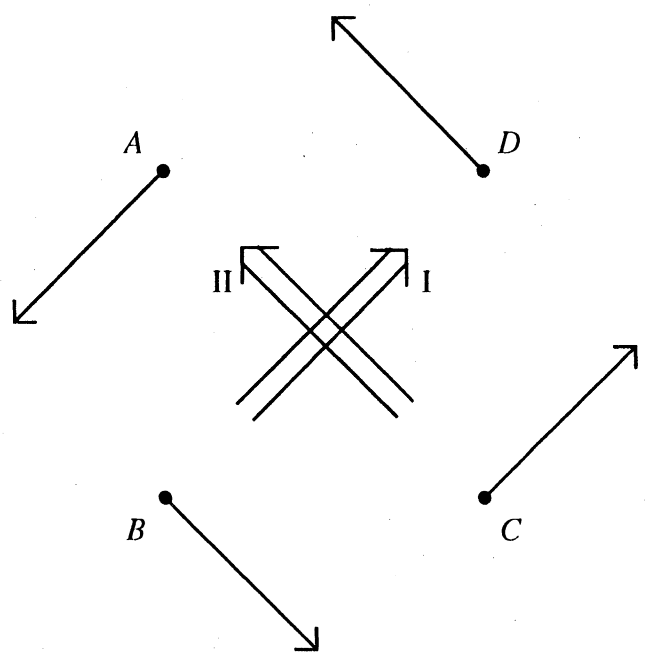

Fig. 1: Objects, asymmetric vectors and principal direction of subject 1 in "joint space".

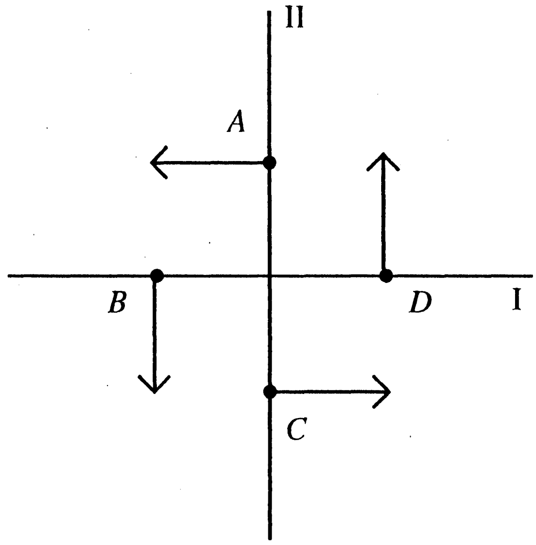

Fig. 2: Objects and asymmetric vectors in "personal space" of subject 1 .

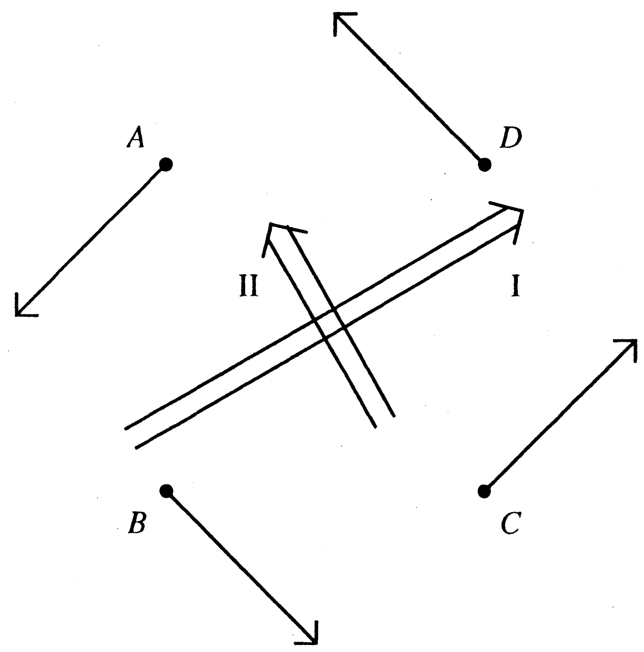

Fig. 3: Objects, asymmetric vectors and principal direction of subject 2 in "joint space".

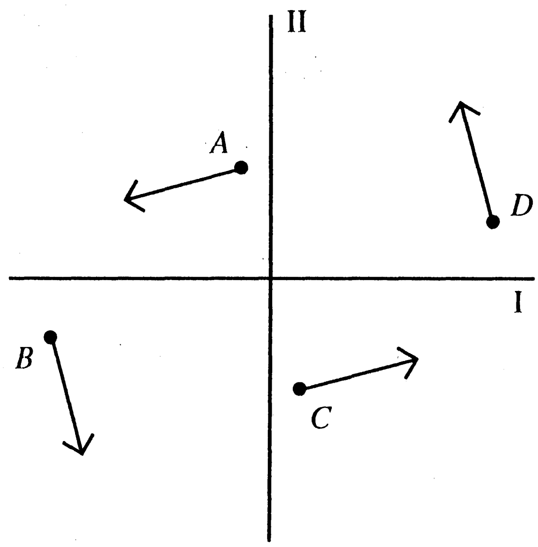

Fig. 4: Objects and asymmetric vectors in "personal space" of subject 2 .

4.2 Interpretation

We can obtain, as a result of analysis, the location coordinates of objects and its asymmetric vectors in the joint space. And also the principal directions of subjects (usually persons) in joint space are obtained. From this, we can calculate the coordinates of objects and asymmetric vectors in each personal space.

Fig. 1, 3 represent the joint space and Fig. 2, 4 represent the personal space of some of the subjects. The dots and thin arrows in the Fig. 1, 2, 3, 4 represent the objects and asymmetric vectors, respectively. The thick white vectors in Fig. 1 and Fig. 3, respectively, shows the principal direction of one subject and some other subject.

We can interpret from the arrows, for example, in the Fig. 1 the object A want to come closer to the object B, on the other hand, the object B intends to go far away from the object A. This means the dissimilarity from the object B to the object A is relatively larger than that from the object A to the object B (See, Yadohisa and Niki, 1994, 1995 for details).

References

[1] J. D. Carroll and P. Arabie, INDCLUS: An individual differences generalization of the ADCLUS model and the MAPCLUS algorithm. Psychometrika, **48** (1983), 157–169.

[2] J. d. Carroll and J. J. Chang, Analysis of individual differences in multidimensional scaling via an N-way generalization of "Eckart-Young" decomposition, Psychometrika, **35** (1970), 283–319.

[3] C. H. Coombs, A Theory of Data, Wiley, New York, 1964.

[4] A. G. Constantine and J. C. Gower, Graphical representation of asymmetric matrices, Appl. Statist. **3** (1978), 279–304.

[5] J. C. Gower, The analysis of asymmetry and orthogonality, Recent Developments in Statistics (Barra, J. R. et al., eds.), North-Holland, Amsterdam, (1977), 109–123.

[6] H. A. L. Kiers, Three-way methods for the analysis of qualitative quantitative two-way data, DSWO Press, Leiden, 1989.

[7] J. B. Kruskal, Nonmetric multidimensional scaling: A numerical method, Psychometrika, **29** (1964), 115–129.

[8] A. F. M. Nierop, The INDRES model: An INDSCAL model with residuals orthogonal to INDSCAL dimensions, Psychometric Methodology (Steyer, R., et al. eds.), Gustar Fischer Verlag, Stuttgart, (1993), 366–370.

[9] N. Niki, Kinetic interpretation of asymmetric proximities, Statistical Methods and Data Analysis (Niki, N., ed.), Scientist Inc., Tokyo, (1990), 29–32.

[10] L. R. Tucker and S. J. Messick, An individual differences model for multidimensional scaling, Psychometrika, **38** (1963), 333–368.

[11] D. G. Weeks, Restricted multidimensional scaling models for asymmetric proximities, Psychometrika, **47** (1982), 201–208.

[12] H. Yadohisa and N. Niki Vector representation of asymmetry in multidimensional scaling, COMPSTAT 1994, Short Communication in Computational Statistics (Dutter, R. and Grossmann, W., eds.), Vienna, (1994), 108–109.

[13] H. Yadohisa and N. Niki, Vector model for Representation of asymmetry in Individual difference scaling, Proceeding of the Eighth Japan and Korea Joint Conference of Statistics, Okayama, (1995), 67–72.

[14] F. W. Young, The general Euclidean model, Research methods for multimode data analysis (Law, H.G., et al. eds), Praeger Scientific, New York, (1984), 440–469.

[15] F. W. Young, Multidimensional Scaling: History, Theory, And Applications, Lawrence Erlbaum Associates, New Jersey, 1987.

[16] G. Young and A. S. Householder, Discussion of a set of points in terms of their mutual distances, Psychometrika, **3** (1983), 19–22.