Lie Groups with Regular Exponential Mapping

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Considerable attention has been devoted in the literature to the problem: for what connected Lie groups G with Lie algebras $\mathfrak{g}$, is the exponential mapping of $\mathfrak{g}$ into G surjective? As is well known, for compact or nilpotent connected Lie groups the answer is affermative. In this paper we shall investigate this problem and prove that if exp is regular at every point in $\mathfrak{g}$ (i. e. $\mathfrak{g}$ is regular in our terminology), then exp is surjective (Theorem 3). Although it has been shown in [2] and [8] that for a solvable and regular Lie algebra exp is surjective, our approach to the problem is different from those and we feel most interest in this point.

It is shown in §1 that a regular Lie algebra is solvable (Theorem 1), and in §2 there is given a necessary and sufficient condition for regularity of a Lie algebra (Theorem 2), which plays an important role in our study. Already several researches along the same lines have been done [2] [6] [7] [8] [9], especially Lemma 2 which is essentially equivalent to Theorem 2 is due to T. Nôno [6]. Finally in §§ 3-4, by means of adjoint representation our problem is reduced to the study of linear Lie algebras, and the analyticity of Lie groups leads to our conclusion (Theorem 3). Furthermore, some related properties of regular Lie algebras (groups) are treated (e. g. Proposition 5).

For notation and terminology, we follow [3] in general except that of regular element. Throughout this paper, only the real and finite dimensional case is treated, and $\mathfrak{g}$ and G denote a real finite dimensional Lie algebra and a corresponding connected Lie group respectively, unless otherwise stated.

§ 1. Properties of regular Lie algebras

Definition. An element $X \in \mathfrak{g}$ is said to be *regular* if the exponential mapping of $\mathfrak{g}$ into G (denoted by exp in the sequel) is regular at the point X, $\mathfrak{g}$ is said to be *regular* if all $X \in \mathfrak{g}$ are regular, and G is said to be *regular* if $\mathfrak{g}$ is regular.

The main result of this section is

THEOREM 1. *A real regular (i. e. exp is a regular mapping) Lie algebra is solvable.*

We recall first a well known fact ([6], Theorem 1, p. 116, and [3], Theorem 1.7, p. 95).

PROPOSITION 1. *An element $X \in \mathfrak{g}$ is regular if and only if adX (ad denotes the adjoint representation of $\mathfrak{g}$) has no such eigenvalues as $2\pi i m$ (m is a non-zero integer).*

From this we see that $\mathfrak{g}$ is regular if and only if for every $X \in \mathfrak{g}$ ad$X$ has no non-zero pure imaginary eigenvalues.

LEMMA 1. *A semisimple Lie algebra* $\mathfrak{g}$ *is not regular.*

PROOF. Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$ determined by a compact real form $\mathfrak{g}_0$ of the complexification $\mathfrak{g}^C$ of $\mathfrak{g}$, i. e.

$$\mathfrak{k}=\mathfrak{g}\cap\mathfrak{g}_0, \qquad \mathfrak{p}=\mathfrak{g}\cap i\mathfrak{g}_0.$$

Let η denote the conjugation of $\mathfrak{g}^C$ with respect to $\mathfrak{g}_0$, and B the Killing form of $\mathfrak{g}^C$. Then the bilinear form on $\mathfrak{g}\times\mathfrak{g}$ defined by

$$B_\eta(X, Y)\equiv-B(X, \eta Y), \qquad X, Y\in\mathfrak{g}$$

is symmetric and positive definite ([3], p. 158). If $X\in\mathfrak{k}$ then

$$B([X, Y], \eta Z)=-B(Y, [X, \eta Z])=-B(Y, \eta[X, Z]),$$

thus we have

$$B_\eta(\mathrm{ad}X(Y), Z)=B_\eta(Y, -\mathrm{ad}X(Z)).$$

This shows that $\mathrm{ad}X$ is represented by a skew symmetric matrix with respect to the metric B_η on $\mathfrak{g}$, therefore the eigenvalues of $\mathrm{ad}X$ are all pure imaginary numbers. By Proposition 1, $\mathfrak{g}$ can not be regular (notice that $\mathfrak{k}\neq\{0\}$).

Remark. This lemma is a simple application of the well known fact for semisimple Lie algebras: with the above notation, if $X\in\mathfrak{p}$ then $\mathrm{ad}X$ is represented by a symmetric matrix, and if $X\in\mathfrak{k}$ a skew symmetric matrix with respect to the metric B_η on $\mathfrak{g}$.

PROOF OF THEOREM 1. Let $\mathfrak{g}$ be a given regular Lie algebra. If $\mathfrak{g}$ is not solvable then, by Levi's theorem, $\mathfrak{g}$ is decomposed into a direct sum $\mathfrak{g}=\mathfrak{s}+\mathfrak{r}$, where $\mathfrak{r}$ is the radical of $\mathfrak{g}$ and $\mathfrak{s}$ $(\neq\{0\})$ is a semisimple subalgebra of $\mathfrak{g}$. If we choose a basis $e_1, \cdots, e_n$ of $\mathfrak{g}$ such that $e_1, \cdots, e_r$ is a basis of $\mathfrak{s}$ and $e_{r+1}, \cdots, e_n$ is a basis of $\mathfrak{r}$, then for each $X\in\mathfrak{s}$ $\mathrm{ad}X$ is represented in terms of the above basis by a matrix of the form

$$\mathrm{ad}X=\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

where A and B are square matrices of order r and $(n-r)$ respectively. By Lemma 1 there exists some $X\in\mathfrak{s}$ for which the eigenvalues of A are all pure imaginary numbers, and not all zero. This contradicts the regularity of $\mathfrak{g}$.

PROPOSITION 2. *An element* $X\in\mathfrak{g}$ *is regular if and only if* $\mathrm{ad}X$ *is a regular element of* $\mathrm{ad}\mathfrak{g}$.

PROOF. Choose a basis $e_1, \cdots, e_n$ of $\mathfrak{g}$ such that $e_{r+1}, \cdots, e_n$ is a basis of the center of $\mathfrak{g}$. Then, the constants c_{ij} determined by means of the relation

$$[X, e_j]=\sum_{i=1}^{n} c_{ij}e_i \qquad j=1, \cdots, n \tag{1}$$

are zeros for $j=r+1, \cdots, n$. On the other hand, $\mathrm{ad}e_1, \cdots, \mathrm{ad}e_r$ is obviously a basis of $\mathrm{ad}\mathfrak{g}$ and for this basis we have

$$[\mathrm{ad}X, \mathrm{ad}e_j]=\sum_{h=1}^{r} c_{hj}\,\mathrm{ad}e_h \qquad j=1, \cdots, r. \tag{2}$$

Comparing the relations (1) and (2), we see that $\mathrm{ad}X$ and $\mathrm{ad}\,(\mathrm{ad}X)$ have the same eigenvalues except zeros. This implies the proposition by Proposition 1.

COROLLARY. *Regularities of* $\mathfrak{g}$ *and ad*$\mathfrak{g}$ *are equivalent.*

PROPOSITION 3. *Every homomorphic image (equivalently every factor algebra) of a regular Lie algebra is also regular.*

The proof is similar to that of Proposition 2 and is omitted.

§ 2. A criterion for regularity

Let $C(\mathfrak{l}, A)$ denote the centralizer of a matrix A in a linear Lie algebra $\mathfrak{l}$, that is

$$C(\mathfrak{l}, A) \equiv \{X \in \mathfrak{l} \mid AX = XA\}.$$

Then we have

THEOREM 2. *Let* $\mathfrak{g}$ *be a real Lie algebra. A necessary and sufficient condition for an element* $X \in \mathfrak{g}$ *to be regular (i. e. exp is regular at* X*) is that*

$$C(\mathrm{ad}\mathfrak{g}, \mathrm{ad}X) = C(\mathrm{ad}\mathfrak{g}, \exp(\mathrm{ad}X)).$$

This theorem follows immediately from Proposition 2 and the following fact due to T. Nôno ([6], Theorem 2, p. 116).

LEMMA 2. *Let* $\mathfrak{l}$ *be a linear Lie algebra. Then an element* $X \in \mathfrak{l}$ *is regular if and only if*

$$C(\mathfrak{l}, X) = C(\mathfrak{l}, \exp X).$$

Let $\mathfrak{z}$ denote the center of $\mathfrak{g}$, and f be any linear mapping of $\mathfrak{g}$ into $\mathfrak{g}$. We put

$$N(\mathfrak{g}, f) \equiv \{Y \in \mathfrak{g} \mid f(Y) \in \mathfrak{z}\}.$$

COROLLARY. *An element* $X \in \mathfrak{g}$ *is regular if and only if*

$$N(\mathfrak{g}, \mathrm{ad}X) = N(\mathfrak{g}, \exp(\mathrm{ad}X) - I) \qquad (I = \textit{identity mapping}).$$

PROOF. If $Y \in N(\mathfrak{g}, \exp(\mathrm{ad}X) - I)$ then $\mathrm{ad}(\exp \mathrm{ad}X(Y) - Y) = 0$. Since $\exp(\mathrm{ad}X)$ is an automorphism of $\mathfrak{g}$ we have

$$\mathrm{ad}(\exp \mathrm{ad}X(Y)) = \exp(\mathrm{ad}X)\ \mathrm{ad}Y \exp(-\mathrm{ad}X),$$

therefore $\mathrm{ad}Y \in C(\mathrm{ad}\mathfrak{g}, \exp(\mathrm{ad}X))$. The converse relation is verified in the same way, thus we have

$$\mathrm{ad}^{-1}(C(\mathrm{ad}\mathfrak{g}, \exp(\mathrm{ad}X))) = N(\mathfrak{g}, \exp(\mathrm{ad}X) - I).$$

Similarly

$$\mathrm{ad}^{-1}(C(\mathrm{ad}\mathfrak{g}, \mathrm{ad}X)) = N(\mathfrak{g}, \mathrm{ad}X).$$

Due to these relations, the corollary is reduced to Theorem 2.

§ 3. Properties of regular Lie groups

THEOREM 3 ([2], Theorem 2, p. 119, and [8], Theorem 1, p. 7). *Let* G *be a real connected regular (i. e. exp is a regular mapping) Lie group with Lie algebra* $\mathfrak{g}$. *Then the exponential mapping of* $\mathfrak{g}$ *into* G *is surjective.*

This is the main theorem in our paper. This section is devoted to some preliminary consideration for the proof and related properties, and in the next section the proof is completed.

PROPOSITION 4. *For a regular Lie algebra* $\mathfrak{g}$, *the exponential mapping of* $ad\mathfrak{g}$ *into the adjoint group Int* $(\mathfrak{g})$ *is injective.*

PROOF. Suppose that $\exp(\mathrm{ad}X)=\exp(\mathrm{ad}Y)$ $(X, Y\in\mathfrak{g})$, then by Theorem 2 ad$X$ and ad$Y$ commute. From this we see that $\exp(\mathrm{ad}(X-Y))=E$. On the other hand, the general solutions of the matrix equation $\exp A=E$ are given by $A=2\pi i\, P^{-1}MP$ where M is any diagonal matrix with integral elements and P is any regular matrix. Consequently, due to the regularity of $\mathfrak{g}$, ad $(X-Y)$ must be zero (Proposition 1).

The following is the key lemma for our proof of Theorem 3.

LEMMA 3. *For a connected regular Lie group G with Lie algebra* $\mathfrak{g}$, *the following properties are equivalent.*

(i) *The exponential mapping of* $\mathfrak{g}$ *into G is surjective.*

(ii) *The exponential mapping of* $ad\mathfrak{g}$ *into Int* $(\mathfrak{g})$ *is surjective.*

PROOF. Consider the commutative diagram,

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\mathrm{ad}"] \arrow[d, "\exp"'] & \mathrm{ad}\mathfrak{g} \arrow[d, "\exp"] \\ G \arrow[r, "\mathrm{Ad}"] & \mathrm{Int}(\mathfrak{g}) \end{tikzcd} .$$

Since Ad is surjective, (i) implies (ii).

Conversely, if (ii) is satisfied then by Proposition 4 exp gives a diffeomorphism of ad$\mathfrak{g}$ onto Int $(\mathfrak{g})$ (see also Corollary in § 1). Thus Int $(\mathfrak{g})$ is simply connected. The kernel of Ad is the center Z of G so follows that Z is connected ([1], p. 59). Furthermore from this fact, we see that Z is the underlying group of the connected Lie subgroup of G corresponding to the center $\mathfrak{z}$ of $\mathfrak{g}$ ([1], p. 125). This means that $\exp\mathfrak{z}=Z$. Finally let $\mathfrak{g}_1$ denote a vector subspace of $\mathfrak{g}$ such that $\mathfrak{g}=\mathfrak{z}+\mathfrak{g}_1$ (direct sum), then as is easily seen from Proposition 4, the set $\{\exp X | X\in\mathfrak{g}_1\}$ is a representative set of G/Z. These last two facts show that any element $a\in G$ is written in the form

$$a=\exp X\exp Y=\exp(X+Y) \qquad X\in\mathfrak{g}_1,\ Y\in\mathfrak{z}.$$

Remark. Repeated application of Lemma 3 gives a proof of the well known theorem: if G is a connected nilpotent Lie group then exp is surjective ([3], p. 229).

LEMMA 4. *For a regular Lie algebra* $\mathfrak{g}$, *the exponential mapping of* $ad\mathfrak{g}$ *into Int* $(\mathfrak{g})$ *is surjective.*

The proof is given in § 4.

The above proof of Theorem 3 gives another proof of the following corollaries.

COROLLARY 1 ([5], Corollary, p. 186). *If* $\mathfrak{g}$ *is a regular Lie algebra, then the adjoint group Int* $(\mathfrak{g})$ *is simply connected.*

COROLLARY 2 ([8], Theorem 2, p. 9, see also [5], Corollary, p. 186). *The center* *of a connected regular Lie group is connected.*

COROLLARY 3 ([2], Theorem 3, p. 120, and [8], Theorem 1, p. 7). *Let G be a simply connected regular Lie group with Lie algebra $\mathfrak{g}$. Then the exponential mapping gives a diffeomorphism of $\mathfrak{g}$ onto G.*

PROOF. Any simply connected solvable Lie group has no nontrivial compact subgroup ([4], p. 138), in particular no periodical one-parameter subgroup. Hence the mapping exp: $\mathfrak{z} \rightarrow Z$, in the proof of Lemma 3, is bijective. We can easily prove from this that exp is a bijective mapping of $\mathfrak{g}$ onto G.

Finally we shall give an application of Corollary 3.

PROPOSITION 5. *Let G be a simply connected regular Lie group with Lie algebra $\mathfrak{g}$. For any $a=\exp X$, $b=\exp Y$ $(X, Y \in \mathfrak{g})$, $ab=ba$ if and only if $[X, Y]=0$.*

PROOF. $[X, Y]=0$ implies obviously $ab=ba$.

Conversely if $ab=ba$ then

$$\exp(\mathrm{ad}\, X)\exp(\mathrm{ad}\, Y)=\mathrm{Ad}(ab)=\exp(\mathrm{ad}\, Y)\exp(\mathrm{ad}\, X)\ .$$

Hence, by the next lemma, $[X, Y]\equiv Z$ is an element of the center of $\mathfrak{g}$, so for sufficiently small t

$$\exp tX \exp tY \exp(-tX)\exp(-tY)=\exp(t^2 Z)\ .$$

Both sides of this equation are entire functions of t, therfore it holds for all t. On putting $t=1$, we find that Z is equal to zero.

LEMMA 5. *Let $\mathfrak{g}$ be a regular Lie algebra. For any $A=\exp(\mathrm{ad}X)$, $B=\exp(\mathrm{ad}Y)$ $(X, Y \in \mathfrak{g})$, $AB=BA$ if and only if $\mathrm{ad}X\ \mathrm{ad}Y=\mathrm{ad}Y\ \mathrm{ad}X$.*

PROOF. If A and B commute, then $\exp(A\mathrm{ad}YA^{-1})=\exp(\mathrm{ad}\,Y)$. Since A is an automorphism of $\mathfrak{g}$, we have $A\ \mathrm{ad}Y\ A^{-1}=\mathrm{ad}AY$, therefore by Proposition 4 $A\ \mathrm{ad}Y\ A^{-1}=\mathrm{ad}Y$. This means the commutativity of $\mathrm{ad}X$ and $\mathrm{ad}Y$ by Theorem 2. The converse is obvious.

§ 4. Proof of Lemma 4

By means of Lie's theorem ([3], p. 134), Theorem 1, and Proposition 1, we can easily prove the following fact.

LEMMA 6. *For any regular Lie algbra $\mathfrak{g}$, we can always choose a basis of $\mathfrak{g}$ with the following property:*

For every $X \in \mathfrak{g}$, $\mathrm{ad}X$ is represented in the terms of this basis by a matrix

$$\mathrm{ad}X=\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ & A_{22} & \cdots & A_{2m} \\ & & \ddots & \vdots \\ 0 & & & A_{mm} \end{pmatrix} \tag{3}$$

where $A_{ii}(1 \leq i \leq m)$ are (1,1) or (2,2) matrices. (The pattern of partition of $\mathrm{ad}X$ into blocks is the same for all $X \in \mathfrak{g}$.) Moreover, each (2,2) matrix A_{ii} is a scalar multiple of a fixed matrix of the form

$$\begin{pmatrix} \mu_i & \nu_i \\ -\nu_i & \mu_i \end{pmatrix} \quad \mu_i \neq 0, \quad \nu_i \neq 0 \ \textit{real}.$$

Now, let $e_1, \cdots, e_n$ be an ordered basis of $\mathfrak{g}$ which satisfies the assertion of the above lemma. The vector subspace $\mathfrak{h}$ spanned by e_1 if the order of A_{11} is 1, and e_1, e_2 if it is 2, is obviously an ideal in $\mathfrak{g}$. Let φ denote the natural homomorphism of $\mathfrak{g}$ onto $\mathfrak{g}_1 \equiv \mathfrak{g}/\mathfrak{h}$, then $\varphi(e_2), \cdots, \varphi(e_n)$ (or, $\varphi(e_3), \cdots, \varphi(e_n)$ in the second case) is a basis of $\mathfrak{g}_1$.

LEMMA 7. *For any $X \in \mathfrak{g}$, corresponding to the representation* (3), *$ad(\varphi X)$ is represented in terms of the above basis by the matrix*

$$ad(\varphi X) = \begin{pmatrix} A_{22} & \cdots\cdots & A_{2m} \\ & \ddots & \vdots \\ 0 & & A_{mm} \end{pmatrix}.$$

The proof is obvious.

Next, select a basis $\overline{A}_1, \cdots, \overline{A}_r$ of $ad\mathfrak{g}_1$, then by the above lemma there exists a corresponding subset $\{A_1, \cdots, A_r\}$ of $ad\mathfrak{g}$. From this we can construct a basis $A_1, \cdots, A_r, B, B_1, \cdots, B_s$ of $ad\mathfrak{g}$ with the following properties.

(i) B is a particular matrix of the form (3) for which $A_{11} \neq 0$, $A_{ij} = 0$ if $i \geq 2$.

(ii) $B_1, \cdots, B_s$ are particular matrices of the form (3) for which $A_{11}=0$, $A_{ij}=0$ if $i \geq 2$.

(It may happen that B or $B_1, \cdots, B_s$ or all of them does not exist, but these cases are treated by similar ways so omitted.)

Now we can prove Lemma 4 by induction on $\dim \mathfrak{g}$. For $\dim \mathfrak{g}=1$ the lemma is trivial. $\mathfrak{g}_1$ is regular due to Proposition 3 and $\dim \mathfrak{g}_1 < \dim \mathfrak{g} (=n)$, so by induction hypothesis the equation

$$\exp(\sum \alpha_1^i \overline{A}_i) \exp(\sum \alpha_2^i \overline{A}_i) = \exp(\sum x^i \overline{A}_i) \tag{4}$$

has a unique analytic solution (see also Corollary in §1 and Proposition 4).

Next, consider the equation

$$\begin{aligned} &\exp(\sum \alpha_1^i A_i + \beta_1 B + \sum \beta_1^j B_j) \exp(\sum \alpha_2^i A_i + \beta_2 B + \sum \beta_2^j B_j) \\ &= \exp(\sum x^i A_i + yB + \sum y^j B_j). \end{aligned} \tag{5}$$

From (4) $x^1, \cdots, x^r$ are uniquely determined. Using this fact we can easily calculate that y is also uniquely determined. Finally for $y^1, \cdots, y^s$, we have a linear equation of the form

$$y^1 C_1 + y^2 C_2 + \cdots + y^s C_s = C \tag{6}$$

where $C_1, \cdots, C_s$, and C are some $(1, n\text{-}1)$ matrices (or, $(2, n\text{-}2)$ matrices in the second case). In this equation (6), $C_1, \cdots, C_s$ are determined only depending on $x^1, \cdots, x^r$, and y so they must be linearly independent at every point. If they are linearly dependent at some point $x_0^1, \cdots, x_0^r$, y_0 then for any $y_0^1, \cdots, y_0^s$ the matrix

$$\exp(\sum x_0^i A_i + y_0 B + \sum y_0^j B_j)$$

has many other expressions, which contradicts Proposition 4. On the other hand

$C_1, \cdots, C_s, C$ are always linearly dependent. In fact, the equation (5) has a unique analytic solution on a neighborhood U of the origin in $\mathrm{ad}\,\mathfrak{g}\times\mathrm{ad}\,\mathfrak{g}$, hence they are linearly dependent on U. This means that they are linearly dependent at every point by analyticity. (Each $(s+1)$ minor is zero on U therefore must be zero identically by the theorem of identity.) Consequently, the equation (6) has a unique analytic solution.

The consideration above shows that $\exp(\mathrm{ad}\,\mathfrak{g})$ is a subgroup of $\mathrm{Int}\,(\mathfrak{g})$, which is an open subgroup due to the regularity of $\mathrm{ad}\,\mathfrak{g}$. An open subgroup is always closed and due to the connectedness of $\mathrm{Int}\,(\mathfrak{g})$ we finally obtain the desired result $\exp(\mathrm{ad}\,\mathfrak{g})=\mathrm{Int}\,(\mathfrak{g})$.

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