Lie Groups with Regular Exponential Mapping II

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This note is a sequel to the previous one [4], in which regular Lie groups (i.e., Lie groups with the property that the exponential mapping is everywhere regular) was investigated. Already several researches along the same lines have been done, and among others the following interesting properties have been found. (As in [4], in this paper, only the finite dimensional and real case is treated, and $\mathfrak{g}$ and G denote a real finite dimensional Lie algebra and a corresponding connected Lie group respectively.) (i) A regular Lie algebra is solvable [4]. (ii) Let G be a solvable and simply connected Lie group, then exp is surjective if and only if G is regular [2] [9]. (iii) If G is regular and simply connected then exp is an analytic diffeomorphism of $\mathfrak{g}$ onto G [2] [9].

In K. T. Chen [1] it has been noticed that a Lie algebra $\mathfrak{g}$ is nilpotent if and only if $P(X, Y)$ is a polynomial function where $P(X, Y)$ is defined locally by $\exp X \exp Y \equiv \exp P(X, Y)$ $X, Y \in \mathfrak{g}$ (the so-called Campbell-Hausdorff formula). We shall prove in section 2 an analogous result that a Lie algebra is regular if and only if $P(X, Y)$ is an entire function (Theorem 2). The proof is based on the above result (iii) and the following property due to T. Nôno [8]: the exponential mapping of $\mathfrak{g}$ into G is not locally injective or locally surjective at any singular point in $\mathfrak{g}$ (Theorem 1). Section 1 is devoted to another proof of this property. Finally, in section 3, a particular type of regular Lie algebras is treated.

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§ 1. Local properties of exponential mapping

First of all we recall some definitions and notation.

An element $A \in \mathfrak{g}$ is said to be ***regular*** (*singular*) if the exponential mapping of $\mathfrak{g}$ into G is regular (singular) at the point A, $\mathfrak{g}$ is called ***regular*** if all $X \in \mathfrak{g}$ are regular, and G is called ***regular*** if $\mathfrak{g}$ is regular (see [4]). Let $C(\mathfrak{g}, A)$ and $C(\mathfrak{g}, a)$, respectively, denote the centralizers of $A \in \mathfrak{g}$ and $a \in G$ in $\mathfrak{g}$, that is

$$C(\mathfrak{g}, A) \equiv \{X \in \mathfrak{g} \mid \mathrm{ad}A(X) = 0\}$$
$$C(\mathfrak{g}, a) \equiv \{X \in \mathfrak{g} \mid \mathrm{Ad}(a)X = X\}.$$

The following criterion for regularity is due to T. Nôno ([7] Theorem 1, p. 163).

LEMMA 1. *An element* $A \in \mathfrak{g}$ *is regular if and only if* $C(\mathfrak{g}, A) = C(\mathfrak{g}, \exp A)$.

From this we have

LEMMA 2. *Let A be an arbitrary fixed regular element of* $\mathfrak{g}$. *Then the general solutions of the equation*

$$\exp Z = \exp A \tag{1}$$

are given by $Z=A+X$, where X satisfies $[A, X]=0$ and $\exp X=e$ (identity element of G).

PROOF. From the equation (1) we have $\mathrm{Ad}(\exp A)\, Z=\mathrm{Ad}(\exp Z)Z=Z$. This means by Lemma 1 that $\mathrm{ad}A(Z)=0$. Put $X=Z-A$, then obviously $[A, X]=0$ and $\exp A=\exp A \exp X$ therefore $\exp X=e$. The converse is obvious.

LEMMA 3. *There exists a symmetric convex open neighborhood U of* 0 *in* $\mathfrak{g}$ *with the following property. For any element A in* $\mathfrak{g}$, *let $R(A)$ and $S(A)$ respectively denote the sets of all regular elements and all singular elements in the neighborhood $A+U$ of A. Then*

(i) *$\exp R(A) \cap \exp S(A)=\phi$,*

(ii) *The restriction of the exponential mapping to $R(A)$ is an analytic diffeomorphism of $R(A)$ onto $\exp R(A)$.*

PROOF. It is possible to select a symmetric convex open neighborhood U of 0 in $\mathfrak{g}$ such that if $X \in 2U$ and $X\neq 0$ then $\exp X\neq e$. Then such a U satisfies the assertion of this lemma. In fact, for any two elements $Z_1=A+X_1$, $Z_2=A+X_2$ in $A+U$, where Z_1 is regular and $Z_1\neq Z_2$, we have $Z_2=Z_1+(X_2-X_1)$, $X_2-X_1 \in 2U$ hence from Lemma 2 $\exp Z_1\neq \exp Z_2$. This means the above lemma.

Remark. If G is a simply connected solvable Lie group then this property holds globally, that is, for $U=\mathfrak{g}$ ([2] Theorem 2, p. 119).

Now, we recall some definitions. A mapping f of a topological space E into a topological space F is said to be *locally injective* at a point a in E, if there exists a neighborhood U of a in E such that f is injective on U, and f is said to be *locally surjective* at a point a in E, if for every neighborhood U of a in E, $f(U)$ is a neighborhood of $f(a)$ in F. As mentioned above, the following property is due to T. Nôno ([8] Theorem 1, p. 318), but the proof in this paper differs somewhat from the original one.

THEOREM 1. *Let G be a connected Lie group with Lie algebra* $\mathfrak{g}$, *and A be a singular element in* $\mathfrak{g}$. *Then the exponential mapping of* $\mathfrak{g}$ *into G is not locally injective or locally surjective at A.*

PROOF. The following proof that exp is not locally injective at A was shown in [7], p. 164. From Lemma 1 there exists an element $Y \in C(\mathfrak{g}, \exp A)-C(\mathfrak{g}, A)$, for such a Y the non-trivial curve $A(t)=\exp(t\,\mathrm{ad}Y)A$ (t: real parameter) is mapped on the single point $\exp A$ by exp.

We shall now prove that exp is not locally surjective at A. Let U be an open neighborhood of 0 in $\mathfrak{g}$ which satisfies the assertion of Lemma 3, and let W be a neighborhood of A in $\mathfrak{g}$ such that its closure $\overline{W}$ is contained in $A+U$ and for some t_0, $A(t_0) \in (A+U)-\overline{W}$. Then we can select a sequence $\{X_n\}$ of regular elements in $(A+U)-\overline{W}$, which converges to $A(t_0)$. The sequence $\{\exp X_n\}$ obviously converges to $\exp A(t_0)=\exp A$. On the other hand, the sets W and $\{X_n\}$ are disjoint and both contained in the neighborhood $A+U$ of A in $\mathfrak{g}$ moreover $\{X_n\}\subset R(A)$, therefore from Lemma 3 follows that $\exp W\cap\exp\{X_n\}=\phi$. These last two facts show that exp W is not a neighborhood of $\exp A$ in G, that is, exp is not locally surjective at A.

§ 2. A characterization of regularity

The main result in this note is

THEOREM 2. *Let G be a real connected Lie group with Lie algebra $\mathfrak{g}$. A necessary and sufficient condition for $\mathfrak{g}$ to be regular is the existence of an analytic function $P(X, Y)$ defined for all $(X, Y)\in\mathfrak{g}\times\mathfrak{g}$ such that $P(0, 0)=0$ and*

$$\exp X \exp Y \equiv \exp P(X, Y) \qquad \text{for } X, Y\in\mathfrak{g}. \tag{2}$$

PROOF. By analyticity (see [3] Lemma 4.3, p. 228) this theorem is equivalent to the existence of a global analytic function $P(X, Y)$ for which (2) is equal to the Campbell-Hausdorff formula on a neighborhood of (0, 0) in $\mathfrak{g}\times\mathfrak{g}$. (This means also that $P(X, Y)$ is uniquely determined only depending on the Lie algebra $\mathfrak{g}$.)

First suppose $\mathfrak{g}$ is regular. Then the exponential mapping of $\mathfrak{g}$ into the corresponding simply connected Lie group $\widetilde{G}$ is an analytic diffeomorphism of $\mathfrak{g}$ onto $\widetilde{G}$ ([2] Theorem 2, p. 119 and [9] Theorem 1, p. 7). Hence the existence of $P(X, Y)$ is obvious, that is, $P(X, Y)\equiv(\exp)^{-1}(\exp X \exp Y)$.

Conversely suppose that there exists an analytic function $P(X, Y)$ which satisfies the identity (2). Then by analyticity $P(X, 0)\equiv X$ for all $X\in\mathfrak{g}$, therefore for an arbitrary fixed X_0 in $\mathfrak{g}$ and any given neighborhood U of X_0 in $\mathfrak{g}$ we can select a neighborhood W of 0 in $\mathfrak{g}$ such that $P(X_0, Y)\in U$ for every $Y\in W$. For such a W we have $\exp X_0 \exp W\subset\exp U$, which implies exp is locally surjective at X_0. Thus, due to Theorem 1, $\mathfrak{g}$ must be regular.

§ 3. Split Lie algebras

LEMMA 4. *Let $\mathfrak{h}$ be a subalgebra of a Lie algebra $\mathfrak{g}$. If an element $A\in\mathfrak{h}$ is regular in $\mathfrak{g}$ then A is regular in $\mathfrak{h}$. In particular if $\mathfrak{g}$ is regular then $\mathfrak{h}$ is regular.*

This is easily seen from the following general property (see also [3] p. 103 (1)). Let M and N be analytic manifolds and Φ be an analytic mapping of M into N. Let S be a submanifold of M ([3] p. 23) and a be a point in S. If Φ: $M\to N$ is regular at a, then the restriction of Φ to S is regular at a. (From [6] Theorem 1, p. 116 this lemma is also obvious.)

Definition. A real square matrix A is called *split* if A has all its eigenvalues real.

LEMMA 5. *Let $\mathfrak{l}$ be a real linear Lie algebra. If every $X\in\mathfrak{l}$ is split then $\mathfrak{l}$ is regular.*

This follows from the above lemma and the following fact (see [6]). Let $\mathfrak{gl}(n, R)$ be the real general linear Lie algebra, then $A\in\mathfrak{gl}(n, R)$ is regular if and only if $\lambda_h-\lambda_k\neq 2\pi i m$ $h, k=1, 2,\cdots, n$ (m is any non-zero integer) where $\lambda_1,\cdots,\lambda_n$ are the eigenvalues of A.

Definition. A real Lie algebra $\mathfrak{g}$ is called *split* if for any $X\in\mathfrak{g}$ ad$X$ is split.

Remark. Split is called *à racines réelles* in M. Saitô [10].

PROPOSITION 1. *A split Lie algebra is regular.*

This is a special case of the above referred property [6] Theorem 1, p. 116.

THEOREM 3. *Let* $\mathfrak{l}$ *be a real linear Lie algebra. If every* $X\in\mathfrak{l}$ *is split then* $\mathfrak{l}$ *is split.*

PROOF. Let $\mathfrak{t}(n, R)$ denote the linear Lie algebra of all real upper triangular matrices of order n. Then $\mathfrak{t}(n, R)$ is split as is shown below. In general, any subalgebra of a split Lie algebra is split, so this theorem follows from the next proposition.

Let $\mathfrak{M}$ be a given set of real square matrices of order n. An interesting question is now: when can $\mathfrak{M}$ be simultaneously triangularized, i. e., exists there a real regular matrix P such that for every $X\in\mathfrak{M}$, $P^{-1}XP$ is an upper triangular matrix? If $\mathfrak{M}$ has this property, then the smallest linear Lie algebra $\mathfrak{l}(\mathfrak{M})$ containing $\mathfrak{M}$ has the same property, hence any $X\in\mathfrak{l}(\mathfrak{M})$ is split. The converse also holds because of Lemma 5 and Lie's theorem (see also the property (i) in introduction). Therefore we have

PROPOSITION 2. *With the notation above,* $\mathfrak{M}$ *is simultaneously triangularized if and only if every* $X\in\mathfrak{l}(\mathfrak{M})$ *is split.*

A real split Lie algebra is analogous to a complex solvable Lie algebra in the following sence (see [2] Corollary, p. 121).

COROLLARY. *A real Lie algebra* $\mathfrak{g}$ *is split if and only if there exists a sequence*

$$\mathfrak{g}=\mathfrak{g}_0\supset\mathfrak{g}_1\supset\cdots\supset\mathfrak{g}_{n-1}\supset\mathfrak{g}_n=\{0\} \qquad (n=dim\ \mathfrak{g})$$

where $\mathfrak{g}_r$ *is an ideal in* $\mathfrak{g}$ *and dim* $\mathfrak{g}_r=n-r$ $(1\leq r\leq n)$.

The proof is obvious from Proposition 2.

We shall conclude with an example. (For notation the reader is referred to [5].)

EXAMPLE. *Let* L *be a real semi-simple Lie algebra and let* $L=R'+L_0'$ *denote an Iwasawa decomposition* ([5] p. 527). *Then* R' *is split.*

PROOF. Comparing the direct sums $L=R'+L_0'$ and $\widetilde{L}=R+\widetilde{L_0'}$ (where $\widetilde{L}$ and $\widetilde{L_0'}$ denote the complexifications of L and L_0' respectively), it is easily seen that $\dim_R R'=\dim_C R$. Hence R is the complexification of R'. On the other hand, for any $x\in R'$, $\mathrm{ad}_R x$ is expressed as a lower triangular matrix with a real diagonal in terms of the ordered basis

$$h_1,\cdots, h_n, e_\alpha, e_\beta,\cdots,e_\rho \qquad \alpha<\beta<\cdots<\rho$$

of R. These facts show R' is split.

For instance, consider the special linear Lie algebra $\mathfrak{sl}(n, R)$. Then we have an Iwasawa decomposition $\mathfrak{sl}(n, R)=\mathfrak{st}(n, R)+\mathfrak{o}(n)$, where $\mathfrak{st}(n, R)$ denotes the set of all real upper triangular matrices of order n with trace 0, and $\mathfrak{o}(n)$ denotes the set of all real skew-symmetric matrices of order n. From this we see that

$\mathfrak{sl}(n, \boldsymbol{R})$ is split, and the direct sum of any two split Lie algebras is obviously split so $\mathfrak{l}(n, \boldsymbol{R})$ is split. (This fact is also seen by a direct calculation.)

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