The Four-Color Problem*

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1. A Ring K{x}

Let $K=GF(p^m)$ be a field made of p^m elements. $K\{x\}$ is a set of polinomials of r variables $x_1, x_2, \ldots, x_r$ whose coefficients are elements of K. The element of $K\{x\}$ is denoted briefly by $f(x)$. The operations in $K\{x\}$ are like as in usual polinomial ring, but x^{p^m} is identified with x, and so $f(x)$ is a polinomial of lower degree than p^m.

$K\{x\}$ is a ring with zero-devisers. For example $x(x^{p^m}-1)=0$. The substitution of elements of K for variables is permitted, because $x^{p^m}=x$ as if x is an element of K. The substitution of an element of $K\{x\}$ for a avariable is also permitted by the next (1.1).

(1.1) $(f(x))^{p^m}=f(x)$ for $f(x)\in K\{x\}$.

Proof. $(f(x))^{p^m}=(\sum_j a(j_1\cdots j_r)\prod_i x_i^{j_i})^{pm}=\sum_j a(j_1\cdots j_r)^{p^m}\prod_i (x_i^{p^m})^{j_i}$

$$=\sum_j a(j_1\cdots j_r)\prod_i x_i^{j_i}=f(x). \qquad \text{q. e. d.}$$

(1.2) Let d be a r-dimensional vector whose i-th component d_i is an element of K, and $Y_d(x)=\prod_{i=1}^{r}\prod_{b_{ij}\neq d_i}(x_i-b_{ij})$ where $\prod_{b_{ij}\neq d_i}$ means the product of p^m-1 linear polinomials x_i-b_{ij} took as constant term b_{ij} all element of K except d_i. Then

$$Y_d(x)\cdot Y_{d'}(x)=0 \quad \text{for} \quad d\neq d', \quad \text{and} \quad (Y_d(x))^2\neq 0.$$

Proof. If $(Y_d(x))^2=0$, then $(Y_d(d))^2=\prod_{i=1}^{r}\prod_{b_{ij}\neq d_i}(d_i-b_{ij})^2=0$: a product of non-zero elements of the field K would be zero. When $d\neq d'$, $Y_d(x)\cdot Y_{d'}(x)$ has a factor $\prod_{b\in K}(x_i-b)=x_i^{p^m}-x_i=0$ for some x_i, where $\prod_{b\in K}(x_i-b)$ means the product of $x-b$ taken as b all element of K. q. e. d.

(1.3) Let $\{d\}$ be the set of all r-dimensional vector of K. Any element of $K\{x\}$ can be expressed as a linear combination of elements of the set $\{Y_d(x);\ d\in\{d\}\}$ with coefficients in K.

Proof. Assume that $\sum_{d\in\{d\}} c_d Y_d(x)=0$ $(c_d\in K)$. Multiplying both sides of this equation by some $Y_{d'}(x)$ and using (1.2), we get $c_{d'}\{Y_{d'}(x)\}^2=0$, and so $c_{d'}=0$. Therefore, $Y_d(x)$ $(d\in\{d\})$ are linearly independent on K. On the other hand, all element of $K\{x\}$ is shown as a linear combination of p^{mr} elements $\prod_{i=1}^{r} x_i^{e_i}$ $(0\leqq e_i\leqq p^m-1)$ with coefficients in K, and $K\{x\}$ is a p^{mr}-dimensional vector space having elements of K as scalar. $\{Y_d(x);$

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$d \in \{d\}\}$ is its base. q.e.d.

(1.4) Any $f(x)$ in $K\{x\}$ can be written in the next form.

$$f(x)=(-1)^r \sum_{d\in\{d\}} f(d)Y_d(x).$$

Proof. We first show that $Y_d(d')=0$ for $d\neq d'$ and $Y_d(d)=(-1)^r$. If $d\neq d'$, some factor x_i-b_{ij} will become a difference of the same element when d' is substituted for x. In case of $d=d'$, the product $\prod_{b_{ij}\neq d_i}(d_i-b_{ij})$ is got by the substitution of d_i for x_i in the equation $\frac{d}{dx_i}\prod_{b_{ij}\in K}(x_i-b_{ij})=\frac{d}{dx}(x_i^{p^m}-x_i)=p^m x_i^{p^m-1}-1$, and equal to -1. Therefore, we get $c_d=(-1)^r f(d)$ by the substitution $x=d$ in the both sides of $f(x)=\sum_{d\in\{d\}} c_d Y_d(x)$.

(1.5) The value of $f(x)$, substituted d ($d\in\{d\}$) for x, is all zero if and only if $f(x)$ is the zero element of $K\{x\}$.

The proof is obvious from (1.4).

2. Conditions of Solvability of the Four-Color Problem

It is easily seen that to prove the four-color conjecture we may do it with only so called cubic graph, in which just three edges meet at every vertex.

We say that a graph is colorable when its faces can be distinguished with four colors, or when a condition equivalent to it is satisfied.

We denote the fields made of two or three elements by k_2 and k_3 respectively, i.e. $k_2=GF(2)$ and $k_3=GF(3)$.

(2.1) A cubic graph G is face-four-colorable if and only if it is edge-three-colorable.

Proof. Suppose that G is face-four-colorable. We name the four colors by 2-dimensional vectors on k_2: (00), (01), (10) and (11). If we give to every edge the sum of vectors given to the two faces which have the edge in common, it is the edge-three-coloration, because the sum of different two vectors is not (00), and three edges which meet at a vertex have different values, for the three faces which have the vertex in common have different values.

Suppose, conversely, that edges of G are colored by (01), (10) and (11). We can color every face one by one, by adding the value of an edge which separates the face from already colored face to that face-color, when occurs no contradiction because (01) +(10)+(11)=(00). q.e.d.

(2.2) A graph G is colorable if and only if non-zero value in k_3 can be given to every vertex such that the sum of these vertex-values around every face is zero in k_3.

Proof. Suppose that the graph is colorable. We determine an order in the three colors of edge. If the order of colors of the three edges meeting at a vertex is clockwise, then we give to the vertex the value -1, and in the inverse case the value $+1$. If a point moves from an edge to the next across a vertex in the way which passes through a periphery of a face clockwise, then the color of the edge, on which the point lies, changes on- or backward in that color-order according as the vertex-value is $+1$ or -1. When the point had passed the way just one round, the color must have returned back to the origin, and so the sum of the vertex-values around the face is zero.

Conversely, if every vertex is valued such, then we may give a color indicated by the vertex-value to every edge, step by step, when no contradiction occurs because the sum of vertex-values around a face is zero.

(2.3) In the former proposition (2.2), the phrase "around every face" may be changed to "around $n-3$ faces except three that have a vertex v_0 in common", where n is the number of faces in G.

Proof. Assume that values $+1$ or -1 are given to all vertex with one exception v_0 such that the sum of them around every $n-3$ faces which has not the vertex v_0 is zero, and name other end points of edges which meet at v_0, v_1, v_2 and v_3. The edge-coloration is got from this vertex-value, which doesn't contradict at every vertex except v_0.

W. T. Tutte proved the next theorem in [1].

Let G be any 4-connected planar graph having at least two edges. Then G has a Hamiltonian circuit. Moreover if no two edges of G have both ends in common, and if E and E' are distinct edges of the same terminal circuit of G, then there is a Hamiltonian circuit of G having both E and E' as edges.

The dual graph of our cubic graph G can be assumed 4-connected, for otherwise the coloration of G is reduced to that of a graph which has fewer faces than G, and, as we want to prove our theorem by induction with respect to the number of faces, we may reject the case when the dual graph of G is three separable. Then the dual graph of G have, from above theorem, a Hamiltonian circuit. Corresponding to this Hamiltonian circuit there is a simple closed curve which cut just two edges of every face of G at mid point of them. We call this circuit "*T-circuit*" of G. *T-circuit* separates G into two parts. Every of them is tree. We name these two trees T_1 and T_2.

From the last part of above Tutte's theorem, it is assured to put that the *T-circuit* cut the edge v_0v_1.

The sum of values of three edges which meet at a vertex is zero, i.e. one value of them is equal to the sum of other two. Therefore, the value of all edge is determined by that of edges, through which the *T-circuit* crosses and which is not v_0v_1. We call them "*initial edges*". This coloration of edges is of cause the same as that determined by the given vertex-value if the values of initial edges are chosen so. The value of v_0v_1 is equal to the sum of v_0v_2 and v_0v_3 because the value of v_0v_1 calculated along the trees T_1 and T_2 must be both equal to the sum of all value of initial edge, and so this coloration doesn't contradict at v_0, too. q.e.d.

We want to prove the colorability of graph by the mathematical induction with respect to the number of faces. If a circuit of the dual graph of G which is not a periphery of a face has three edges, then the colorability of G can be reduced to that of a graph with fewer faces. (Upon which is touched already in (2.3).) Similarly is also a graph with triangular or tetrahedral faces reducible. Therefore we treat a graph which is not so.

Such a graph must have a pentagonal face. Although its proof is easy, we will state it briefly. Let n, e and v be the number of faces, edges and vertices of a cubic graph respectively. Then $3v=2e$. From this and the *Euler's relation*: $n+v=e+2$, we get $v=2n-4$ and $e=3n-6$, but, if every face had six or more edges, the number of the edges would be $3n$ or more because every edge belongs to only two faces, contrarily to $e=3n-6$.

Here we define the "color-characteristic formula" (which is abbreviated to *ccf*) of a cubic graph. $CG(n)$ denote a cubic graph with n faces. A_1 is a matrix on k_3 of $(n-3)\times(2n-5)$ type, whose columns correspond to vertices except v_0 and rows of A_1 correspond to faces which have not the vertex v_0. The (ij)-element of A_1 is 1 when the vertex which corresponds to the j-th column is on the face which corresponds to the i-th row, and is 0 in the other case.

$CG(n)$ has a *T-circuit* as shown in the Fig. 1. Let the two trees which is made by this *T-circuit* T_1 and T_2, and T_1 contains the edge v_0v_1, the existence of which is assured by the last half part of "*Tutte's theorem*".

The square matrix A of order $n-3$ which is consist of the columns which correspond to the vertices in T_1 is regular on k_3, because, if a linear combination of rows of A is a zero-vector, then the coefficient of the row whose element in the column of v_1 is 1 must be zero, and so the coefficient next to it in T_1 must be also zero, and so on. (see Fig. 1)

Let x_1 be a vector of order $2n-5$ whose components are variables, then the condition of colorability of $CG(n)$ is the existence of a solution of the simultaneous linear equation $A_1x_1=0$ whose component is all non-zero. [(2.3)]

Let B be a matrix made of columns of A_1 which is not in A, and $x_1=\begin{pmatrix}x\\y\end{pmatrix}$, where x is a vector of order $n-3$, then the solution of $A_1x_1=(A,\ B)\begin{pmatrix}x\\y\end{pmatrix}=0$ is $x=A^{-1}By$, where y is an arbitrary vector on k_3 of order $n-2$, whose components are called variables. The square of the product of the variables and the components of $A^{-1}By$, linear combinations of variables, is called the color-characteristic formula (*ccf*) and represented by $F_{CG(n)}(y)$.

(2.4) The condition of colorability of $CG(n)$ is that $F_{CG(n)}(y)$ is not the zero-element of $k_3\{y\}$.

(2.5) If we treat just as above, omitting two faces which have a vertex v_0 in common, instead of omitting three faces which has v_0 in common, then we get the same formula as $F_{CG(n)}(y)$.

Proof. We join a row which corresponds to the new face and a column which corresponds to the vertex v_0. In the column of v_0, only one element which is in the new row is 1 and others are zero. Therefore the component of the same y can be taken as variables, by which the value of vertices in T_1, other than v_0, is represented in the same form. We write the formula thus obtained $y_0^2 F_{CG(n)}(y)$. $y_0^2 F_{CG(n)}(y) - F_{CG(n)}(y)$ is zero for values of y which make $F_{CG(n)}(y)$ non-zero, by (2.3), and of cause for the values which make it zero, and so equal to zero by (1.5). q.e.d.

(2.6) Notation is all as above. Let a variable which corresponds to v_1 be y_1. If we don't multiply the factor y_1^2 when we make the *ccf* of $CG(n)$ and put $y_1=0$, then we get a *ccf* of a cubic graph $CG(n-1)$, which is got from $CG(n)$ by taking off the edge v_0v_1.

Proof. We can take a *T-circuit* such that y_1 is an independent variable. Name the square of product of linear form corresponding to all vertex except v_0 and v_1 $F(y)$. The variable y_1 is used in it. If we remove two faces from $CG(n-1)$ such that remaining faces are same as that of $CG(n)$ stated above, and make a product of square of linear form, which are determined such that the sum of them around every face there remained is zero, using the same variables as $CG(n)$, then the linear forms used in it are got by substituting 0 for y_1 in that of $CG(n)$. Therefore this *ccf* of $CG(n-1)$ is got by putting $y_1=0$ in $F(y)$.

N.B. It is only when the independent variables (i.e. the *T-circuit*) are suitably took that the formula got above is a *ccf* of $CG(n-1)$ as is seen in (2.5), but two *ccf* with respect to different variables are changed each other by regular linear transformation, and the vanishing of one is followed by other's vanishing.

3. Condition of a Not Colorable Graph with Fewest Faces

Suppose that $CG(n)$ is the cubic graph which has fewest faces in what is not colorable. We want to show that the existence of such a graph $CG(n)$ implies a contradiction. Let vertices of a pentagonal face F_0 of $CG(n)$ be v_j $(0 \leqq j \leqq 4)$, which lie in the order of suffix. Three faces F_0, F_1 and F_2 have the same vertex v_1 in common. F_1, F_2, F_3, F_4 and F_5 are the five neighbouring faces of F_0, the order of their position is the same as their suffix. v_0 is a common vertex of F_0, F_1 and F_2. (see Fig. 2) We can take a *T-circuit* which passes through F_1, F_2, F_0, F_3, F_4 and F_5 in this order, whose existence is secured by the last half part of *Tutte's theorem*, applying it to the dual graph of what is got from $CG(n)$ by removal of edges which separate the six faces F_j $(0 \leqq j \leqq 5)$ each other.

Let $F(y)$ be the product of all linear form of variables for every vertex. $F^2(y)$ is the *ccf* of $CG(n)$. Only four factors of $F(y)$ have the term of y_1 whose coefficient is not zero. (We represent them by y_1, y_1+a, $-y_1+b$ and y_1+c, and suppose that they correspond to v_1, v_2, v_3 and v_4 respectively.)

Proof. Elements of A is as in Fig. 3. The reason of it can be seen in Fig. 2. Because $A^{-1}A=E$, the k-th row of A^{-1} corresponding to v, other than v_2, v_3, and v_4, must be as in Fig. 3. Therefore, in the linear form of y, which express x_k and is k-th row of $-A^{-1}By$, the coefficient of y_1 is zero. q. e. d.

From our assumption that our $CG(n)$ is not colorable

$$F(y)=(y_1+a)(-y_1+b)(y_1+c)y_1f=0, \qquad \cdots\cdots\cdots\cdots(1)$$

where f is the product of linear formulas for vertices other than v_1, v_2, v_3 and v_4. The coefficient of y_1 and y_1^2 in (1) must be zero in $k_3\{y\}$:

$$-abcf+af-bf+cf=0,$$

$$-f+bcf-caf+abf=0. \qquad \cdots\cdots\cdots\cdots(2)$$

The graph $CG(n-1)$ made from $CG(n)$ removing the edge v_0v_1 has a *ccf* got from $(y_1+a)^2(y_1+b)^2(y_1+c)^2f^2$ putting $y_1=0$, namely $F_{CG(n-1)}(y)=a^2b^2c^2f^2$.

It is seen from (2) that $f=0$ for the value of variables which make two of a, b and c zero, while none of a, b and c can be zero when other two and f are not zero by (2.3). That is that the value of variables which make $a=0$ or $b=0$ or $c=0$ make also $f=0$, and by (1.3), $a^2f=b^2f=c^2f=f$. Hence $F_{CG(n-1)}(y)=f^2$.

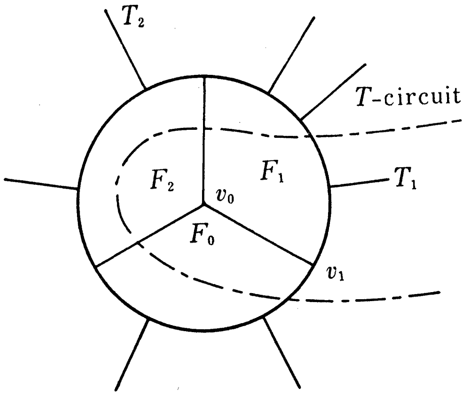

Fig. 1.

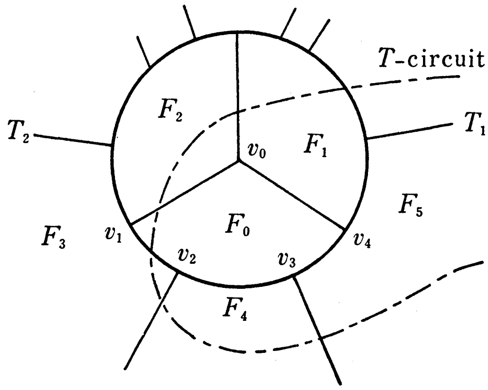

Fig. 2.

$$
\begin{array}{c}
\mathrm{A}^{-1}\\
\begin{array}{r|c|c|c|c|}
\hline
v_4\longrightarrow & 1 & -1 & 1 & \\ \hline
v_3\longrightarrow & 0 & 1 & -1 & \\ \hline
v_2\longrightarrow & 0 & 0 & 1 & \\ \hline
 & \vdots & \vdots & \vdots & \\ \hline
v\longrightarrow & 0 & 0 & 0 & ****** \\ \hline
 & \vdots & \vdots & \vdots & \\ \hline
 & 0 & 0 & 0 & \\ \hline
\end{array}\\
\quad\uparrow\ \ \uparrow\ \ \uparrow\\
\quad F_5\ F_4\ F_3
\end{array}
\qquad
\begin{array}{c}
\mathrm{A}\\
\begin{array}{r|c|c|c|c|}
\hline
F_5\longrightarrow & 1 & 1 & 0 & ****** \\ \hline
F_4\longrightarrow & 0 & 1 & 1 & ****** \\ \hline
F_3\longrightarrow & 0 & 0 & 1 & ****** \\ \hline
 & 0 & 0 & 0 & \\
 & \vdots & \vdots & \vdots & \\
 & 0 & 0 & 0 & \\ \hline
\end{array}\\
\quad\uparrow\ \ \uparrow\ \ \uparrow\\
\quad v_4\ \ v_3\ \ v_2
\end{array}
\qquad
\begin{array}{c}
\mathrm{B}\\
\begin{array}{r|c|c|}
\hline
F_5\longrightarrow & 0 & \\ \hline
F_4\longrightarrow & 0 & \\ \hline
F_3\longrightarrow & 1 & \\ \hline
 & 0 & \\
 & \vdots & \\
 & 0 & \\ \hline
\end{array}\\
\quad\uparrow\\
\quad v_1
\end{array}
\qquad
\begin{array}{c}
\mathrm{y}\\
\begin{array}{|c|}
\hline
y_1 \\ \hline
y_2 \\ \hline
y_3 \\ \hline
\vdots \\
\\ \hline
\end{array}
\end{array}
$$

Fig. 3.

Fig. 3 shows that f is determined from lower $n-6$ rows of A, which means that $F_{CG(n-1)}(y)$ is determined from the relation of vertices and faces of $CG(n)$ except F_j $(0 \leqq j \leqq 5)$. The same conclusion is reduced for cubic graph $CG(n-1)$ which is got from $CG(n)$ removing any edge of F_0, i.e. they must have the same $ccf\,f^2$.

For example, in the $CG(n-1)$ in which the face F_0 and F_2 is fused in one, the face-color of F_2 (or F_0) is different from that of F_3, F_4, F_5 and F_1, which are determined uniquely for any value of variables that make $f \neq 0$. Considering other $CG(n-1)$, it is concluded that thus determined colors of F_j $(1 \leqq j \leqq 5)$ are different each other. It is obviously a contradiction.

Therefore there is no graph which is not colorable.

Reference

[1] W. T. Tutte, A Theorem on Planar Graph, Trans. Am. Math. Soc. 82, 99-116 (1956).