DESIGN OF FIXED-POINT SMOOTHER BASED ON INNOVATIONS THEORY FOR WHITE GAUSSIAN PLUS COLORED OBSERVATION NOISE

Seiichi NAKAMORI

(Received October 1, 1989)

Abstract

This paper designs a new fixed-point smoother based on the innovations theory for white Gaussian plus colored observation noise in linear continuous systems. The signal to be estimated is nonstationary or stationary stochastic. The proposed fixed-point smoothing algorithm calculates estimates sequentially by using the following information. (1) The autocovariance information of the signal plus colored observation noise. (2) The crosscovariance information between the signal and the observed value. (3) The observed value. The autocovariance function of (1) and the crosscovariance function of (2) are assumed to be expressed in the semi-degenerate kernel forms. Some numerical simulation results show that the presented fixed-point smoother is feasible.

1. Introduction

Kalman filter is applicable to an estimation problem of a signal which is observed with additive white Gaussian observation noise. The Kalman filter needs information of a state-space model of the signal (Kalman, 1960). Kailath (1968) considers linear least-squares filtering problems by introducing an innovations theory in continuous systems. The innovations theory plays an important role on the estimations in linear and nonlinear stochastic systems. There is an approach which does not utilize the state-space model except for observed values and covariance information of signal and noise on the estimation problems of the stochastic signal in linear systems (Casti, Kalaba and Murthy, 1972, Nakamori and Sugisaka, 1977). Casti etc. (1972) present a Cauchy system by applying an invariant imbedding method to a Wiener-Hopf integral equation with a displacement kernel. The kernel in the Wiener-Hopf integral equation represents an autocovariance function of observed data, and least-squares estimation problems by the Wiener-Hopf integral equation have been researched extensively in the communication theory (Trees, 1968). Since the displacement kernel has a specific integral functional form, the estimation treated by Casti etc. (1972) is extended by Nakamori etc. (1977) to more general case by assuming that the autocovariance function of the signal is expressed in a semi-degenerate kernel form. The semi-degenerate kernel is suitable for expressing auto-

Department of Technology (Electrical Technology), Faculty of Education, Kagoshima University, 1-20-6, Kohrimoto, Kagoshima 890, JAPAN

covariance functions of general nonstationary or stationary stochastic processes. Then the estimation algorithm, which is derived from the Wiener-Hopf integral equation with the semi-degenerate kernel, is widely applied to estimation problems of general stochastic processes.

Kailath (1976) shows in his book (see pages 123–125) a filtering algorithm, which includes a differential equation for an autocorrelation function of a filtering estimate, under a prescribed autocovariance function of the observed value. The algorithm by Kailath (1976) still needs information of a system matrix in the state-space model. In the approach by Casti etc. (1972) and Nakamori etc. (1977), the covariance information, which is approximated by some functions such as normalized orthogonal functions etc., is used directly in the estimation algorithms. An example for predicting air pollution levels of SO_2 in Ohita city, Japan, is implemented by using a covariance information of SO_2 levels (Nakamori and Hataji, 1980). This example makes us understand how the approach in Nakamori etc. (1977, 1980) is practically effective.

This paper presents a new fixed-point smoothing algorithm based on the innovations theory for the white Gaussian plus colored observation noise. The necessary information for this estimator is as follows. (1) The autocovariance information of the signal plus colored observation noise. (2) The crosscovariance information between the signal and the observed value. (3) The observed value. It is assumed that covariance functions of (1) and (2) are expressed in the semi-degenerate kernel forms. The treatment of the estimation problems here is a natural extension of the Wiener filter (Kailath, 1974) to the fixed-point smoothing problems using covariance information for the white Gaussian plus colored observation noise. There exist some differences between the presented fixed-point smoother based on the innovations theory and that in Nakamori (1988). They are summarized as follows.

(1) The presented fixed-point smoothing algorithm has quite different forms from that in Nakamori (1988).

(2) Number of differential equations included in the presented fixed-point smoothing algorithm is reduced in comparison with that in the fixed-point smoother (Nakamori, 1988) for the white Gaussian plus colored observation noise.

Some digital simulation results show that the presented fixed-point smoothing algorithm is feasible.

2. Linear least-squares estimation problems

The observation equation is given by

$$y(t)=x(t)+v_c(t)+v(t),\ z(t)=x(t)+v_c(t), \tag{1}$$

where $y(t)$ is an n-dimensional observed value, $x(t)$ is a zero-mean signal, $v_c(t)$ is a zero-mean colored observation noise and $v(t)$ is a zero-mean white Gaussian observation noise. It is assumed that $x(t)$ and $v_c(t)$ are nonstationary or stationary. The autocovariance function of the white Gaussian observation noise is given by

$$E[v(t)v^T(s)]=R\delta(t-s) \tag{2}$$

There are uncorrelation properties between the signal $x(.)$, the colored observation noise $v_c(.)$ and the white Gaussian observation noise $v(.)$ as

$$E[x(t)v_c^T(s)]=0,\ E[x(t)v^T(s)]=0,\ E[v_c(t)v^T(s)]=0,\ 0\leqq s,t\leqq T, \tag{3}$$

in the estimation problems by Trees (1968) and Nakamori (1988). The proposed fixed-point smoothing algorithm can be applied to both cases when the signal and the colored observation noise are correlated or uncorrelated. Let a fixed-point smoothing estimate $\hat{x}(t,T)$ of $x(t)$ at a fixed-point t be given by

$$\hat{x}(t,T)=\int_0^T g(t,s)\nu(s)ds \tag{4}$$

as a linear integral transform of an innovations process $\nu(s)$ (Kailath, 1968), $0\geqq s,\ t\leqq T$, which is expressed by

$$\nu(s)=y(s)-\hat{z}(s). \tag{5}$$

Here, $\hat{z}(s)$ is a filtering estimate of $z(s)$ at time s. Let us consider a linear least-squares smoothing problem which minimizes a cost function

$$J=E[\|x(t)-\hat{x}(t,T)\|^2]. \tag{6}$$

Minimizing the cost function of (6) leads to a Wiener-Hopf integral equation

$$E[x(t)\nu^T(s)]=\int_0^T g(t,s')\ E[\nu(s')\nu^T(s)]ds,\ 0\leqq s\leqq t\leqq T. \tag{7}$$

It is known that the variance of the innovations process is equal to that of the white Gaussian observation noise particularly in linear continuous systems (Kailath, 1968).

$$E[\nu(t)\nu^T(s)]=R\delta(t-s) \tag{8}$$

Substituting (8) into (7), one has

$$E[x(t)\nu^T(s)]=g(t,s)R. \tag{9}$$

From (9), one finds that the optimal impulse response function, which satisfies (7), is given by

$$g(t,s)=E[x(t)\nu^T(s)]R^{-1} \tag{10}$$

(Sage, 1971). It is a problem to formulate the statistical quantity $E[x(t)\nu^T(s)]$. As noticed from (5) for the innovations process, the expression for the filtering estimate $\hat{z}(s)$ is needed. In section 3, the filtering algorithm for $\hat{z}(t)$ is shown. Then sequential algorithm for the optimal impulse response function $g(t,s)$ is presented in section 4.

It is a specific characteristic to introduce the autocovariance function of $z(t)$ expressed by $K_z(t,s)=A(t)B^T(s)$ for $0\leqq s\leqq t$ and $K_z(t,s)=B(t)A^T(s)$ for $0\leqq t\leqq s$, where $A(t)$ and $B(s)$ are bounded $n\times n'$ matrices. This is a reasonable expression and $K_z(t,s)$ in this form is called the semi-degenerate kernel. Also, the crosscovariance function $K_{xy}(t,s)$ between the signal $x(t)$ and the observed value $y(s)$ is assumed to be expressed by $K_{xy}(t,s)=\alpha(t)\beta^T(s)$

for $0 \leqq s \leqq t$ and $K_{xy}(t,s) = \varepsilon(t)\zeta^T(s)$ for $0 \leqq t \leqq s$, where $\alpha(t)$, $\beta(s)$, $\varepsilon(t)$ and $\zeta(s)$ are bounded $n \times m'$ matrices.

3. Filtering algorithm for $\hat{z}(t)$

In [Theorem 1], the filtering algorithm for $\hat{z}(t)$ is presented. If $\hat{z}(t)$ is evaluated, the innovations process $\nu(t)$ is calculated by (5).

[Theorem 1]

If the autocovariance function of the signal $z(t)$ is represented by $K_z(t,s) = A(t)B^T(s)$ for $0 \leqq s \leqq t$, then the linear least-squares filtering estimate $\hat{z}(t)$ is calculated by Eqs. (11)～(14) sequentially (Nakamori etc., 1977).

$$\hat{z}(t) = A(t)e(t) \tag{11}$$

$$de(t)/dt = J(t,t)(y(t) - \hat{z}(t)),\ \textit{Initial condition}: e(0) = 0 \tag{12}$$

$$J(t,t)R = (B^T(t) - r(t)A^T(t)) \tag{13}$$

$$dr(t)/dt = J(t,t)(B(t) - A(t)r(t)),\ \textit{Initial condition}: r(0) = 0 \tag{14}$$

Proof of [Theorem 1] is shown in Nakamori etc. (1977).

The innovations process is given by (5) and the filtering estimate is calculated by the Cauchy system of [Theorem 1] using the observed values and the covariance information of $z(t)$ and $v(t)$. It is interesting to note that the innovations process, which is white Gaussian with the variance R, is yielded from the observed value. This is called the whitening filter (Trees, 1968), since the input to the filter is the observed value and its output is the white Gaussian innovations process.

4. Algorithms for g (t,s) and $\hat{x}(t,T)$

In [Theorem 2], algorithms for the optimal impulse response function $g(t,s)$ and the fixed-point smoothing estimate $\hat{x}(t,T)$ are presented by using covariance information.

[Theorem 2]

If the autocovariance function of $z(t)$ is given by $K_z(t,s) = A(t)B^T(s)$ for $0 \leqq s \leqq t$ and the crosscovariance function $K_{xy}(t,s)$ between the signal $x(t)$ and the observed value $y(s)$ is assumed to be expressed by $K_{xy}(t,s) = \alpha(t)\beta^T(s)$ for $0 \leqq s \leqq t$ and $K_{xy}(t,s) = \varepsilon(t)\zeta^T(s)$ for $0 \leqq t \leqq s$, then the optimal impulse response function $g(t,s)$ is calculated by Eqs. (15)～(20) sequentially.

$$g_1(t,s): g(t,s)\ \textit{for}\ 0 \leqq s \leqq t$$

$$g_1(t,s) = \alpha(t)(\beta^T(s) - q^T(s)A^T(s))R^{-1} \tag{15}$$

$$dq(t)/dt = J(t,t)\ (\beta(t) - A(t)q(t)),\ \textit{Initial condition: } q(0)=0 \tag{16}$$

$g_2(t,s)$: $g(t,s)$ *for* $0 \leqq t \leqq s$

$$g_2(t,s) = (\varepsilon(t)\zeta^T(s) - \alpha(t)D^T(s,t)A^T(s) - \varepsilon(t)E^T(s,t)A^T(s))R^{-1} \tag{17}$$

$$\partial D(T,t)/\partial T = -J(T,T)A(T)D(T,t) \tag{18}$$

$$dD(t,t)/dt = J(t,t)\ (\beta(t) - A(t)D(t,t)),\ \textit{Initial condition: } D(0,0)=0 \tag{19}$$

$$\partial E(T,t)/\partial T = J(T,T)\ (\zeta(T) - A(T)E(T,t)),\ \textit{Initial condition: } E(t,t)=0 \tag{20}$$

$J(t,t)$ is calculated by (13) and (14).

Also, the fixed-point smoothing estimate $\hat{x}(t,T)$ is calculated by Eqs. (11)~(14), (18), (19), (20) and (21) sequentially.

$\hat{x}(t,T)$: *Fixed-point smoothing estimate*

$$\partial \hat{x}(t,T)/\partial T = \varepsilon(t)\ (\zeta^T(T) - E^T(T,t)A^T(T))R^{-1}(y(T) - \hat{z}(T)) - \alpha(t)D^T(T,t)A^T(T)R^{-1}(y(T) - \hat{z}(T)) \tag{21}$$

$\hat{x}(t,t)$: *Filtering estimate*

$$\hat{x}(t,t) = \alpha(t)Q(t) \tag{22}$$

$$dQ(t)/dt = (\beta^T(t) - q^T(t)A^T(t))R^{-1}(y(t) - \hat{z}(t)),\ \textit{Initial condition: } Q(0)=0 \tag{23}$$

Here, $\hat{z}(T)$ and $\hat{z}(t)$ are calculated by using the algorithm for the filtering estimate in [Theorem 1].

(Proof)

In the calculation of the optimal impulse response function $g(t,s)$ given by (10), the statistical quantity $E[x(t)\nu^T(s)]$ is necessary. From (5) $E[x(t)\nu^T(s)]$ is developed as

$$\begin{aligned} E[x(t)\nu^T(s)] &= E[x(t)(y(s) - \hat{z}(s))^T] \\ &= E[x(t)y^T(s)] - E[x(t)\hat{z}^T(s)]. \end{aligned} \tag{24}$$

In Nakamori etc. (1977), the linear least-squares filtering estimate $\hat{z}(t)$ is fomulated as

$$\hat{z}(t) = \int_0^t h(t,s)y(s)ds. \tag{25}$$

Substituting (25) into (24), one has

$$E[x(t)\nu^T(s)] = E[x(t)y^T(s)] - \int_0^s E[x(t)y^T(s')]h^T(s,s')ds'. \tag{26}$$

From Nakamori etc. (1977), one knows that

$$h(s,s') = A(s)J(s,s'). \tag{27}$$

Let us derive at first $g_1(t,s)$ which denotes $g(t,s)$ for $0 \leqq s \leqq t$. If one substitutes $K_{xy}(t,s) = \alpha(t)\beta^T(s)\ (=E[x(t)y^T(s)])$ for $0 \leqq s \leqq t$ and (27) into (26), one obtains

$$E[x(t)\nu^T(s)] = \alpha(t)(\beta^T(s) - \int_0^s \beta^T(s')J^T(s,s')ds'A^T(s)). \tag{28}$$

If one introduces the function, which satisfies

$$q(t) = \int_0^t J(t,s')\beta(s')ds', \tag{29}$$

one can rewrite (28) as

$$E[x(t)\nu^T(s)] = \alpha(t)(\beta^T(s) - q^T(s)A^T(s)). \tag{30}$$

Substituting (30) into (10) yields (15).

Differentiating (29) with respect to t, one has

$$\begin{aligned} dq(t)/dt &= J(t,t)\beta(t) + \int_0^t \partial J(t,s')/\partial t \beta(s')ds' \\ &= J(t,t)(\beta(t) - A(t)q(t)) \end{aligned} \tag{31}$$

by using (29) and an identity (Nakamori etc., 1977)

$$\partial J(t,s)/\partial t = -J(t,t)A(t)J(t,s). \tag{32}$$

The initial condition on the differential equation (31) for the function $q(t)$, at $t=0$, is $q(0)=0$ from (29). It is noted that the function $q(t)$ is equal to the function $r(t)$ (see (16) and (14)), when the signal $x(t)$ is observed with only additive white Gaussian noise $v(t)$.

Secondly, let us derive an expression of $g(t,s)(=g_2(t,s))$ for $0 \leqq t \leqq s$. In the calculation of the optimal impulse response function $g_2(t,s)$, the statistical quantity $E[x(t)\nu^T(s)]$ should be developed as seen from (10). From (26), $E[x(t)\nu^T(s)]$ for $0 \leqq t \leqq s$ is also written as

$$E[x(t)\nu^T(s)] = E[x(t)y^T(s)] - \int_0^s E[x(t)y^T(s')]h^T(s,s')ds'. \tag{33}$$

If one notes that $K_{xy}(t,s) = \alpha(t)\beta^T(s)$ for $0 \leqq s \leqq t$ and $K_{xy}(t,s) = \varepsilon(t)\zeta^T(s)$ for $0 \leqq t \leqq s$, one can express $g_2(t,s)$ as follows from (27) and (33).

$$\begin{aligned} g_2(t,s) &= \{E[x(t)y^T(s)] - \int_0^s E[x(t)y^T(s')]h^T(s,s')ds'\}R^{-1} \\ &= \{\varepsilon(t)\zeta^T(s) - \int_0^t E[x(t)y^T(s')]h^T(s,s')ds' \\ &\quad - \int_t^s E[x(t)y^T(s')]h^T(s,s')ds'\}R^{-1} \\ &= \{\varepsilon(t)\zeta^T(s) - \int_0^t \alpha(t)\beta^T(s')J^T(s,s')A^T(s)ds' \\ &\quad - \int_t^s \varepsilon(t)\zeta^T(s')J^T(s,s')A^T(s)ds'\}R^{-1} \end{aligned} \tag{34}$$

If one introduces functions

$$D(s,t)=\int_0^t J(s,s')\beta(s')ds' \tag{35}$$

and

$$E(s,t)=\int_t^s J(s,s')\zeta(s')ds', \tag{36}$$

one can rewite (34) as

$$g_2(t,s)=[\varepsilon(t)\zeta^T(s)-\alpha(t)D^T(s,t)A^T(s)-\varepsilon(t)E^T(s,t)A^T(s)]R^{-1}. \tag{37}$$

If one differentiates (35) with respect to s, one obtains

$$\begin{aligned}\partial D(s,t)/\partial s&=\int_0^t \partial J(s,s')/\partial s\beta(s')ds'\\&=-\int_0^t J(s,s)A(s)J(s,s')\beta(s')ds'\\&=-J(s,s)A(s)D(s,t)\end{aligned} \tag{38}$$

by using (35) and an identity $\partial J(s,s')/\partial s=-J(s,s)A(s)J(s,s')$ (Nakamori etc., 1977). An initial condition on the partial differential equation (38) for $D(s,t)$ at $s=t$ is $D(t,t)$. $D(t,t)$ is expressed by

$$D(t,t)=\int_0^t J(t,s')\beta(s')ds' \tag{39}$$

from (35). If one differentiates (39) with respect to t, one obtains

$$\begin{aligned}dD(t,t)/dt&=J(t,t)\beta(t)+\int_0^t \partial J(t,s')/\partial t\beta(s')ds'\\&=J(t,t)\beta(t)-J(t,t)A(t)\int_0^t J(t,s')\beta(s')ds'\\&=J(t,t)(\beta(t)-A(t)D(t,t)),\ D(0,0)=0,\end{aligned} \tag{40}$$

from (39) and the identity $\partial J(s,s')/\partial s=-J(s,s)A(s)J(s,s')$ (Nakamori etc., 1977).

If one differentiates (36) with respect to s, one obtains

$$\begin{aligned}\partial E(s,t)/\partial s&=J(s,s)\zeta(s)+\int_t^s \partial J(s,s')/\partial s\zeta(s')ds'\\&=J(s,s)\zeta(s)-J(s,s)\int_t^s A(s)J(s,s')\zeta(s')ds'\\&=J(s,s)(\zeta(s)-A(s)E(s,t)),\ E(t,t)=0,\end{aligned} \tag{41}$$

from (36) and the identity $\partial J(s,s')/\partial s=-J(s,s)A(s)J(s,s')$.

From (4), (15) and (17), the fixed-point smoothing estimate is written as

$$\hat{x}(t,T)=\int_0^t g_1(t,s)\nu(s)ds+\int_t^T g_2(t,s)\nu(s)ds$$

$$=\alpha(t)\int_0^t(\beta^T(s)-q^T(s)A^T(s))R^{-1}\nu(s)ds+$$
$$\varepsilon(t)\int_t^T(\zeta^T(s)-E^T(s,t)A^T(s))R^{-1}\nu(s)ds-$$
$$\alpha(t)\int_t^T D^T(s,t)A^T(s)R^{-1}\nu(s)ds. \tag{42}$$

If one introduces a function $Q(t)$ given by

$$Q(t)=\int_0^t(\beta^T(s)-q^T(s)A^T(s))R^{-1}\nu(s)ds, \tag{43}$$

one can rewrite (42) as

$$\hat{x}(t,T)=\alpha(t)Q(t)+\varepsilon(t)\int_t^T(\zeta^T(s)-E^T(s,t)A^T(s))R^{-1}\nu(s)ds-$$
$$\alpha(t)\int_t^T D^T(s,t)A^T(s)R^{-1}\nu(s)ds. \tag{44}$$

If one differentiates (44) with respect to T and uses the expression for the innovations process of (5), one obtains the partial differential equation (21) for the fixed-point smoothing estimate $\hat{x}(t,T)$. If one differentiates (43) with respect to t, one readily obtains the differential equation (23) for $Q(t)$. An initial condition on $Q(t)$ at $t=0$ is $Q(0)=0$ from (43). One finds that the initial condition on the partial differential equation (21) at $T=t$ is $\hat{x}(t,t)=\alpha(t)Q(t)$, which denotes the filtering estimate of $x(t)$ (Q.E.D.).

5. Existence of smoothing estimate in presented smoother

From (13) and (16), one obtains

$$dq(t)/dt=(B^T(s)-r(t)A^T(t))R^{-1}(\beta(t)-A(t)q(t)),\ q(0)=0. \tag{45}$$

(45) looks like the Riccati type differential equation, which appeared in Nakamori etc.(1978) for the white Gaussian observation noise. The existence and uniqueness for the solution of the Riccati type differential equation (14) are already proved by Nakamori etc.(1978). In Kailath (1976), it is indicated that the solution of a Riccati type nonlinear differential equation exists in filtering problems of linear systems. This differential equation calculates the autocorrelation function of the filtering estimate in the Kalman filter. The existence of the solution is ensured by the following two points. (1) The upper bound for the autocorrelation function of the filtering estimate is the autocorrelation function of the signal, since the filtering error covariance function of the signal is a nonnegative definite matrix. (2) The lower bound for the autocorrelation function of the filtering estimate is a null matrix. In this section, the existence for the solution of (45) is considered by introducing a smoothing error covariance function.

The smoothing error covariance function $P(t,T)$ is defined as

$$P(t,T)=E[(x(t)-\hat{x}(t,T))(x(t)-\hat{x}(t,T))^T], \tag{46}$$

where t is the fixed-point. From an orthogonal projection lemma (46) is written as

$$\begin{aligned} P(t,T)&=E[(x(t)-\hat{x}(t,T))x^T(t)] \\ &=E[x(t)x^T(t)]-E[\hat{x}(t,T)\hat{x}^T(t,T)]. \end{aligned} \tag{47}$$

The smoothing error covariance function $P(t,T)$ and the autocorrelation function of the smoothing estimate $\hat{x}(t,T)$ are nonnegative definite matrices. Then

$$0 \leqq E[\hat{x}(t,T)\hat{x}(t,T)] \leqq E[x(t)x^T(t)]=K_x(t,t). \tag{48}$$

Here, $K_x(t,t)$ represents the autocorrelation function of the signal $x(t)$. Along the discussin by Kailath (1976), it is found that the solution of the Riccati type differential equation (45) exists for the bounded $n \times n$ matrix $K_x(t,t)$.

Now, it is interesting to derive an equation for calculating $P(t,T)$. Substituting (4) into (47), one obtains

$$\begin{aligned} P(t,T)&=K_x(t,t)-\int_0^T\int_0^T g(t,s)E[\nu(s)\nu^T(s')]g^T(t,s')ds'ds \\ &=K_x(t,t)-\int_0^T g(t,s)Rg^T(t,s)ds \end{aligned} \tag{49}$$

by noting that the variance of the innovations process is R. If one differentiates (49) with respect to T and uses the expression for $g(t,T)(=g_2(t,T))$

$$g_2(t,T)=(\varepsilon(t)\zeta^T(T)-\alpha(t)D^T(T,t)A^T(T)-\varepsilon(t)E^T(T,t)A^T(T))R^{-1} \tag{50}$$

from (17), one obtains a partial differential equation

$$\begin{aligned} \partial P(t,T)/\partial T=-&(\varepsilon(t)\zeta^T(T)-\alpha(t)D^T(T,t)A^T(T)-\varepsilon(t)E^T(T,t)A^T(T))R^{-1} \\ &(\zeta(T)\varepsilon^T(t)-A(T)D(T,t)\alpha^T(t)-A(T)E(T,t)\varepsilon^T(t)) \end{aligned} \tag{51}$$

with an intial condition $P(t,0)=K_x(t,t)$.

For the case when $T=t$, the filtering error covariance function satisfies

$$\begin{aligned} dP(t,t)/dt=&F(t)P(t,t)+P(t,t)F^T(t)-\alpha(t)(\beta^T(t)-q^T(t)A^T(t))R^{-1} \\ &(\beta(t)-A(t)q(t))\alpha^T(t),\ P(0,0)=K_x(0,0), \end{aligned} \tag{52}$$

from (15) and (49), provided that the relationship

$$d\alpha(t)/dt=F(t)\alpha(t)\ (or\ \partial K_x(t,s)/\partial t=F(t)K_x(t,s)) \tag{53}$$

is valid. Therefore, the smoothing and filtering error covariance functions are calculated by (51) and (52) respectively. The filtering error covariance function $P(t,t)$ is also used as an initial value instead of $P(t,0)=K_x(t,t)$ in computing the partial differential equation (51) for the smoothing error covariance function $P(t,T)$.

6. Comparison of presented fixed-point smoother with previous one

Let us compare the presented fixed-point smoother with that in Nakamori (1989). The fixed-point smoothing algorithm is summarized in [Theorem 3].

[Theorem 3]

If the autocovariance function $K_x(t,s)$ of $x(t)$ is expressed by $G(t)H^T(s)$ for $0 \leqq s \leqq t$, and an autocovariance function $K_c(t,s)$ of the colored noise $v_c(t)$ by $I(t)L^T(s)$ for $0 \leqq s \leqq t$, the fixed-point smoothing estimate $\hat{x}(t,T)$ is calculated by the following Cauchy system. Here, it is assumed that $G(t)$ and $H(s)$ are bounded $n \times i$ matrices, and $I(t)$ and $L(s)$ are bounded $n \times j$ matrices. Here, it is also assumed that the signal $x(t)$ is uncorrelated with the colored observation noise $v_c(s)$ for $0 \leqq s,t < \infty$.

$\hat{x}(t,T)$: *Fixed-point smoothing estimate*

$$\partial \hat{x}(t,T)/\partial T = h(t,T,T)(y(T)+G(T)f(T)+I(T)S(T)),\ \hat{x}(t,0)=0 \tag{54}$$

Initial condition of $\hat{x}(t,T)$ *at* $T=t$: $\hat{x}(t,t)$

$\hat{x}(t,t)$: *Filtering estimate*

$$\hat{x}(t,t) = -G(t)f(t) \tag{55}$$

$$df(t)/dt = N(t,t)(y(T)+G(T)f(T)+I(T)S(T)),\ f(0)=0 \tag{56}$$

$$dS(T)/dT = M(T,T)(y(T)+G(T)f(T)+I(T)S(T)),\ S(0)=0 \tag{57}$$

$$h(t,T,T) = (H(t)G^T(T) - U(t,T)G^T(T) - W(t,T)I^T(T))R^{-1} \tag{58}$$

$$\partial U(t,T)/\partial T = h(t,T,T)(H(t)+G(T)Y(T)+I(T)b(T)),\ U(t,0)=0 \tag{59}$$

Initial condition of $U(t,T)$ *at* $T=t$: $U(t,t) = -G(t)Y(t)$

$$\partial W(t,T)/\partial T = h(t,T,T)(L(T)+G(T)O(T)+I(T)Z(T)),\ W(t,0)=0 \tag{60}$$

Initial condition of $W(t,T)$ *at* $T=t$: $W(t,t) = -G(t)O(t)$

$$dY(T)/dT = N(T,T)(H(T)+G(T)Y(T)+I(T)b(T)),\ Y(0)=0 \tag{61}$$

$$dO(T)/dT = N(T,T)(L(T)+G(T)O(T)+I(T)Z(T)),\ O(0)=0 \tag{62}$$

$$N(T,T) = (-H^T(T) - Y(T)G^T(T) - O(T)I^T(T))R^{-1} \tag{63}$$

$$db(T)/dT = M(T,T)(H(T)+G(T)Y(T)+I(T)b(T)),\ O(0)=0 \tag{64}$$

$$dZ(T)/dT = M(T,T)(L(T)+G(T)O(T)+I(T)Z(T)),\ Z(0)=0 \tag{65}$$

$$M(T,T) = (-L^T(T) - b(T)G^T(T) - Z(T)I^T(T))R^{-1} \tag{66}$$

The fixed-point smoothing algorithm in [Theorem 3] is derived from a well-known integral equation (Trees, 1968) which an optimal impulse response function $h(t,s,T)$ satisfies.

$$h(t,s,T)R=K_x(t,s)-\int_0^T h(t,s',T)(K_x(s',s)+K_c(s',s))ds' \tag{67}$$

The numbers of the differential equations of the fixed-point smoother in [Theorem 3] are $n(i+j+1)$ for the partial differential equations and $i+j+(i+j)^2$ for the ordinary differential equations. Those of the presented fixed-point smoother are $n+2n'm'$ for the partial differential equations and $(m'+n')(1+n')+n'm'$ for the ordinary differential equations. The number of the ordinary differential equations in the presented fixed-point smoother is less than that in [Theorem 3] by n'^2 when $i=j=m'=n'$, and this fastens a computation time for the optimal fixed-point smoothing estimate. Both smoothers in the current approach and that in [Theorem 3] are optimal in the linear least-squares sense. The presented fixed-point smoother is applicable to the case when the signal $x(t)$ is correlated with the colored noise $v_c(s)$. However, the fixed-point smoother in [Theorem 3] can not be applied to this case from a theoretical point of view.

7. A numerical simulation example

This section shows a numerical simulation example for the presented fixed-point smoother. Signal $x(t)$ is generated by

$$dx(t)/dt=-kx(t)+u(t),\ E[u(t)u(s)]=2kp\delta(t-s),\ E[x(0)^2]=p,\ k=5,\ P=10. \tag{68}$$

The autocovariance function of the signal $x(t)$ is expressed by $K(t,s)=pe^{-k(t-s)}$ (Baggeroer, 1970) for $0\leqq s\leqq t$. Also, the colored noise process is a Wiener process which is generated by

$$dv_c(t)/dt=w(t),\ E[w(t)w(s)]=\sigma^2\delta(t-s),\ E[v_c(0)^2]=0,\ \sigma^2=10. \tag{69}$$

The autocovariance function of the colored noise process is expressed by $K_c(t,s)=\sigma^2 s$ (Baggeroer, 1970) for $0\leqq s\leqq t$. Hence, the autocovariance function of $z(t)(=x(t)+v_c(t))$ is

$$\begin{aligned}K_z(t,s)&=K_x(t,s)+K_c(t,s)\\&=A(t)B^T(s)\\&=[pe^{-kt}\ \sigma^2]\begin{bmatrix}e^{ks}\\ s\end{bmatrix},\ 0\leqq s\leqq t,\end{aligned} \tag{70}$$

since $E[x(t)v_c(s)]=0$. From (70), one finds that $A(t)=[pe^{-kt}\ \sigma^2]$ and $B(s)=[e^{ks}\ s]$. Also, $K_{xy}(t,s)$ is expressed by

$$\begin{aligned}K_{xy}(t,s)&=K_x(t,s)\\&=\alpha(t)\beta(s)\\&=pe^{-kt}e^{ks},\ 0\leqq s\leqq t,\end{aligned} \tag{71}$$

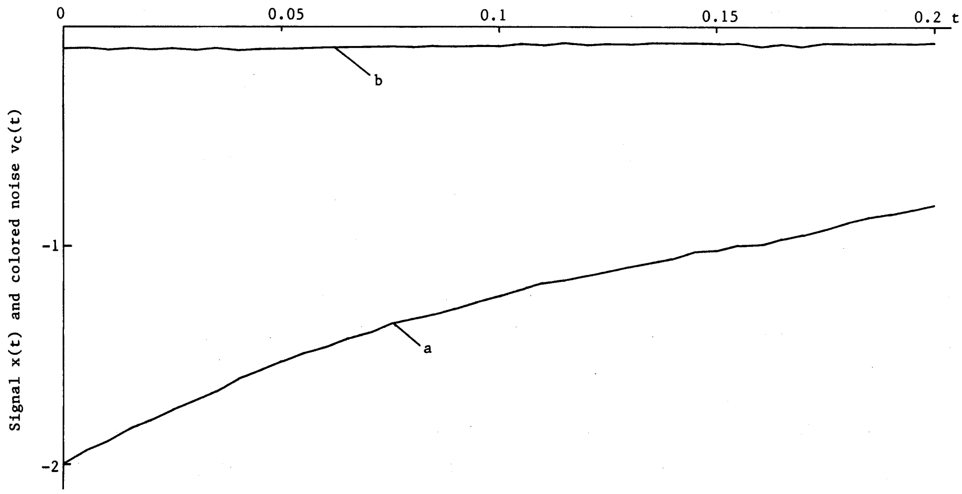

Fig. 1 Signal $x(t)$ and colored noise process $v_c(t)$ vs. t.
a... Signal process $x(t)$.
b... Colored noise process $v_c(t)$ for $v_c(0)=-0.1$ when the colored noise parameter is $\sigma^2=10$.

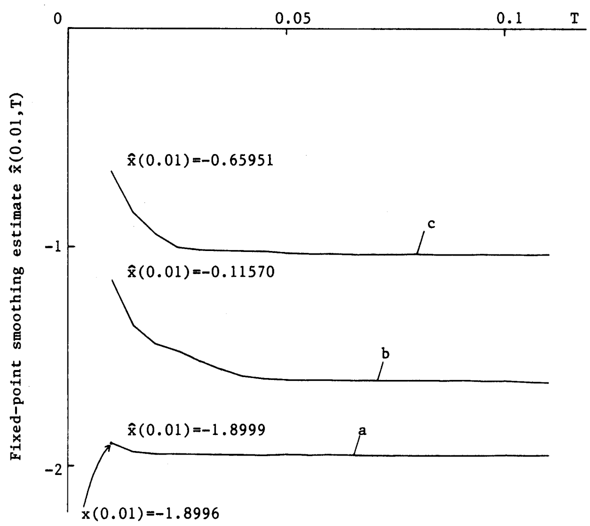

Fig. 2 Fixed-point smoothing estimate $\hat{x}(0.01, T)$ vs. T for $v_c(0)=-0.1$ when the colored noise parameter is $\sigma^2=10$.
a... $\hat{x}(0.01, T)$ for white Gaussian observation noise $N(0, 0.1^2)$.
b... $\hat{x}(0.01, T)$ for white Gaussian observation noise $N(0, 0.3^2)$.
c... $\hat{x}(0.01, T)$ for white Gaussian observation noise $N(0, 0.5^2)$.

since $E[x(t)v_c(s)]=0$ and $E[x(t)v(s)]=0$. Substituting $A(t)$, $B(t)$, $\alpha(t)$, $\beta(t)$, $\varepsilon(t)(=\beta(t))$ and $\zeta(t)(=\alpha(t))$ into [Theorem 2], the fixed-point smoothing estimate $\hat{x}(t,T)$ is calculated. Fig.1 illustrates the signal process (graph a) and the colored noise process (graph b) for $\sigma^2=10$ and the initial value $v_c(0)=-0.1$. Fig.2 illustrates the fixed-point smoothing estimate $\hat{x}(0.01, T)$ vs. T when the initial value of the colored noise is $v_c(0)=-0.1$ and the colored noise parameter is $\sigma^2=10$. Graph a illustrates the fixed-point smoothing estimate $\hat{x}(0.01, T)$ for the white Gaussian observation noise $N(0, 0.1^2)$. Similarly, graphs b and c illustrate $\hat{x}(0.01,\ T)$ for the white Gaussian observation noises $N(0,\ 0.3^2)$ and $N(0,\ 0.5^2)$ respectively. Fig.3 and Fig.4 illustrate the fixed-point smoothing estimates $\hat{x}(0.03,\ T)$ and $\hat{x}(0.05,\ T)$ vs. T respectively when the initial value of the colored noise is $v_c(0)=-0.1$ and the colored noise parameter is $\sigma^2=10$. In Fig.3 and Fig.4, graphs a, b and c depict the fixed-point smoothing estimates for the white Gaussian observation noises $N(0,\ 0.1^2)$, $N(0,\ 0.3^2)$ and $N(0,\ 0.5^2)$. The fixed-point smoothing estimates start with the filtering estimates at their fixed-points. From Fig.3, Fig.4 and Fig.5,

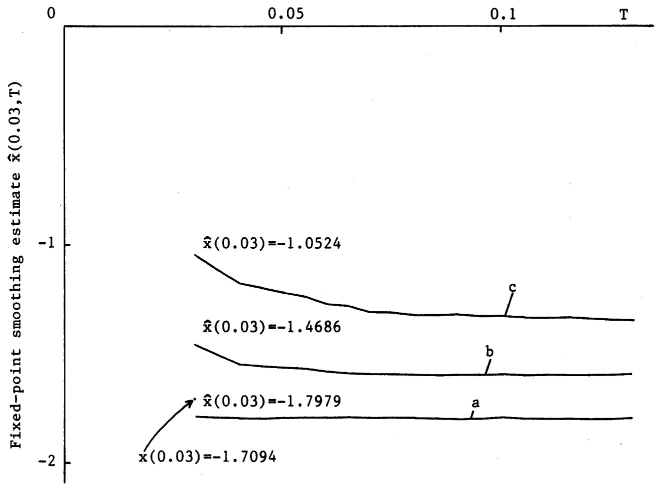

Fig. 3 Fixed-point smoothing estimate $\hat{x}(0.03,\ T)$ vs. T for $v_c(0)=-0.1$ when the colored noise parameter is $\sigma^2=10$.
a... $\hat{x}(0.03,\ T)$ for white Gaussian observation noise $N(0,\ 0.1^2)$.
b... $\hat{x}(0.03,\ T)$ for white Gaussian observation noise $N(0,\ 0.3^2)$.
c... $\hat{x}(0.03,\ T)$ for white Gaussian observation noise $N(0,\ 0.5^2)$.

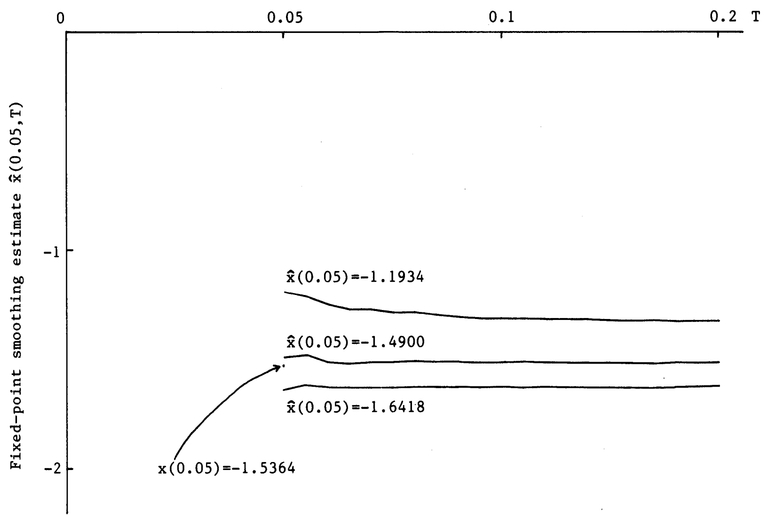

Fig. 4 Fixed-point smoothing estimate $\hat{x}(0.05,\ T)$ vs. T for $v_c(0)=-0.1$ when the colored noise parameter is $\sigma^2=10$.
a... $\hat{x}(0.05,\ T)$ for white Gaussian observation noise $N(0,\ 0.1^2)$.
b... $\hat{x}(0.05,\ T)$ for white Gaussian observation noise $N(0,\ 0.3^2)$.
c... $\hat{x}(0.05,\ T)$ for white Gaussian observation noise $N(0,\ 0.5^2)$.

one finds that the filtering and fixed-point smoothing estimates have a tendency to approach to the true value of the signal $x(t)$ gradually as t and T become large.

8. Conclusions

Some numerical results have shown that the original fixed-point smoothing algorithm proposed in this paper is feasible. Main advantage of the current estimator is that one needs not identify the state-space model in estimating the stochastic signal except the covariance information and the observed value.

REFERENCES

BAGGEROER, B., 1970, State Variables and Communication Theory (MIT Press).
CASTI, J. L., KALABA, R., and MURTHY V. K., 1972, IEEE Trans. Inf. theory, 18, 515.
KAILATH, T., 1968, IEEE Trans. Autom. Control, 13, 645.
KAILATH, T., 1974, IEEE Trans. Inf. Theory, 20, 146.
KAILATH, T., 1976, Lectures on Linear Least-Squares Estimation (Springer-Verlag).
KALMAN, R. E., 1960, Trans. ASME, J. Basic Eng., 82D, 34.
NAKAMORI, S., and SUGISAKA, M., 1977, Automatica, 13, 623.
NAKAMORI, S., and SUGISAKA, M., 1978, Systems and Control, 22, 111.
NAKAMORI, S., and HATAJI, A., 1980, Electronics and Communications in Japan, 63–A, 29.
NAKAMORI, S., 1988, Bulletin of the Faculty of Education, Kagoshima University, Natural Science, 40, 85.
SAGE, A.P., and MELSA, J.L., 1971, Estimation Theory with Applications to Communications and Control (McGraw-Hill).
TREES, H. L., 1968, Detection, Estimation, and Modulation Theory; Part 1 (John Wiley & Sons).