A class of random fractal tessellations in hyperbolic planes

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1 Introduction

In his famous Essay Mandelbrot (1982) has presented various fractal models for the Universe. He and his predecessors have demanded that these models satisfy the two conditions which on the surface are contradictory each other. The one of these is that the mass $M(\rho)$ in a sphere with radius ρ and center at the Earth grows as ρ^D when ρ tends to infinity. Here D is a fraction such that $0 \leq D \leq 3$, which Mandelbrot call the fractal dimension of the Universe. The other condition is that the mass distribution in the Universe satisfies some cosmographic principle, which roughly states that to every observer at any position, the mass distribution has the same appearance. Mandelbrot has found that in order to satisfy both conditions, it is necessary to introduce randomness into fractal models.

Although these models have great values both theoretically and practically, it seems to the present author that they have an unnecessary restriction. Mandelbrot's study and later studies (for these see Falconer (1993)) have confined themselves to fractal models in Euclidean spaces. In Euclidean spaces, among various types of fractals, the most simple are self-similar ones. On the contrary, in hyperbolic spaces, it is impossible to consider similarity. As is well-known, the existence of similar sets is equivalent to the axiom of parallelism (As for the hyperbolic geometry, consult, for example, Fenchel (1989)). How we define fractals in hyperbolic spaces ?

In this paper we present a class of random tessellations in hyperbolic planes, and show that they have a fractal property. To put it more explicitly, we construct random tessellations with unbounded domains which are determined by ultraparallel straight lines. In a special case these tessellations reduce to non-random ones which are composed of mutually congruent domains. Imagine that the mass lies uniformly on lines which are boundaryies of constituent damains of a tessellation, and interiors of these domains are void of the mass. Let $M(\rho)$ be the total mass in a disk with radius ρ and center at some point. Then our main theorem roughly states that the expectation of $M(\rho)$ behaves as $\mathrm{e}^{D\rho}$ as ρ tends to infinity, where D is a fraction such that $0 \leq D \leq 1$. Thus we observe a somewhat peculiar phenomenon that tessellation which is composed of strictly or statistically congruent domains exhibit a fractal behaviour.

In Section 2 we first present the definition of random tessellations with which we concern ourselves throughout the paper. And after preparing several lemmas, we offer a heuristic argument which derives an infinite series that approximates the expectation of $M(\rho)$. In Section 3 we study asymptotic behaviour of this series in a special case. In Section 4, based on the result established in the previous section, we prove our main theorem. Before Section 5, we do not pay any attention to any cosmographic principle. In Section 5 we construct tessellations with a cosmographic principle whose composing domains are statistically congruent. Especially we offer non-random tessellations whose domains are strictlty congruent.

2 Definitions and preliminaries

Random fractal tessellations which we consider in this paper will be constructed by generating ultraparallel lines according to a branching stocahstic process. Thus we introduce a branching stocahstic process on $\{0, 1, 2, \ldots\}$. We represnt a realization of this process by a tree, whose nodes are finite sequences of positive integers $\{1, 2, 3, \ldots\}$. We denote this random tree by $\mathbf{T}$. Now, let $\mathbf{i}$ be a node of $\mathbf{T}$ and let $N_{\mathbf{i}}$ be the number of outgoing edges from the node $\mathbf{i}$. Particularly when $\mathbf{i}$ is the root node of $\mathbf{T}$, we denote this number by $N_{\emptyset}$. We assume that

(A1) all $N_{\mathbf{i}}$ are mutually independent and idetically distributed.

We denote this common probability distribution by $Q = \{q_n : n = 0, 1, 2, \ldots\}$. We allow the possibility that $N_{\mathbf{i}} = 0$, that is, $q_0 > 0$.

Now we go into the realm of the hyperbolic geometry a little while. Let $\mathbf{D}$ be the Poincaré disk and $\partial\mathbf{D}$ be the boundary of $\mathbf{D}$. Furthermore, let

H be the half-plane $\{x+iy : y>0\}$ and $l_\emptyset$ be the line $\{x+iy : y=0\}$. In $\mathbf{D}$, a line represented by a circle which is orthogonal to $\partial\mathbf{D}$. Denote by $l(\alpha,\theta)$ the line whose two points of infinity are $e^{i(\theta+\alpha)}$ and $e^{i(\theta-\alpha)}$. Thus α is the parallel angle at the origin (the center of $\mathbf{D}$). Consider the translation which moves the line $l_\emptyset$ to the line $l(\alpha,\theta)$. There are infinitely many such translations. Out of these we adopt the translation $\phi=\phi(\,\cdot\,;\alpha,\theta)$ whose inverse is expressed as

$$\phi^{-1}(z)=ie^{-i\theta}\,\frac{z-z_0}{1-\overline{z_0}z},\quad z\in\mathbf{D}\ ,$$

where

$$z_0=\frac{1-\sin\alpha}{\cos\alpha}e^{i\theta}\ .$$

In order to state the manner of generating lines explicitly, we introduce a family of probability distributions $\{Q_n : n=1,2,\ldots\}$ where each Q_n is a distribution on $\{(\alpha_1,\theta_1,\ldots,\alpha_n,\theta_n) : 0<\alpha_j<\frac{\pi}{2}, 0<\theta_j<\pi \text{ for every } j\}$. Lines generated according to Q_n lie in the half-plane H. In the following we only consider the case that these generated lines are mutually ultraparallel. Thus we assume that for each n

(A2) the support of Q_n is contained in
$\{(\alpha_1,\theta_1,\ldots,\alpha_n,\theta_n) : 0<\theta_1-\alpha_1<\theta_1+\alpha_1<\cdots<\theta_n-\alpha_n<\theta_n+\alpha_n<\pi\}$.

We turn to define tessellations which are determined by ultraparallel lines. We generate these lines in the following manner :

1. First we generate $N_\emptyset$ lines according to the probability distribution Q and the family of probability distributions $\{Q_n : n=1,2,\ldots\}$. We denote one of the resulting lines by $l(\alpha_{i_1},\theta_{i_1})$.

2. Suppose that a line $l(\alpha_{i_1i_2\ldots i_{k-1}},\theta_{i_1i_2\ldots i_{k-1}})$ has already been generated. Then we generate $N_{i_1i_2\ldots i_{k-1}}$ lines. Then we translate these lines by the translation $\phi(\,\cdot\,;\alpha_{i_1i_2\ldots i_{k-1}},\theta_{i_1i_2\ldots i_{k-1}})$. We denote one of these lines by $l(\alpha_{i_1i_2\ldots i_{k-1}i_k},\theta_{i_1i_2\ldots i_{k-1}i_k})$.

3. We repeat the procedure stated in step 2 indefinitely.

As soon as we have generated infinitely many ultralparallel lines

$$\{l(\alpha_{\mathbf{i}},\theta_{\mathbf{i}}) : \mathbf{i}\in\mathbf{T}\},$$

we obtain a tessellation with unbounded domains.

Now we prepare several lemmas concering lines in the hyperbolic plane.

Lemma 1. If the line $l(\alpha_{i_1 i_2 \dots i_{k-1} i_k}, \theta_{i_1 i_2 \dots i_{k-1} i_k})$ is a translate of a line $l(\alpha_{i_k}, \theta_{i_k})$ by the translation $\phi(\ \cdot\ ; \alpha_{i_1 i_2 \dots i_{k-1} i_k}, \theta_{i_1 i_2 \dots i_{k-1} i_k})$, then

$$\tan \alpha_{i_1 i_2 \dots i_{k-1} i_k} = \frac{\sin \alpha_{i_1 i_2 \dots i_{k-1}} \sin \alpha_{i_k}}{\cos \alpha_{i_k} + \cos \alpha_{i_1 i_2 \dots i_{k-1}} \sin \theta_{i_k}} .$$

Proof Lemma 1. Denote $\alpha_{i_1 i_2 \dots i_{k-1}}, \theta_{i_1 i_2 \dots i_{k-1}}, \alpha_{i_1 i_2 \dots i_{k-1} i_k}, \theta_{i_1 i_2 \dots i_{k-1} i_k}, \alpha_{i_k}$ and θ_{i_k} by $\alpha, \theta, \alpha', \theta', \alpha_0$ and θ_0 respectively. In $\mathbf{D}$ lines $l(\alpha_0, \theta_0)$ and $l(\alpha', \theta')$ are represented by the equations

$$|z|^2 - (\overline{c_0} z + c_0 \overline{z}) + 1 = 0 \quad \text{and} \quad |z|^2 - (\overline{c'} z + c' \overline{z}) + 1 = 0$$

respectively, where

$$c = \frac{1}{\cos \alpha_0} e^{i\theta_0} \quad \text{and} \quad c = \frac{1}{\cos \alpha'} e^{i\theta'} .$$

Then, because $l(\alpha', \theta')$ is a translate of $l(\alpha_0, \theta_0)$ by the translation $\phi(\ \cdot\ ; \alpha, \theta)$, we can derive

$$c' = -ie^{i\theta} \cdot \frac{2ir + c_0 + \overline{c_0}(ir)^2}{1 + \overline{c_0} \cdot ir + c_0 \cdot \overline{ir} + |ir|^2} , \tag{1}$$

where

$$r = \frac{1 - \sin \alpha}{\cos \alpha} . \tag{2}$$

Using (1) and (2), after an elementary calculation, we obtain

$$\frac{1}{\alpha'} = |c'| = \frac{\cos \alpha_0 + \cos \alpha \sin \theta_0}{\sqrt{\cos^2 \alpha \cos^2 \alpha_0 + \sin^2 \alpha + 2 \cos \alpha \cos \alpha_0 \sin \theta_0 + \cos^2 \alpha \sin^2 \theta_0}} .$$

From this it follows that

$$\tan \alpha' = \frac{\sin \alpha_0 \sin \alpha}{\cos \alpha + \cos \alpha_0 \sin \alpha} ,$$

which is the result we have to prove.

Lemma 2. Denote the hyperbolic distance between $l_\emptyset$ and $l(\alpha, \theta)$ by $d(l_\emptyset, l(\alpha, \theta))$. Then

$$\cosh d(l_\emptyset, l(\alpha, \theta)) = \frac{\sin \theta}{\sin \alpha} .$$

Proof of Lemma 2. Let u_1, v_1 be points of infinity of l_1, and u_2, v_2 be those of l_2. Denote the cross ratio of four points u_1, v_1, u_2, v_2 by r. In the hyperbolic geometry it is known that if two lines l_1 and l_2 are ultraparallel, then $\cosh d(l_1, l_2) = \frac{1+r}{|1-r|}$. In order to prove the lemma, it is sufficient to put $u_1 = 1, v_1 = -1, u_2 = e^{i(\theta+\alpha)}$, and $v_2 = e^{i(\theta-\alpha)}$.

Let D_ρ be the disk with radius ρ and with center at the origin, where ρ denotes the hyperbolic distance. Denote the length of a line segment by $m(\cdot)$.

Lemma 3.

$$m\left(l(\alpha,\theta) \cap D_\rho\right) = 2\log\left(\cosh\rho\sin\alpha + \sqrt{\cosh^2\rho\sin^2\alpha - 1}\right) .$$

Proof of Lemma 3. Without loss of generality we suppose that $\theta = 0$. In $\mathbf{D}$ the line $l(\alpha,\theta)$ can be represented by the equation $|z|^2 - (\overline{c}z + c\overline{z}) + 1 = 0$, where $c = 1/\cos\alpha$. Moreover, the circle $C_\rho = \partial D_\rho$ can be represented by an Euclidean circle with center at the origin and radius $r = \tanh\frac{\rho}{2}$. Then, letting two points where $l(\alpha,\theta)$ and C_ρ intersect be $re^{\pm i\omega}$, we have

$$\cos\omega = \frac{1+r^2}{2r}\cos\alpha = \coth\rho\cos\alpha. \tag{3}$$

Now, from the hyperbolic geometry, we borrow the knowledge that for two points z_1 and z_2 in $\mathbf{D}$, the hyperbolic distance between these points is given by

$$\log\frac{1+\left|\dfrac{z_1 - z_2}{1-\overline{z_1}z_2}\right|}{1+\left|\dfrac{z_1 - z_2}{1-\overline{z_1}z_2}\right|}.$$

Then, putting $z_1 = re^{i\omega}$ and $z_2 = re^{-i\omega}$, and substituting (3), we can complete the proof.

In this paper we concern ourself with the total length of the portions of lines $\{l(\alpha_{\mathbf{i}}, \theta_{\mathbf{i}}) : \mathbf{i} \in \mathbf{T}\}$ inside the disk D_ρ, that is,

$$M(\rho) = \sum_{\mathbf{i}\in\mathbf{T}} m\left(l(\alpha_{\mathbf{i}}, \theta_{\mathbf{i}}) \cap D_\rho\right).$$

We are interested in asymptotic behaviour of $\mathrm{E}\left(M(\rho)\right)$ as ρ tends to infinity, where $\mathrm{E}(\cdot)$ denotes the expectation, and particularly in comparison with the area of D_ρ. Now it is known that the area of D_ρ is given by $2\pi(\cosh\rho - 1)$, which grows approximately as $\frac{1}{2}e^\rho$ as ρ tends to infinity. Thus it seems reasonable to investigate asymptotic behaviour of $\log\mathrm{E}\left(M(\rho)\right)$ instead of

$\mathrm{E}\,(M(\rho))$. Define the functions $f(t)$ and $f_0(t)$ as

$$f(t) = \begin{cases} 2\log\left(t + \sqrt{t^2 - 1}\right) & \text{for } t \geq 1, \\ 0 & \text{for } t < 1 \end{cases}$$

and

$$f_0(t) = \begin{cases} \log t & \text{for } t \geq 1, \\ 0 & \text{for } t < 1 \end{cases} .$$

Then, by the usual argument in the calculus, we can show that there is a constant K such that

$$2\, f_0(t) \leq f(t) \leq K\, f_0(t).$$

Thus, if we put

$$M_0(\rho) = \sum_{\mathbf{i} \in \mathbf{T}} f_0(\cosh\rho \sin\alpha_{\mathbf{i}}),$$

we have

$$2\, M_0(\rho) \leq M(\rho) \leq K\, M_0(\rho).$$

Accordingly, it is sufficient to study asymptotic behaviour of $\log \mathrm{E}\,(M_0(\rho))$.

Now we give a following heuristic argument which will be rigorously proved later under appropriate assumptions :

1. From *Lemma* 1 it follows that
$$\tan\alpha_{i_1 i_2 \ldots i_k} \leq \tan\alpha_{i_1 i_2 \ldots i_{k-1}} \cdot \frac{\sin\alpha_{i_k}}{\sin\theta_{i_k}} .$$
2. Accordingly, since $\sin\alpha_{i_k}/\sin\theta_{i_k} < 1$, we can expect $\alpha_{i_1 i_2 \ldots i_k} \to 0$ as $k \to \infty$.
3. Thus, when $k \to \infty$,
$$\sin\alpha_{i_1 i_2 \ldots i_k} \sim \tan\alpha_{i_1 i_2 \ldots i_k} \sim \sin\alpha_{i_1 i_2 \ldots i_{k-1}} \cdot \frac{\sin\alpha_{i_k}}{\cos\alpha_{i_k} + \sin\theta_{i_k}},$$
where the notation " $\sim$ " means "both sides are asymptotically equal".

Based on these observations, in the remainder of this section, we offer a rough estimate for $\mathrm{E}\,(M_0(\rho))$.

Before we set about this task, we prepare some notations. Let $\mathbf{T}_k$ be the set of nodes of $\mathbf{T}$ with length k . Let $\mathcal{F}_0$ be the trivial σ -fields, and given $\mathcal{F}_{k-1}$, define

$$\mathcal{F}_k = \sigma\left(\mathcal{F}_{k-1} \cup \{N_{\mathbf{i}}, l(\alpha_{\mathbf{i}}, \theta_{\mathbf{i}}) : \mathbf{i} \in \mathbf{T}_{k-1}\}\right).$$

Then we may expect

$$\begin{aligned}
&\mathrm{E}\left(\sum_{i_1 i_2 \ldots i_k} f_0\left(\cosh\rho \sin\alpha_{i_1 i_2 \ldots i_k}\right) \mid \mathcal{F}_{k-1}\right) \\
&\sim \mathrm{E}\left(\sum_{i_1 i_2 \ldots i_k} f_0\left(\cosh\rho \sin\alpha_{i_1 i_2 \ldots i_{k-1}} \cdot \frac{\sin\alpha_{i_k}}{\cos\alpha_{i_k} + \sin\theta_{i_k}}\right) \mid \mathcal{F}_{k-1}\right) \\
&= \sum_{i_1 i_2 \ldots i_{k-1}} \mathrm{E}\left(\sum_{j=1}^{N} f_0\left(\cosh\rho \sin\alpha_{i_1 i_2 \ldots i_{k-1}} \cdot \frac{\sin A_j^{(N)}}{\cos A_j^{(N)} + \sin\Theta_j^{(N)}}\right)\right),
\end{aligned}$$

where N is a random variable with probability distribution Q, and when $N = n$, $(A_1^{(n)}, \Theta_1^{(n)}, \ldots, A_n^{(n)}, \Theta_n^{(n)})$ is a random vector with probability distribution Q_n.

Now we introduce a random vector $(\Lambda_1^{(n)}, \ldots, \Lambda_n^{(n)})$ by setting

$$\Lambda_j^{(n)} = \frac{\sin A_j^{(n)}}{\cos A_j^{(n)} + \sin\Theta_j^{(n)}}$$

for $j = 1, \ldots, n$, and denote its probability distribution by P_n. Moreover, we define an operator Λ by

$$(\Lambda f_0)(t) = \mathrm{E}\left(\sum_{j=1}^{N} f_0(t\ \Lambda_j^{(N)})\right).$$

Then we obtain the following

$$\begin{aligned}
&\mathrm{E}\left(\sum_{\mathbf{i}\in\mathbf{T}_k} f_0\left(\cosh\rho \sin\alpha_{\mathbf{i}}\right) | \mathcal{F}_{k-1}\right) \\
&\sim \sum_{\mathbf{i}\in\mathbf{T}_{k-1}} (\Lambda f_0)(\cosh\rho \sin\alpha_{\mathbf{i}}).
\end{aligned} \tag{4}$$

Applying (4) k times, we can get

$$\mathrm{E}\left(\sum_{\mathbf{i}\in\mathbf{T}_k} f_0\left(\cosh\rho \sin\alpha_{\mathbf{i}}\right)\right) \sim (\Lambda^k f_0)(\cosh\rho). \tag{5}$$

Accordingly, by a heuristic argument, we have derived

$$\mathrm{E}\left(M_0(\rho)\right) \sim \sum_{k=0}^{\infty} (\Lambda^k f_0)(\cosh\rho).$$

In the next section we will investigate asymptotic behaviour of this infinite series.

3 Asymptotic behaviour of an approximated expectation of the mass distribution

Let $\{p_j : j = 1, 2, \ldots, m\}$ be positive numbers, $\{\lambda_j : j = 1, 2, \ldots, m\}$ be positive numbers such that $\lambda_j < 1$ $(j = 1, 2, \ldots, m)$, and define an operator Λ by

$$(\Lambda f)(t) = \sum_{j=1}^{m} p_j \; f(\lambda_j t), \tag{6}$$

where

$$f(t) = \begin{cases} \log t & \text{for } t \geq 1 \\ 0 & \text{for } t < 1 \end{cases} . \tag{7}$$

In this section we study asymptotic behaviour of an infinite series

$$F(t) = \sum_{k=0}^{\infty} (\Lambda^k f)(t) \; . \tag{8}$$

as t tends to infinity. In turn, as will be seen later in this section, in order to study asymptotic behaviour of the infinite series (8), we have to know asymptotic behaviour of the following integral

$$(9) \quad I(t) = I(t;c) \;=\; (2\pi)^{-\frac{m-1}{2}} \int_0^{\infty} \cdots \int_0^{\infty} z^c \, \frac{z^{z+\frac{1}{2}}}{\prod_j x_j^{x_j+\frac{1}{2}}} \prod_j p_j^{x_j} \cdot f\left(t \prod_j \lambda_j^{x_j} \right) \prod_j dx_j \; ,$$

where $z = \sum_{j=1}^{m} x_j$, c is a constant and the index j of every product in (9) runs over $\{1, 2, \ldots, m\}$.

In the integal (9) we change variables as

$$\begin{cases} x_j &= \; z \; u_j \qquad\qquad (j = 1, 2, \ldots, m-1) \\ x_m &= \; z \; \left(1 - \sum' u_j\right) \end{cases} ,$$

where the sum $\sum'$ is taken over $\{1, 2, \ldots, m-1\}$. Then, since the Jacobian

$$\frac{\partial \left(x_1, x_2, \ldots, x_m\right)}{\partial \left(z, u_1, \ldots, u_{m-1}\right)} = z^{m-1},$$

we have

$$I(t) \;=\; (2\pi)^{-\frac{m-1}{2}} \int_0^{\infty} dz \int \cdots \int_D \left(\prod_j u_j \right)^{-\frac{1}{2}} \left(\frac{\prod_j p_j^{u_j}}{\prod_j u_j^{u_j}} \right)^{z} \cdot f\left(t \left(\prod_j \lambda_j^{u_j} \right)^{z} \right) \; z^{\frac{m-1}{2}+c} {\prod_j}' du_j \; ,$$

where $u_m = 1 - \sum' u_j$ and

$$D = \{(u_1, u_2, \ldots, u_{m-1}) : \sum' u_j \leq 1\}.$$

Now, using the vector notation $\mathbf{u} = (u_1, u_2, \ldots, u_{m-1})$, we introduce the following functions

$$\begin{aligned} h(\mathbf{u}) &= -\sum_{j=1}^{m} u_j \log u_j, \\ a(\mathbf{u}) &= \sum_{j=1}^{m} a_j u_j, \quad a_j = \log \frac{1}{\lambda_j}, \end{aligned}$$

and

$$b(\mathbf{u}) = \sum_{j=1}^{m} b_j u_j, \quad b_j = \log \frac{1}{p_j} .$$

Then $I(t)$ can be expressed as

$$\begin{aligned} I(t) &= (2\pi)^{-\frac{m-1}{2}} \int_0^\infty dz \int_D \cdots \int \left(\prod_j u_j \right)^{-\frac{1}{2}} d'\mathbf{u} \\ & \quad \int_0^\infty e^{z(h(\mathbf{u})-b(\mathbf{u}))} f\left(te^{-za(\mathbf{u})}\right) z^{\frac{m-1}{2}+c} \, dz , \end{aligned}$$

where $d'\mathbf{u} = \prod'_j du_j$.

Moreover we introduce the functions

$$\mu(\mathbf{u}) = \frac{h(\mathbf{u}) - b(\mathbf{u})}{a(\mathbf{u})}$$

and

$$k(\mathbf{u}) = (2\pi)^{-\frac{m-1}{2}} \left(\prod_j u_j \right)^{-\frac{1}{2}} a(\mathbf{u})^{-\frac{m+1}{2}-c} .$$

Then, after the change of variable as $z = \frac{1}{a(\mathbf{u})} \log \frac{1}{y}$, we have

$$\begin{aligned} (10) \qquad I(t) &= \int_D \cdots \int k(\mathbf{u}) \, d'\mathbf{u} \\ & \quad \cdot \int_0^1 f(ty) \left(\log \frac{1}{y} \right)^{\frac{m-1}{2}+c} \frac{dy}{y^{1+\mu(\mathbf{u})}} . \end{aligned}$$

At this point we prepare several lemmas.

Lemma 1. Let μ and δ be positive real constants, and let

$$g(t) = g(t; \mu, \delta) = \int_0^1 f(tx) \left(\log \frac{1}{x} \right)^{\delta} \frac{dx}{x^{1+\mu}}.$$

Then, as $t \to \infty$,

$$g(t) = t^{\mu} (\log t)^{2+\delta} \cdot \frac{1}{\mu^2 (\log t)^2} \left(1 - (\mu \log t + 1) \mathrm{e}^{-\mu \log t}\right) + \epsilon(t; \mu, \delta),$$

where

$$|\epsilon(t; \mu, \delta)| \le t^{\frac{\mu}{2}} (\log t)^{2+\delta}.$$

Proof of Lemma 1. Changing variable as $tx = y$, we have

$$\begin{aligned} g(t) &= t^{\mu} \int_0^t f(y) (\log t - \log y)^{\delta} \frac{dy}{y^{1+\mu}} \\ &= t^{\mu} \int_1^t \log y (\log t - \log y)^{\delta} \frac{dy}{y^{1+\mu}}. \end{aligned}$$

Again changing variable as $\log y = z \log t$, we have

$$g(t) = t^{\mu} (\log t)^{2+\delta} \int_0^1 z(1-z)^{\delta} \, \mathrm{e}^{-\mu z \log t} \, dz.$$

Then, noting that

$$\int_{\frac{1}{2}}^1 z(1-z)^{\delta} \, \mathrm{e}^{-\mu z \log t} \, dz \le \int_{\frac{1}{2}}^1 z \, \mathrm{e}^{-\mu z \log t} \, dz \le \frac{1}{2} t^{-\frac{\mu}{2}},$$

we get

$$g(t) = t^{\mu} (\log t)^{2+\delta} \int_0^1 z \mathrm{e}^{-\mu z \log t} \, dz + \epsilon(t; \mu, \delta).$$

Since

$$\int_0^1 x \mathrm{e}^{-\nu x} dx = \frac{1}{\nu^2} \left(1 - (\nu + 1) \mathrm{e}^{-\nu}\right),$$

where ν is any positive constant, the proof of lemma is completed.

Lemma 2. In the domain D, the function $\mu(\mathbf{u})$ has the unique maximum μ_{max} at a point $\mathbf{u}_0$. This maximum μ_{max} is the unique root of the equation

$$\sum_{j=1}^{m} p_j \lambda_j^{\mu} = 1,$$

and the point $\mathbf{u}_0$ can be determined by

$$u_j = p_j \lambda_j^{\mu_{max}}.$$

Proof of Lemma 2. Regard the function μ as a function of variables $\tilde{\mathbf{u}} = (u_1, u_2, \ldots, u_m)$ with the constraint $\sum_{j=1}^{m} u_j = 1$, and consider the function

$$\tilde{\mu}(\tilde{\mathbf{u}}) = \mu(\tilde{\mathbf{u}}) - \gamma \cdot \left(\sum_{j=1}^{m} u_j - 1 \right),$$

where γ is a positive constant. Letting $\dfrac{\partial \tilde{\mu}}{\partial u_j} = 0$ for all $j = 1, 2, \ldots, m$, we have

$$(1 + \log u_j + b_j) a(\tilde{\mathbf{u}}) + a_j (h(\tilde{\mathbf{u}}) - b(\tilde{\mathbf{u}})) + \gamma a(\tilde{\mathbf{u}})^2 = 0 \tag{11}$$

Multiplying (11) by u_j and summing over $j = 1, 2, \ldots, m$, we get

$$\gamma = -\frac{1}{a(\tilde{\mathbf{u}})} .$$

Putting this into (11), we can deduce that in the interior of the domain D there exists only one extream point $\tilde{\mathbf{u}}$ which satisfies a system of equations

$$u_j = p_j \lambda_j^{\mu(\tilde{\mathbf{u}})}. \tag{12}$$

Since this extream point lies on the hyperplane $\sum_{j=1}^{m} u_j = 1$, the extream value μ has to satisfy the equation

$$\sum_{j=1}^{m} p_j \lambda_j^{\mu} = 1. \tag{13}$$

It remains to show that this extream value is really the maximum. For this purpose, it is sufficient to prove that at this extream point which satisfies (12), the matrix

$$\left(-\frac{\partial^2 \mu}{\partial u_i \partial u_j} \right)_{1 \le i,j \le m-1}$$

is positive definite.
Derivating the function $\mu(\mathbf{u})$ two times and substituting (12), we have

$$\frac{\partial^2 \mu}{\partial u_i \partial u_j} = -\frac{1}{a(\mathbf{u})} \left(\frac{1}{u_m} + \delta_{ij} \frac{1}{u_i} \right) . \tag{14}$$

where δ_{ij} denotes the Kronecker delta. Then we can easily show that the matrix

$$\left(-\frac{\partial^2 \mu}{\partial u_i \partial u_j} \right)_{1 \le i,j \le m-1}$$

is positive definite. Thus the proof is completed.

Returning to the integral (10), we can rewrite it as

$$I(t) = \int_{D} \cdots \int k(\mathbf{u})\ g(t; \mu(\mathbf{u}), \frac{m-1}{2} + c)\ d'\mathbf{u},$$

where $g(t;\cdot,\cdot)$ is the function introduced in *Lemma* 1.
Decompose the domain D into a domain

$$D_1 = \{\mathbf{u} \in D : \mu(\mathbf{u}) \geq \frac{\mu_{max}}{2}\}$$

and its complement $D \backslash D_1$. Then we easily have the following estimates.

Lemma 3.　For any sufficiently large t,

$$\begin{aligned} I(t) \quad &\leq \quad \frac{4}{\mu_{max}^2}(\log t)^{\frac{m-1}{2}} \int_{D_1} \cdots \int k(\mathbf{u}) t^{\mu(\mathbf{u})} d'\mathbf{u} \\ &\qquad + \mathrm{O}\left(t^{\frac{\mu_{max}}{2}}(\log t)^{\frac{m+3}{2}}\right) \end{aligned}$$

and

$$\begin{aligned} I(t) \quad &\geq \quad \frac{1}{2\mu_{max}^2}(\log t)^{\frac{m-1}{2}} \int_{D_1} \cdots \int k(\mathbf{u}) t^{\mu(\mathbf{u})} d'\mathbf{u} \\ &\qquad - \mathrm{O}\left(t^{\frac{\mu_{max}}{2}}(\log t)^{\frac{m+3}{2}}\right) \end{aligned}$$

Proof of Lemma 3.　Put $\delta = \frac{m-1}{2} + c$. Since

$$\frac{1-(x+1)\mathrm{e}^{-x}}{x^2} \leq \min\left\{\frac{1}{x^2}, \frac{1}{2}\right\} \quad \text{for } x > 0,$$

from *Lemma* 1 it follows that

$$\begin{aligned} I(t) \quad \leq \quad & (\log t)^{2+\delta} \int_{D} \cdots \int k(\mathbf{u})\ t^{\mu(\mathbf{u})} \cdot \frac{1}{\mu(\mathbf{u})^2(\log t)^2}\left(1 - (\mu(\mathbf{u}) \log t + 1)\mathrm{e}^{-\mu(\mathbf{u}) \log t}\right) \\ & + (\log t)^{2+\delta} \int_{D} \cdots \int k(\mathbf{u})\ t^{\frac{\mu(\mathbf{u})}{2}}\ d'\mathbf{u} \\ \leq \quad & (\log t)^{2+\delta} \int_{D_1} \cdots \int k(\mathbf{u})\ t^{\mu(\mathbf{u})} \cdot \frac{1}{\mu(\mathbf{u})^2(\log t)^2}\ d'\mathbf{u} \\ & + (\log t)^{2+\delta} \int_{D \backslash D_1} \cdots \int k(\mathbf{u})\ t^{\mu(\mathbf{u})} \cdot \frac{1}{2}\ d'\mathbf{u} \\ & + t^{\frac{\mu_{max}}{2}} (\log t)^{2+\delta} \int_{D} \cdots \int k(\mathbf{u})\ d'\mathbf{u} \\ \leq \quad & \frac{4}{\mu_{max}^2}(\log t)^{\delta} \int_{D_1} \cdots \int k(\mathbf{u})\ t^{\mu(\mathbf{u})}\ d'\mathbf{u} \end{aligned}$$

$$+\frac{3}{2}t^{\frac{\mu_{max}}{2}}(\log t)^{2+\delta}\int\cdots\int_{D} k(\mathbf{u})\ d'\mathbf{u}.$$

On the other hand, since

$$\frac{1-(x+1)\mathrm{e}^{-x}}{x^2} \geq \frac{1}{2x^2} \text{ for any sufficiently large } x>0,$$

we have

$$\begin{aligned}
I(t) \ &\geq\ (\log t)^{2+\delta}\int\cdots\int_{D} k(\mathbf{u})\ t^{\mu(\mathbf{u})}\cdot\frac{1}{\mu(\mathbf{u})^2(\log t)^2}\left(1-(\mu(\mathbf{u})\log t+1)\mathrm{e}^{-\mu(\mathbf{u})\log t}\right) \\
&\quad -(\log t)^{2+\delta}\int\cdots\int_{D} k(\mathbf{u})\ t^{\frac{\mu(\mathbf{u})}{2}}\ d'\mathbf{u} \\
&\geq\ (\log t)^{2+\delta}\int\cdots\int_{D_1} k(\mathbf{u})\ t^{\mu(\mathbf{u})}\cdot\frac{1}{2\mu(\mathbf{u})^2(\log t)^2}\ d'\mathbf{u} \\
&\quad -t^{\frac{\mu_{max}}{2}}(\log t)^{2+\delta}\int\cdots\int_{D} k(\mathbf{u})\ d'\mathbf{u} \\
&\geq\ \frac{1}{2\mu_{max}^2}(\log t)^{\delta}\int\cdots\int_{D_1} k(\mathbf{u})\ t^{\mu(\mathbf{u})}\ d'\mathbf{u} \\
&\quad -t^{\frac{\mu_{max}}{2}}(\log t)^{2+\delta}\int\cdots\int_{D} k(\mathbf{u})\ d'\mathbf{u}.
\end{aligned}$$

Thus the proof is completed.

Lemma 4. As $t\to\infty$,

$$\int\cdots\int_{D_1} k(\mathbf{u})\ t^{\mu(\mathbf{u})}\ d'\mathbf{u} \sim a(\mathbf{u}_0)^{-1-c}\ t^{\mu_{max}}\ (\log t)^{-\frac{m-1}{2}}$$

Proof of Lemma 4. Denote by $J(t)$ the integral with which we have to concern ourself. Let $\mathbf{u}_0=(u_1^0,u_2^0,\dots,u_{m-1}^0)$ be the point at which the function μ attains its maximum. Obviously $\mathbf{u}_0\in D_1$. Since the function μ is twice continuously differentiable, in a neighbourhood of $\mathbf{u}_0$ it can be expanded as

$$\mu(\mathbf{u})=\mu_{max}-\frac{1}{2}\sum_{i,j=1}^{m-1} t_{ij}(u_i-u_i^0)(u_j-u_j^0)+\cdots,$$

where

$$t_{ij}=-\left(\frac{\partial^2\mu}{\partial u_i\partial u_j}\right)_{\mathbf{u}_0}.$$