Estimation of Multivariate Signal Using Covariance Information In Linear Discrete-Time Systems

Seiichi NAKAMORI

(Received 30 September, 1996)

Abstract

This paper proposes estimation algorithms for the filter and the fixed-point smoother which are suitable for recursive estimation of multivariate signal. The signal is observed with additive white Gaussian noise. The algorithms use the covariance information of the signal and the observation noise.

A numerical simulation example is shown to examine if the algorithms are valid.

1. Introduction

The Kalman filter [1] assumes full knowledge of the state-space model, which generates the signal process, in signal estimation problems. This paper presents an alternative estimation technique that estimates the multivariate signal recursively in terms of updated observed value by extending the recursive Wiener filter [2] in continuous-time systems to the filter and the fixed-point smoother in linear discrete-time systems. The estimators use the information of the system matrix Φ, the observation matrix H and the autovariance function of the state variable, $K_x(k, k)$, for the state-space model of the signal. We show that these quantities are realized from the autocovariance data of the signal generated by the AR model. From Ref. [3] , $K_x(k, k)$ is evaluated by the autocovariance data of the signal of finite number. As a consequence, we are able to estimate the multivariate signal from the knowledge of the autocovariance function of the signal, the variance of the observation noise and the observed value. Also, we show, by appropriate choice of H, that some elements of Φ contain the AR parameters obtained by solving the multivariate Yule-Walker equations for the signal process via the AR model.

Department of Technology (Electrical Technology), Faculty of Education,
Kagoshima University, 1-20-6, Kohrimoto, Kagoshima 890, Japan

2. Least-Squares Estimation Problems in Linear Discrete-Time Systems

Let an observation equation be given by $y(k)=Hx(k)+v(k)$, $z(k)=Hx(k)$, where $y(k)$ is an m-dimensional observed value, H is an $m \times m \cdot n$ observation matrix, $z(k)$ is a signal, $x(k)$ is a zero-mean state-variable and $v(k)$ is white Gaussian observation noise with the variance R as $E[v(k)v^T(s)]=R\delta_D(k-s)$. Here, the symbol "$T$" represents transpose and $\delta_D(k-s)$ the Kronecker Delta function, which satisfies $\delta_D(k-s)=1$ for $k=s$ and $\delta_D(k-s)=0$ for $k \neq s$. It is assumed that $x(k)$ and $v(s)$ are uncorrelated: $E[x(k)v^T(s)]=0$, $0 \leqq s$, $k < \infty$. Let us assume that the fixed-point smoothing estimate $\hat{x}(k, L)$ of $x(k)$ at the fixed-point k is expressed by

$$\hat{x}(k, L) = \sum_{i=1}^{L} h(k, i, L)\, y(i), \tag{1}$$

where $h(k, i, L)$ is referred to as the impulse response function. Minimizing the mean-square value of the fixed-point smoothing error $x(k)-\hat{x}(k, L)$, $J(L)=E\{[x(k)-\hat{x}(k, L)]^T[x(k)-\hat{x}(k, L)]\}$, we obtain the Wiener-Hopf equation [4] :

$$E[x(k)y^T(s)] = \sum_{i=1}^{L} h(k, i, L)\, E[y(i)y^T(s)]. \tag{2}$$

Let $K_{xy}(k, s)$ denote the crosscovariance function of $x(k)$ with $y(s)$ and $K_z(k, s)$ the autocovariance function of $z(k)$. From the statistical assumptions for the signal and observation noise, we optain

$$h(k, s, L)\, \mathrm{R} = K_{xy}(k, s) - \sum_{i=1}^{L} h(k, i, L)\, K_z(i, s) \tag{3}$$

which the optimal impulse response function $h(k, i, L)$ satisfies in linear least-squares smoothing problems. Here, $K_z(k, s)$ is expressed by

$$K_z(k, s) = H\Phi^{k-s}K_{xy}(s, s)\,1(k-s) + K_{xy}^T(k, k)(\Phi^T)^{s-k}H^T 1(s-k),$$
$$\Gamma(s, s) = \Phi K_{xy}(s, s),\; K_{xy}(s, s) = K_x(s, s)\, H^T. \tag{4}$$

Φ represents the stable system matrix of the state-space modal for $x(k)$, $K_x(s, s)$ the autovariance function of $x(k)$ and $1(k-s)$ the unit step function.

3. Recursive Least-Squares Algorithms for Filtering and Fixed-Point Smoothing Estimates

[Theorem 1] shows the recursive least-squares algorithms for the filtering and fixed-point smoothing estimates.

[Theorem 1]

Let the autocovariance function $K_z(k, s)$ be given by (4), let $K_x(k, s)$ be the autocovariance function of $x(k)$ and let the variance of white Gaussian observation noise be R. Then, the recursive least-squares algorithms for the filtering and fixed-point smoothing estimates consist of (5)−(10) in linear discrete-time systems.

Fixed-point smoothing estimate of $x(k)$: $\hat{x}(k, L)$

$$\hat{x}(k, L)=\hat{x}(k, L-1)+h(k, L, L)\,[y(L)-H\Phi\hat{x}(L-1, L-1)] \tag{5}$$

$$h(k, L, L)=[K_x(k, k)\,(\Phi^T)^{L-k}H^T-q(k, L-1)\,\Phi^T H^T]\,[R+HK_x(L, L)H^T- H\Phi B(L-1)\,\Phi^T H^T]^{-1} \tag{6}$$

$$q(k, L)=q(k, L-1)\Phi^T+h(k, L, L)\,H[K_x(L, L)-\Phi B(L-1)\,\Phi^T],\quad q(L, L)=B(L) \tag{7}$$

Filtering estimate of $x(L)$: $\hat{x}(L, L)$

$$\hat{x}(L, L)=\Phi\hat{x}(L-1, L-1)+G(L)\,[y(L)-H\Phi\,\hat{x}(L-1, L-1)],\quad \hat{x}(0, 0)=0 \tag{8}$$

$$B(L)=\Phi B(L-1)\Phi+G(L)\,H[K_x(L, L)-\Phi B(L-1)\,\Phi^T],\quad B(0)=0 \tag{9}$$

$$G(L)=[K_x(L, L)H^T-\Phi B(L-1)\,\Phi^T H^T]\,[R+HK_x(L, L)H^T-H\Phi B(L-1)\,\Phi^T H^T]^{-1} \tag{10}$$

Proof of [Theorem 1] is omitted.

4. Realization Using Autocovariance Data of Signal

Let us estimate H, Φ and $K_x(k, k)$, which are used in [Theorem 1], from the autocovariance data of the signal $z(k)$. We assume the wide-sense stationarity as $K_z(k, s)=K_z(k-s)$ and $K_x(k, s)=K_x(k-s)$ for the autocovariances.

Let the m-dimensional signal $z(k)$ be generated by the multivariate AR model of order n :

$$z(k)=-\sum_{i=1}^{n} a_i z(k-i)+e(k),\ E[e(k)e^T(s)]=\sigma\,\delta_D(k-s). \tag{11}$$

In the linear least-squares estimation problem considered, we place restrictions on the AR model the minimum phase condition. That is, all roots of the characteristic equation det $(I+a_1 z^{-1}+a_2 z^{-2}+\cdots+a_n z^{-n})=0$ must lie inside the unit circle in the z plane [5].

Let the observation matrix in the observation equation $y(k)=z(k)+v(k)$, $z(k)=Hx(k)$, for the state-variable vector $x(k)=[x_1(k)\ x_2(k)\cdots x_{mn}(k)]^T$, be given by $H=[\mathbf{I}\ \ \mathbf{0}]$. Here, H consists of the identity matrix of order m and m×m・(n−1) zero matrix. The AR model is expressed in the state space form of $x(k+1)=\Phi x(k)+\xi(k)$, $\xi(k)=[0\ 0\cdots I]^T$, $E[\zeta(k)\ \zeta^T(k)]=\sigma$, $\zeta(k)=e(k+n)$ with

$$\Phi=\begin{bmatrix} 0 & I & \cdots\cdots & 0 \\ 0 & 0\ I\ 0\cdots & & 0 \\ \cdots & \cdots & \cdots & \\ 0 & \cdots\cdots & 0 & I \\ -a_n & -a_{n-1}\cdots & -a_2 & -a_1 \end{bmatrix}, \tag{12}$$

where I represents m×m identity matrix. It follows from Ref. [3] that $K_x(k,\ k)$ is given by $K_x(k,\ k)=[\Phi K_x^{-1}(k,\ k)\Phi^T]^{-1}$. Hence, $K_x(k,\ k)$ is represented with its matrix elements as $K_x(k,\ k)=E[x(k)\ x^T(k)]$

$$=\begin{bmatrix} K_z(0) & K_z^T(1) & \cdots & K_z^T(n-1) \\ K_z(0) & K_z(0) & \cdots & K_z^T(n-2) \\ \cdots & \cdots & \cdots & \cdots \\ K_z(n-2) & \cdots & K_z(0) & K_z^T(1) \\ K_z(n-1) & K_z(n-2) & \cdots & K_z(0) \end{bmatrix}. \tag{13}$$

The n・m square matrix $K_x(k,\ k)$ is referred to as the Hankel matrix. For the Hankel matrix with rank m・n, m・n dimensional realization for $z(k)$ exists [6]. The AR parameters are calculated by solving the Yule-Walker equations $K_z(1-i)\ a_1^T+K_z(2-i)\ a_2^T+\cdots+K_z(n-1-i)\ a_{n-1}^T+K_z(n-i)\ a_n^T=-K_z(-i)$, $i=1,\ 2,\ \cdots$, n. Henceforce, if we substitute H, Φ *and* $K_x(k,\ k)$ thus evaluated form the autocovariance function of the signal with the variance R of the observation noise and the observed value into [Theorem 1], we can calculate the filtering and fixed-point smoothing estimates as illustrated in section 5.

5. A Digital Simulation Example

Let the signal be generted by the multivariate AR model of order $n=2$ in (11). Here,

$$a_1=\begin{bmatrix}-0.7 & -0.1\\ -0.3 & -0.4\end{bmatrix},\quad a_2=\begin{bmatrix}0.1 & 0\\ -0.1 & -0.1\end{bmatrix},\quad \sigma=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}.$$

$$H=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{bmatrix},\quad x(k)=[x_1(k)\ x_2(k)\ x_3(k)\ x_4(k)]^T,$$

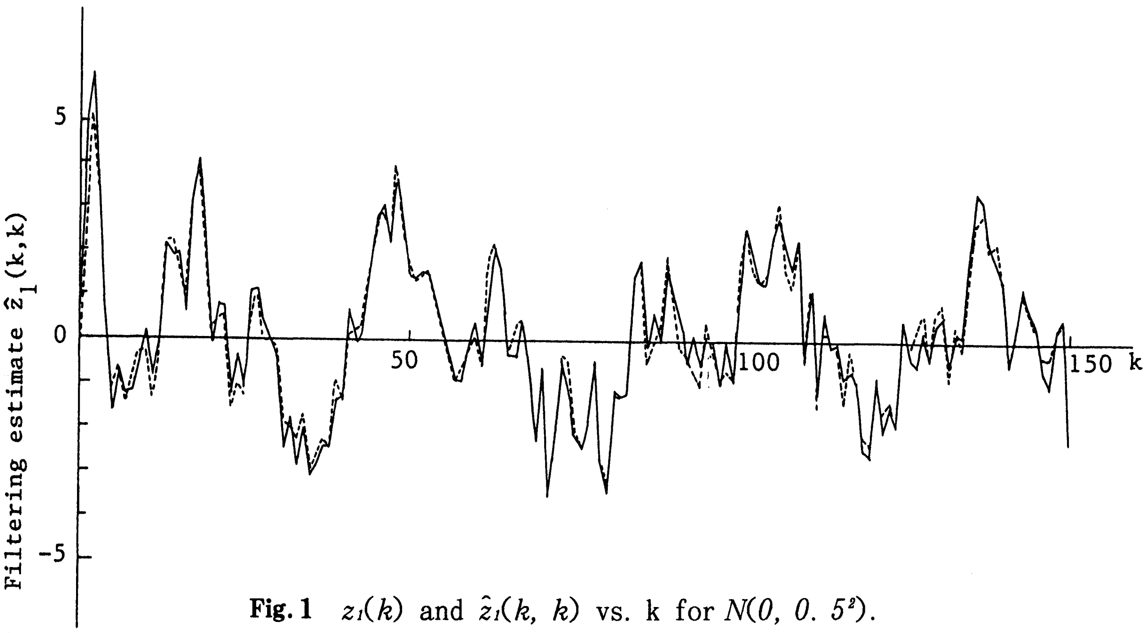

Fig. 1 $z_1(k)$ and $\hat{z}_1(k, k)$ vs. k for $N(0, 0.5^2)$.

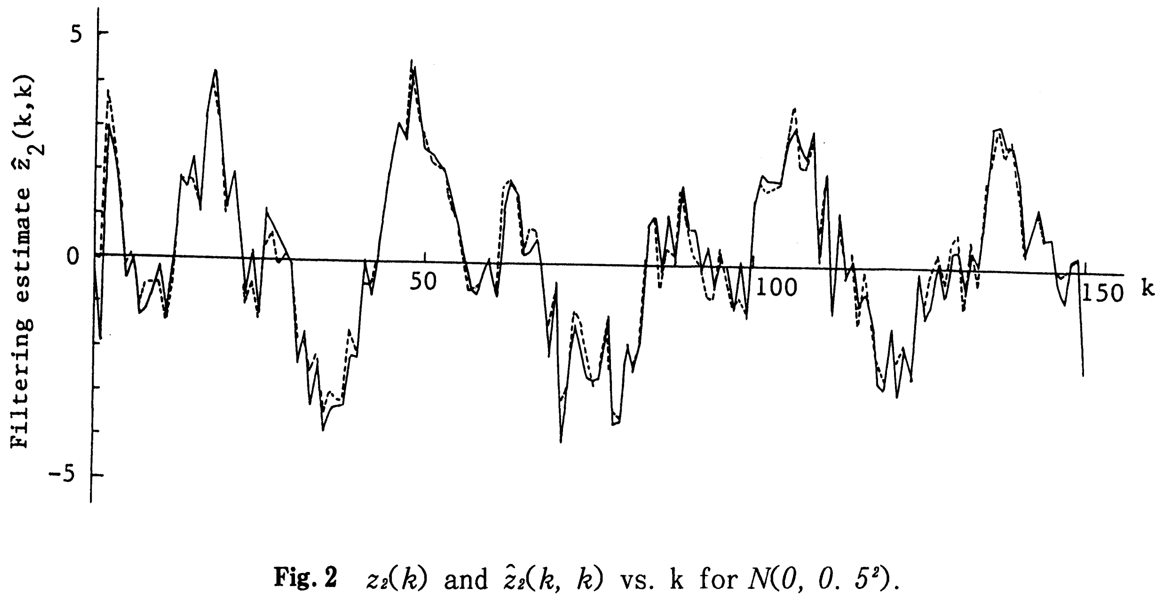

Fig. 2 $z_2(k)$ and $\hat{z}_2(k, k)$ vs. k for $N(0, 0.5^2)$.

$v(k)=[v_1(k)\ \ v_2(k)]^T$. Fig.1 illustrates the signal $z_1(k)$ (*solid line*) and its filtering estimate $\hat{z}_1(k, k)$ (*dotted line*) vs. k for the white Gaussian cbservation noise $N(0,\ 0.5^2)$. Fig.2 illustrates the signal $z_2(k)$ (*solid line*) and its filtering estimate $\hat{z}_2(k, k)$ (*dotted line*) vs. k for the observation noise $N(0,\ 0.5^2)$. Fig.3 illustrates the fixed-point smoothing

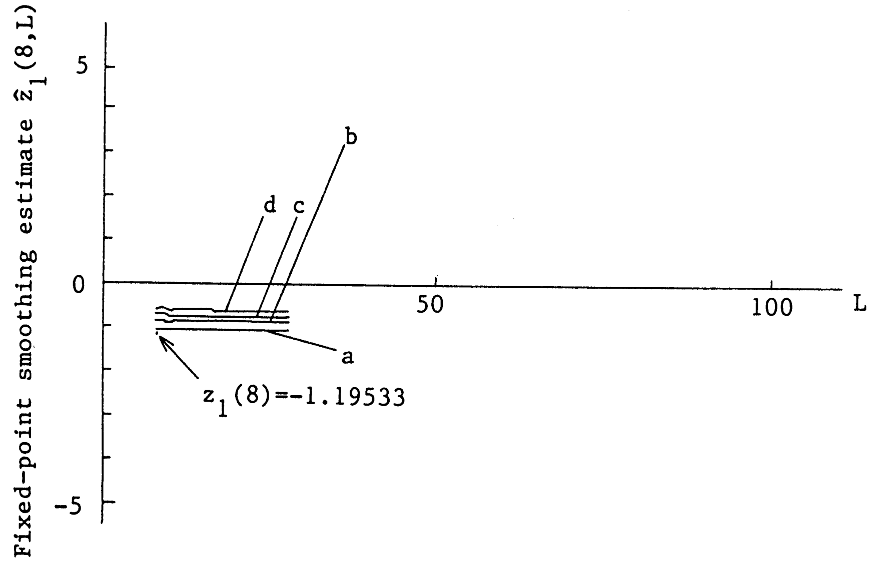

Fig. 3 $\hat{z}_1(8,\ L)$ vs. L, $9 \leqq L \leqq 28$.

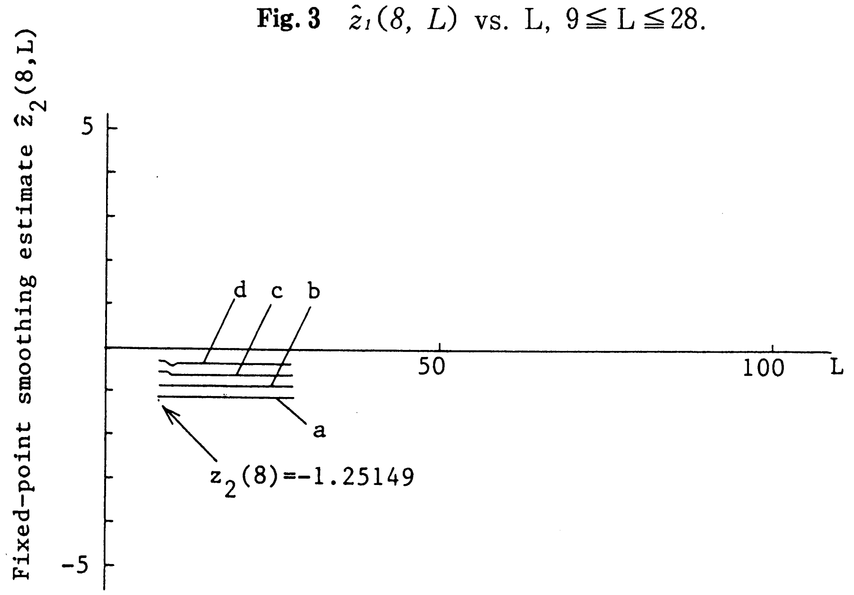

Fig. 4 $\hat{z}_2(8,\ L)$ vs. L, $9 \leqq L \leqq 28$.

estimate $\hat{z}_1(8,\ L)$ of $z_1(8)$ at the fixed-point k=8 vs. $L, 9 \leqq L \leqq 28$. Similarly, Fig.4 illustrates the fixed-point smoothing estimate $\hat{z}_2(8,\ \mathrm{L})$ of $z_2(8)$ *vs.* $L,\ 9 \leqq L \leqq 28$. *Graphs* (*a*), (*b*), (*c*) and (*d*) in Figs.3 and 4 illustrate $\hat{z}_1(8,\ \mathrm{L})$ and $\hat{z}_2(8,\ L)$ for $N(0,\ 0.1^2)$, $N(0,\ 0.3^2)$, $N(0,\ 0.5^2)$ and $N(0,\ 0.7^2)$ respectively. Table 1 shows the mean-square values (*M. S. V.*) of the

White Gaussian observation noise	M.S.V. of filtering error		M.S.V. of fixed-point smoothing error	
	M.S.V. of $z_1(k)-\hat{z}_1(k,k)$	M.S.V. of $z_2(k)-\hat{z}_2(k,k)$	M.S.V. of $z_1(k)-\hat{z}_1(k,k+L)$, $L=1,2,\cdots,20$	M.S.V. of $z_2(k)-\hat{z}_2(k,k+L)$, $L=1,2,\cdots,20$
$N(0,0.1^2)$	8.1293×10^{-3}	9.12001×10^{-3}	7.7911×10^{-3}	8.99787×10^{-3}
$N(0,0.3^2)$	0.0696955	0.0616346	0.0637725	0.0605362
$N(0,0.5^2)$	0.179085	0.155135	0.160711	0.151362
$N(0,7^2)$	0.310622	0.278445	0.274731	0.268542

Table 1 M. S. V. of the filtering and fixed-point smoothing errors.

filtering errors $z_1(k)-\hat{z}_1(k, k)$, $z_2(k)-\hat{z}_2(k, k)$ and the fixed-point smoothing errors $z_1(k)-\hat{z}_1(k, k+L)$, $z_2(k)-\hat{z}_2(k, k+L)$, $1\leqq L\leqq 20$, for $N(0, 0.1^2)$, $N(0, 0.3^2)$, $N(0, 0.5^2)$ and $N(0, 0.7^2)$. The M. S. V. of the filtering errors and the smoothing errors are calculated by

$$\sum_{k=1}^{200}(z_i(k)-\hat{z}_i(k, k))^2/200, \text{ and } \sum_{j=1}^{20}\sum_{k=1}^{200}(z_i(k)-\hat{z}_i(k, k+j))/4000, \quad i=1, 2.$$

Table 1 shows that the M. S. V. for the filtering and fixed-point smoothing errors become small as the noise variance decreases, and the M. S. V. of the smoothing errors are smaller slightly than those of the filtering errors.

6. Conclusions

This paper has proposed the technique, which estimates the multivariate signal in recursive least-squares estimation problems, from the autocovariance data of the signal, the variance of the observation noise and the observed value.

A numerical simulation example has shown that the estimation technique by use of the covariance information is feasible.

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