

An Analysis of Non-Repayable Loans

Shunsuke SHIRAISHI* and Jingkai WANG**

- * First author, Department of Information Management, Faculty of Applied Information Science, Hiroshima Institute of Technology, Hiroshima 7315193, Japan; s.shiraishi.wx@it-hiroshima.ac.jp;
- ** Co-first author and corresponding author, Graduate School of Humanities and Social Sciences, Kagoshima University, Kagoshima 8900065, Japan; kyogaiw@leh.kagoshima-u.ac.jp.

Abstract

When people take out a loan, they believe they can repay it. However, when the loan amount and interest rate are both high, repayment may not be possible even with a substantial fixed payment. This paper illustrates how this situation can occur, using a famous Japanese comic strip (manga) as a case study. The story tells the borrower is told that he can pay off the loan if he makes a 197-month-long repayment, which is not true. We show this by using the present value of perpetual bonds.

Key Words: loan amortization, perpetual bonds, interest, payment

1. Introduction

Loans, a form of debt, are classified into three types: investment-type loans, such as scholarships; fixed-payment loans, such as housing loans; and high-interest consumer loans, such as credit card loans, which have become particularly popular in recent years [5]. Once a borrower takes out a high-interest consumer loan, they can become trapped in a cycle of debt, often culminating in personal bankruptcy. This occurs because, similar to scholarship and mortgage repayments, credit card loans are typically repaid through fixed monthly payments, which can give the borrower a false sense of security. The high interest rate is the critical difference between a credit card loan and a scholarship or mortgage. The interest rate on revolving payments, which is the repayment method of credit card loans, is typically as high as 15%.

Let us quote a scene from the popular Japanese manga “*Gambling Apocalypse: Kaiji*,” which illustrates a predatory lending scheme that traps borrowers [1]. The story revolves around Kaiji, the debtor, and Endo, the loan shark.

Endo: “900 Yen an hour at the convenience store, 9 hours a day, 26 days a month...comes to 216,000 Yen a month. Out of that you could probably afford to pay about 100,000 Yen, if you’re lucky. At that rate, how long do you think it would take to pay us back?”

Kaiji: “Hmm...How long?”

Endo: “100,000 a month on 6.68 million at 18% annuity will take ... 197 months. In other words... 16 years and 5 months.”

While this might seem shocking to Kaiji, the reality is even more horrifying. In the next section, we will explore just how much worse the situation becomes, delving deeper into the depths of despair.

2. The Proof of Impossibility

Endo’s calculation is incorrect. Let’s use the PMT function in Excel to calculate the monthly payment required to repay 6,680,000 over 197 months with equal payments of principal and interest [3] [4]. Note that the annual interest rate of 18% is converted to a monthly interest rate of 1.5% by simple interest calculation. The PMT function of Excel shows

$$\text{PMT}(0.015, 197, 6680000, 0, 0) = -105,834.$$

Therefore, 100,000 yen is insufficient to repay the loan. Consider a perpetual bond with a fixed amount $A = 100,000$ yen. Its present value X is,

$$X = \frac{100,000}{0.015} \approx 6,666,667 < 6,680,000,$$

which will never exceed the borrowed amount of 6,680,000 yen. In other words, the loan cannot be paid off even with perpetual payments of 100,000 yen. As calculated by the PMT function, about 5,000 yen, albeit a small amount, is insufficient to pay off the loan in a finite period (in this case, 197 months). Theoretically, we can show the following facts.

[Theorem 1]

Let the loan amount be X yen, the fixed repayment amount be A yen, and the borrowing rate be r . The necessary and sufficient condition for the loan not to be repaid in full within a finite period is that $X \geq A/r$. In other words, the condition for the loan can be repaid in a finite period is $X < A/r$.

[Proof.]

Suppose $X \geq A/r$. Let S_n be the present value of loan payments over a finite period of n months, where

$$S_n = \frac{A}{(1+r)} + \frac{A}{(1+r)^2} + \cdots + \frac{A}{(1+r)^n} = \frac{A}{r} \left(1 - \frac{1}{(1+r)^n}\right).$$

Therefore, $X \geq A/r > S_n$, indicating that it is impossible to pay off the loan in a finite period.

Conversely, suppose that $X < A/r$. For $\varepsilon = A/r - X > 0$, we have

$$S_n = \frac{A}{r} - \frac{A}{r(1+r)^n} = X + \varepsilon - \frac{A}{r(1+r)^n}.$$

For sufficiently large n , we have

$$\varepsilon - \frac{A}{r(1+r)^n} \geq 0,$$

so, we have $S_n \geq X$. Thus, it is proved that the loan can be paid off in a finite period.

Next, assume that it is impossible to pay off the loan within a finite period. Then for any n , we have $X > S_n$. Thus, we can say that:

$$X \geq \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{A}{r} \left(1 - \frac{1}{(1+r)^n}\right) = \frac{A}{r}.$$

Conversely, assume the loan can be paid off in a finite period. Then for some n , we can say that:

$$S_n \geq X \text{ and } A/r > S_n \geq X.$$

□

We can see the result of the Theorem from the following estimation. If Kaiji borrows 6,680,000 yen and fails to repay the loan for one month, interest of 100,200 yen (= 6,680,000 × 0.015) will be added to the repayment amount. At this point, even if Kaiji repays 100,000 yen, the remaining 200 yen will be added to the principal, so the repayment amount will still be 200 yen short. The shortfall in principal repayment accumulates as follows: it is increased by the interest rate $r = 0.015$ to 203 yen, 206 yen, and 209 yen, respectively. In the final month (197th month), the shortfall amounts to 3,701 yen (= 200 × (1 + 0.015)¹⁹⁶). This is consistent with the fact that, as Wang and

Hata [8] showed, when the beginning balance in period i is X_i , the amount of repayment to the principal follows the formula (1).

$$A - rX_{i+1} = (1 + r)(A - rX_i) \quad (1)$$

Specifically, if $A = 105,834$ yen, the principal repayments will increase by the interest rate $r = 0.015$ to 5,634 yen, 5,718 yen, and 5,804 yen, respectively. In the final month (197th month), the repayment to the principal is 104,270 yen, which means that almost the entire total repayment amount of 105,834 yen is applied to the repayment of the principal.

From equation (1), we can state the following. If $(A - rX_i) > 0$ in a given month i , then $(A - rX_{i+1}) > 0$ in the subsequent months $i + 1$, and the amount of repayment to the principal increases by the factor of $1 + r$, indicating that repayment proceeds smoothly. If $A > rX$, then $(A - rX_1) > 0$, which is consistent with Theorem 1. On the other hand, if $A \leq rX$, then $(A - rX_1) \leq 0$ and $(A - rX_i) \leq 0$, indicating that the borrower is in so-called “debt hell.”

3. Compound Interest Conversion and Simple Interest Conversion

In this paper, the annual interest rate is converted to a monthly interest rate using simple interest. That is, when the annual interest rate is r_{year} , the monthly interest rate is calculated as $r_{month-simple} = r_{year}/12$. In contrast, when converting the annual interest rate to a monthly rate using compound interest, the monthly interest rate, $r_{month-compound}$, must satisfy the equation $(1 + r_{month-compound})^{12} = 1 + r_{year}$ [2] [6] [7]. Therefore,

$$r_{month-compound} = \sqrt[12]{1 + 0.18} - 1 = 0.01388843$$

in the example in this paper. Using this interest rate which is $r_{month-compound}$, the loan can be paid off within a finite period (191 months) if 100,000 yen is repaid per month in this example.

In general, $r_{month-simple} > r_{month-compound}$ is shown as follows. By

$$1 + r_{month-simple} = 1 + r_{year}/12 \text{ and } (1 + r_{month-compound})^{12} = 1 + r_{year},$$

we obtain the following equations:

$$\begin{aligned} \left(1 + \frac{r_{year}}{12}\right)^{12} &= 1 + {}_{12}C_1 \frac{r_{year}}{12} + {}_{12}C_2 \left(\frac{r_{year}}{12}\right)^2 + \dots + \left(\frac{r_{year}}{12}\right)^{12} \\ &> 1 + {}_{12}C_1 \frac{r_{year}}{12} = 1 + r_{year}, \end{aligned}$$

where we use the binomial theorem in the first equation. Therefore, we can conclude that:

$$1 + r_{month-simple} = 1 + r_{year}/12 > (1 + r_{year})^{1/12} = 1 + r_{month-compound}.$$

4. Approximate Repayment Period

In this section, we provide an approximation of the required repayment period. We conduct the approximation using two different methods, yielding the same result. Consider n satisfying the following condition:

$$X = \frac{A}{r} \left(1 - \frac{1}{(1+r)^n} \right),$$

n is the required period for repayment. Thus, we obtain the following equations:

$$(1+r)^n = \frac{A}{A-rX} \text{ and } n \log(1+r) = \log A - \log(A-rX).$$

Here, using the approximate formulas

$$\log(1+r) \approx r \text{ and } \log A - \log(A-rX) \approx \frac{rX}{A-rX},$$

we obtain the following approximate formula for the conclusion:

$$n \approx \frac{X}{A-rX}.$$

On the other hand, using the approximate formula “ $(1+r)^n \approx 1+nr$ ” for the formula as

$$X = \frac{A}{r} \left(1 - \frac{1}{(1+r)^n} \right),$$

we obtain the following approximate formula as

$$A - rX \approx \frac{A}{1 + nr}.$$

Therefore, the approximations

$$1 + nr \approx \frac{A}{A - rX} \text{ and } nr \approx \frac{rX}{A - rX},$$

yield the same conclusion approximation as

$$n \approx \frac{X}{A - rX}.$$

5. Summary

We have shown that in the case of high-interest consumer loans, such as credit card loans, it is sometimes impossible to pay off the loan in full even though a substantially large fixed payment is made. This fact should be a matter of caution, especially for young people likely to use credit card loans in the future.

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