ON THE EMBEDDING PROBLEM FROM THE VIEWPOINT OF DIFFERENTIAL TOPOLOGY

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1. What is the embedding problem?

After the establishment of the concept of manifold, the embedding problem of manifolds has long been one of the main topics in topology.

Definition of immersion Let M^m and N^n be differentiable connected manifolds without boundary, of dimenssions m and n, respectively. M^m is usualy assumed to be compact. A differentiable map $f : M^m \to N^n$ is called an immersion if the rank df_x of its differetial df at x is equal to $m(= \dim M^m)$ for each $x \in M^m$.

Definition of embedding A map $f : M^m \to N^n$ is called an embedding if f is an immersion and if $f : M^m \to f(M^m)$ is a homeomorphism.

Definition of homotopy A map $f : M \to N$ is homotopic to $g : M \to N$ if there exists a continuous map $F : M \times [0,1] \to N$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for each $x \in M$. In other words, there exists a continuous 1-parameter family $\{f_t\}_{t\in[0,1]}$ such that $f_0 = f$ and $f_1 = g$. Both F and $\{f_t\}_{t\in[0,1]}$ are called a homotopy of f to g

Definition of isotopy Two embeddings $f, g : M \to N$ are said to be isotopic if there exists a differentiable map $F : M \times [0,1] \to N$ such that $F(x,0) = f(x)$, $F(x,1) = g(x)$ and $F(\ ,t) : M \to N$ $(x \mapsto F(x,t))$ is an embedding for each $t \in [0,1]$. In other words, there exists a differentiable 1-parameter family of embeddings $\{f_t\}_{t\in[0,1]}$ such that $f_0 = f$ and $f_1 = g$.

Both relations "homotopic" and "isotopic" are equivalence relations.

Existence problem For a continuous (or differentiable) map $f : M^m \to N^n$, does there exist an embedding $g : M^m \to N^n$ homotopic to f ? If $N^n = R^n$, then any map of $M^m \to R^n$ is nullhomotopic. Thus the existence problem is replaced with the embeddability of M^m into R^n.

For an embedding $f : M \to N$, we denote by $\mathrm{Emb}[M,N]_{[f]}$ the set of isotopy classes of embeddings homotopic to f. If $N = R^n$, we use $\mathrm{Emb}[M,R^n]$ instead.

Classification problem (or Enumeration problem) For an embedding $f : M \to N$,
(1) determine the set $\mathrm{Emb}[M,N]_{[f]}$,

(2) characterize each element of $\mathrm{Emb}[M, N]_{[f]}$ by using characteristic classes, numbers and some other topological invariants of M, N and f.

2. From Whitney to Haefliger

H. Whitney proved the following theorem in [20]:

Theorem 2.1 (Whitney). *(1) If $n \geq 2m+1$, then each map $f : M^m \to N^n$ is homotopic to an embedding.*
(2) If $n \geq 2m+2$, then any two embeddings homotopic to $f : M^m \to N^n$ are isotopic.

In the case where $N^n = R^n$, more strict results hold.

Theorem 2.2. (1)(**Whitney**) *If $n \geq 2m$, then there exists an embedding of M^m to R^n.*
(2)(**Wu**) *If $n \geq 2m+1$, then any two embeddings of M^m to R^n are isotopic.*

Both theorems are best-possible in a sense.

Let $n = 2^{r(1)} + \cdots + 2^{r(s)}$, $0 \leq r(1) < \cdots < r(s)$ be the dyadic expansion of n and let $\alpha(n) = s$.

Immersion conjecture For each n-manifold M^n, does there exist an immersion of M^n to $R^{2n-\alpha(n)}$?

Embedding conjecture For each n-manifold M^n, does there exist an embedding of M^n to $R^{2n-\alpha(n)+1}$? The immersion conjecture has been proved affirmatively by R. Cohen[4], while the embedding conjecture may be an obsolete word.

A. Haefliger generalized Whitney's theorem (Theorem 2.1) in two ways [6] and [8].

Theorem 2.3 (Haefliger). *Let $f : M^m \to N^n$ be $(k+1)$-connected, that is, $f_\# : \pi_i(M) \to \pi_i(N)$ is injective for $i \leq k$ and surjective for $i = k+1$.*
(1) If $2n \geq 3(m+1)$ and $n \geq 2m-k$, then f is homotopic to an embedding.
(2) If $2n > 3(m+1)$ and $n > 2m-k$, then any two embeddings homotopic to f are isotopic.

Remark. This theorem for $k = -1$ coincides with Theorem 2.1.

A map $h : X \times X \to Y \times Y$ is called Z_2-equivariant if $h(x_1, x_2) = (y_1, y_2)$ implies $h(x_2, x_1) = (y_2, y_1)$ and ΔX denotes the diagonal $\{(x, x) \mid x \in X\}$ of $X \times X$.

Theorem 2.4 (Haefliger). *Let M^m and N^n be manifolds of dimension m and n, respectively .*
(1) Assume that $2n \geq 3(m+1)$. Then $f : M \to N$ is homotopic to an embedding if and only if there exists a Z_2-equivariant homotopy $F_t : M \times M \to N \times N, (t \in [0,1])$ such that $F_0 = f \times f$ and $F_1^{-1}(\Delta N) = \Delta M$.
(2) Assume that $2n > 3(m+1)$ and two embeddings $f, g : M \to N$ are homotopic, e.g., by a homotopy $\{f_u\}_{u \in [0,1]}$. Then they are isotopic if and only if there exists a 2-parameter family of Z_2-equivariant maps $\{F_{u,v}\}_{u,v \in [0,1]}$ such that $H_{u,0} = f_u \times f_u$ and $H_{u,1}^{-1}(\Delta N) = \Delta M$.

Furthermore, Haefliger[7] considered obstructions for a map $f : M^m \to N^n$ to be homotopic to an embedding. Let $\sum_{i\geq 0} w_i(f) = w(f) = \bar{w}(M^m)f^*(w(N^n)), (\bar{w}(M^m) = w(M^m)^{-1})$ and let $U_{M^m} \in H^m(M^m \times M^m; Z_2)$ and $U_{N^n} \in H^n(N^n \times N^n; Z_2)$ be the Z_2-Thom classes (or the diagonal classes) of M^m and N^n, respectively.

Theorem 2.5 (Haefliger). *If $f : M^m \to N^n$ is homotopic to an embedding, then*
(1) $U_M(1 \times w_{n-m}(f)) + (f \times f)^* U_N = 0$, *and*
(2) $w_i(f) = 0$ *for* $i > n - m$.

Remark. If f is homotopic to an immersion, then (2) holds, thus (1) is essential for embeddings. Haefliger assumed that both M^m and N^n are compact, but the compactness of N^n can be dropped.

These theorems are very important in the study of embddings. However they have not been applied to the embedding problem sufficciently in the cases where N^n is not euclidean spaces. For example, the next theorem[24] could be proved in 1962.

Theorem 2.6 (Yasui). *Each map $f : M^n \to RP^{2n}$ $(n \geq 3)$ is homotopic to an embedding by* Theorem 2.3, *and this is best-posible in the sense that for each integer n, there exists an n-manifold M_0^n and a map $f_{2n-1} : M_0^n \to RP^{2n-1}$ such that f_{2n-1} is not homotopic to an embedding.*

3. After Haefliger's work

In 1970's, Haefliger's two methods were reconsidered by J.-P. Dax[5], H. A. Salomonsen[14] and L. L. Larmore[9], [10]. For an embedding $f : M \to N$, Let

$$\pi_1(\mathrm{Map}(M, N), \mathrm{Emb}(M, N), f) = \mathrm{Emb}[M, N]_f,$$

where $\mathrm{Map}(M, N)$ and $\mathrm{Emb}(M, N)$ denote the spaces of all continuous/differentiable maps and all embeddings of M to N, respectively. For $[f_t] \in \mathrm{Emb}[M, N]_f$ and $[g_t] \in \pi_1(\mathrm{Map}(M, N), f)$, let $[g_t][f_t]$ be the composite of homotopies $[g_t]$ and $[f_t]$. Then the group $\pi_1(\mathrm{Map}(M, N), f)$ acts on $\mathrm{Emb}[M, N]_f$ from the left. It is known that

$$\mathrm{Emb}[M, N]_f / \pi_1(\mathrm{Map}(M, N), f) = \mathrm{Emb}[M, N]_{[f]}$$

They investigated the set $\mathrm{Emb}[M, N]_f$. Roughly speaking, Dax and Salomonsen improved the first method of Haefliger(Theorem 2.3). They approximate a given map by a generic map and consider eliminating its singular point set. So thier method is called the singularity method. Larmore used the second method of Haefliger(Theorem 2.4) to convert the embedding problem into the lifting problem of fibrations. That is why his method is called the classical method.

Dax[5] treated and calculated, e.g.,

$$\mathrm{Emb}[M^m, N^n]_f \otimes Q, \quad \mathrm{Emb}[M^m, R^n] \otimes Q$$
$$\mathrm{Emb}[M^m, S^1 \times R^{n-1}]_f, \quad \mathrm{Emb}[S^m, S^1 \times R^{n-1}]_{[f]},$$

while Larmore[9], [10] determined, e.g.,

$$\mathrm{Emb}[S^n, M^{2n+1}]_f, \quad \mathrm{Emb}[S^n, M^{2n}]_f \text{ for simply connecte } M, \quad \mathrm{Emb}[S^n, RP^{2n}]_f.$$

B.-H. Li and his studens, e.g., P. Zhang [11], [12], [25] followed Dax, and Yasui[23] followed Larmore. They treated or determined, e.g., the following sets:

$$\mathrm{Emb}[M^n, N^{2n+1}]_f, \quad \mathrm{Emb}[S^n, RP^{2n-2}]_f, \quad \mathrm{Emb}[M^n, N^{2n}]_f, \quad \mathrm{Emb}[M^n, RP^{2n}]_f$$
$$\mathrm{Emb}[M^n, CP^n]_f, \quad \mathrm{Emb}[RP^n, CP^n]_{[f]}, \quad \mathrm{Emb}[CP^n, CP^{2n}]_{f]}$$

Unfortunately, both singularity method and classical method are, so far, not successful in the existence problem. In the classical method, one needs the obstruction theory in pair fibrations, while the canonical element to determine the obstruction is uncertain in the singularity method. Thus in both methods, the study of the obstructions to embeddings is impending.

4. Approaches from cobordism theory

After Thom's paper [18], not only manifolds but also maps, immersions and embeddings have been studied from the point of view of cobordism theory.

Definition of cobordism of manifolds Let M_0 and M_1 be n-dimensional compact manifolds. M_0 is cobordant to M_1 if there exists an $(n+1)$-dimensional compact manifold W with boundary $M_0 \cup M_1$ (disjoint union). (W, M_0, M_1) is called a cobordism. One can easily define its oriented version.

Problem For an n-manifold M^n, determine the lowest dimension of euclidean space in which an n-manifold cobordant to M^n is embedded.

As for this problem, R. L. W. Brown[2] showed

Theorem 4.1 (Brown). *Each n-manifold M^n is cobordant to an n-manifold which is embedded in $R^{2n-\alpha(n)+1}$. Further, for each n, there exists an n-manifold M^n such that any manifold cobordant to M^n is not embedded in $R^{2n-\alpha(n)}$.*

I. Takada[17] improved this theorem for orientable manifolds.

Theorem 4.2. *Each orientable n-manifold M^n is cobordant to an n-manifold which is embedded in $R^{2n-\alpha(n)-\min\{\alpha(n),\nu(n)\}}$, where the integer $\nu(n)$ is determined by $n = (2m+1)2^{\nu(n)}$.*

The oriented version of Brown's theorem might not have been discussed.

Definitions of cobordism of maps (1)(Stong's sense) Let $f_i : M_i^m \to N_i^n$ $(i = 0, 1)$ be maps. f_0 is said to be cobordant to f_1 if there exist cobordisms (W, M_0, M_1) and (V, N_0, N_1) and a map $F : W \to V$ such that $F|M_i = f_i$ $(i = 0, 1)$.
(2) Let $f_i : M_i^m \to N^n$ $(i = 0, 1)$ be maps. f_0 is said to be cobordant to f_1 if there exists a cobordism (W, M_0, M_1) and a map $F : W \to N \times [0, 1]$ such that $F|M_i = f_i : M_i \to N \times \{i\} \approx N$ $(i = 0, 1)$.
(3) Let $f_i : M_i^m \to N^n$ $(i = 0, 1)$ be maps. f_0 is said to be cobordant to f_1 if there exists a cobordism (W, M_0, M_1) and a map $F : W \to N$ such that $F|M_i = f_i$.

Existence problem Let $f : M^m \to N^n$ be a map. Does there exist an embedding cobordant to f ?

R. L. W. Brown[3] contributed the existence problem in Stong's sense.

Theorem 4.3 (Brown). *A map $f : M^m \to N^n$ $(n > m)$ is cobordant to an embedding if and only if the following conditions hold:*

(1) *All Stiefe-Whitney numbers of f involving $w_i(f)$ for $i > n-m$ are zero.*

(2) *All Stiefel-Whitney numbers of the type $\langle f^*w_I(N)f^*f_*w_{I_1}(M)\cdots f^*f_*w_{I_{r-1}}(M)w_{I_r}(M),[M]\rangle$ are equal to $\langle f^*w_I(N)w_{I_1}(M)\cdots w_{I_r}(M)w_{n-m}(f)^{r-1},[M]\rangle$.*

Here $w_I(V)$ for $I = (i(1),\ldots,i(p))$ means $w_1(V)^{i(1)}\cdots w_p(V)^{i(p)}$.

For a map $f : M^m \to N^n$ between compact manifolds, let

$$f_* : H^i(M;Z_2) \underset{\cong}{\xrightarrow{\cap[M]}} H_{m-i}(M;Z_2) \xrightarrow{f_*} H_{m-i}(N;Z_2) \underset{\cong}{\xleftarrow{\cap[N]}} H^{n-m+i}(N;Z_2)$$

and let

$$\theta^*(f) = w_{n-m}(f) + f^*f_*(1) \in H^{n-m}(M;Z_2).$$

M. A. Aguilar and G. Pastor[1] determined more explicit conditions in some cases.

Theorem 4.4 (Aguilar and Pastor). *Let $n \geq 2k+1$ $(k = 1,2)$. Then $f : M^n \to N^{2n-k}$ $(k = 1,2)$ is cobordant to an embedding if and only if*

(1) $w_1(M)\theta^*(f) = 0$ *for* $k = 1$,

(2) $w_1(M)w_{n-1}(f) = 0$, $w_1(M)^2\theta^*(f) = w_2(M)\theta^*(f) = 0$ *for* $k = 2$.

A neccessary and sufficient condition that $f : M^n \to N^{2n-3}$ is cobordant to an embedding will be determined by a way similar to [1] but by a tedious calculation. However, the following proposition is easily obtained.

Proposition 4.5 (Yasui). *Let $n \geq 2k+1$. Then $f : M^n \to N^{2n-k}$ is cobordant to an embedding if one of the following conditions is satisfied:*

(1) $w_i(M) = 0$ *for* $1 \leq i \leq k$,

(2) $k = 3$, $w_{n-i}(f) = 0$ $(i = 1,2)$ *and* $\theta^*(f) = 0$,

(3) $4 \leq k \leq 7$, $w_{n-i}(f) = 0$ $(0 < i < k)$, $\theta^*(f) = 0$ *and either* $w_2(M) = 0$ *or* $w_1(M)^2 = 0$.

Definitions of cobordism of embeddings Let $f_i : M_i^m \to N_i^n$ $(i = 0,1)$ be an embeddings.

(1) We denote $f_0 \sim f_1$ if f_0 is cobordant to f_1 in Stong's sense, where the map $F : W \to V$ is also an embedding. Let

$$E\mathfrak{N}(m,n) \overset{\text{def}}{=} \{f \mid f : M^m \to N^n \text{ embedding}\}/\sim.$$

This set is known to be an abelian group.

(2) Assume that $N_0 = N_1 = N$. We denote $f_0 \sim' f_1$ if f_0 is cobordant to f_1, where the map $F : W \to N \times [0,1]$ is an embedding. Let

$$E\mathfrak{N}_m(N^n) = \{f \mid f : M^m \to N^n \text{ embedding}\}/\sim',$$

and $E\Omega_m(N^n)$ its oriented version.

The following theorem is proved by using the Thom-Pontrjagin construction:

Theorem 4.6. (1) $E\mathfrak{N}(m,n) \cong \mathfrak{N}_n(MO(n-m))$, *the bordism group of the Thom space of the universal* $(n-m)$*-plane bundle.*
(2) $E\Omega_m(R^n) = \pi_n(MSO(n-m))$, $E\mathfrak{N}_m(R^n) = \pi_n(MO(n-m))$.

G. Pastor[13], A. Szücs[16], M. Yu Zvagel'skii[26] et al. determined some of the groups in Theorem 4.6 more explicitly, e.g.,

$$E\Omega_n(R^{2n-k})\ (k \le 2), \quad E\Omega_5(R^{5+k}) \text{ for all } k,$$
$$E\mathfrak{N}_m(R^n) \text{ mod finite 2-primary groups.}$$

The embedding problem is still developing, though it is slow, both in the classical approaches and approaches from cobordism theory.

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