Bull. of Yamagata Univ., Nat. Sci., Vol. 11, No. 1, Jan. 1984

Notes on enumerating embeddings of unorientable n-manifolds in Euclidean $(2n-1)$-space for odd n

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(Received August 24, 1983)

Abstract

Denote by $[M \subset R^m]$ the set of isotopy classes of embeddings of an n-manifold M in Euclidean m-space. In topology, the computation of this set is an interesting subject. The set $[M \subset R^{2n-1}]$ has been studied when n is even or M is orientable [15]. Hence, in this article, we shall study the set $[M \subset R^{2n-1}]$ for an n-manifold M for which n is odd and M is unorientable. Further we compute $[P(m, n) \subset R^{2m+4n-1}]$ for the Dold manifold of type (m, n) of dimension $m+2n$, both m and n being odd.

§1. Introduction

Throughout this note, "n-manifold" and "embedding" will mean closed connected differentiable manifold of dimension n and differentiable embedding, respectively. Denote $[M \subset R^m]$ the set of isotopy classes of embeddings of a manifold M in Euclidean m-space R^m. The set $[M \subset R^{2n-1}]$ for an n-manifold M has been investigated when n is even or M is orientable [15]. In this note the set $[M \subset R^{2n-1}]$ is studied for an n-manifold M for which n is odd and M is unorientable, under the following condition :

Condition (1.1). *$H_2(M; Z)$ is isomorphic to a direct sum of some copies of the group of order 2, Z_2.*

Theorem A. *Let n be odd and $n > 6$ and assume that an unorientable n-manifold M satisfies Condition (1.1) above. Then there is a bijection*

$$[M \subset R^{2n-1}] = (1-t^*)[H^{n-1}(M; Z) \otimes H^{n-1}(M; Z)] \times H^{n-2}(M; Z_2)$$
$$\times H_1(M; Z) \times H^{2n-1}(M^*; Z_2)/Sq^2 p_2 H^{2n-3}(M^*; Z[v]),$$

where the map $t: M \times M \to M \times M$ is defined by interchanging factors, the space $M^* = (M \times M - \Delta M)/Z_2$ (ΔM is the diagonal in $M \times M$) is the reduced *symmetric product of M, v is the first Stiefel-Whitney class of the double covering* $M \times M - \Delta M \longrightarrow M^*$, *and* $Z[x]$ *for* $x \in H_1(X; Z_2)$ *is a sheaf of coefficients over X, locally isomorhic to Z, twisted by x. The following information is sufficient to determine* $H^{2n-1}(M^*; Z_2)/Sq^2\tilde{\rho}_2 H^{2n-3}(M^*; Z[v])$:

(i) *the integral cohomology groups* $H^i(M; Z)$ *for* $n-3 \leq i \leq n$,

(ii) *the actions of* Sq^2 *on* $H^i(M; Z_2)$ *for* $i = n-3$, $n-2$,

(iii) *the actions of* $w_1(M)$ *on* $H^i(M; Z_2)$ *for* $i = n-3$, $n-2$.

Remark. If n is odd and $n > 4$, then any n-manifold can be embedded in Euclidean $(2n-1)$-space, cf. [8]. Moreover the group $H^{2n-1}(M^*; Z_2)$ is isomorphic to $H^{n-1}(M; Z_2)$ by [11, § 2].

The Dold manifold $P(m,n)$ of type (m,n) of dimension $m+2n$, both m and n being odd, satisfies the condition (1.1) above (see § 4).

Theorem B. *Assume that both m and n are odd and that* $m+2n > 6$. *Then*

$$\# [P(m,n) \subset R^{2m+4n-1}] = \begin{cases} 8 & \text{if } m \geqq 3, \\ \infty & \text{if } m = 1, \end{cases}$$

where $\# S$ *denotes the cardinality of the set S.*

Remark. For all the other Dold manifolds $P(m,n)$ with $m+2n > 5$, it has been proved that

$\# [P(m,n) \subset R^{2m+4n-1}]$

$= 16$ *if* $n \equiv 3(4)$, *either* $m = 2$ *or* $m \equiv 0(4)$ *and* $m > 0$,

$= 8$ *if* $m \equiv 0(2)$, $n \equiv 1(4)$, $m > 0$ *or if* $m \equiv 2(4)$, $n \equiv 3(4)$, $m \geqq 4$,

$= 4$ *if* $m \geqq 2$, $n \equiv 0(2)$, $n > 0$ *or if* $m \equiv 3(4)$, $n = 0$,

$= 2$ *if* $m = 1$, $n \equiv 0(2)$, $n > 0$ *or if* $m \not\equiv 3(4)$, $m \neq 2^r$, $n = 0$,

$= \infty$ *if* $m = 0$.

In fact $[P(m,n) \subset R^{2m+4n-1}]$ for $m,n > 0$ with $m \equiv 0(2)$ or $n \equiv 0(2)$ has been proved in [15, Proposition 5] and that for $m = 0$ or $n = 0$ has been given, e.g., in [4, Theorem (2.4)], [1, p.299], [8, Theorem 0.1] and [13, Theorem C], because $P(m,0)$ and $P(0,n)$ are the real and the complex projective spaces, respectively.

As for the existence of embeddings of $P(m,n)$ in Euclidean $(2m+4n-1)$-space, we have the following

Theorem C. *Assume that* $m+2n > 4$. *Then the Dold manifold* $P(m,n)$ *of type* (m,n) *is embedded in Euclidean* $(2m+4n-1)$*-space if and only if* $(m,n) \neq (2^r, 0)$ *for* $r \geqq 3$.

This note is essentially a sequel to the paper entitled "Enumerating embeddings of n-manifolds in Euclidean $(2n-1)$-space" [15]. Thus we shall use the same definitions and notations as those of [15].

The remainder of this note is organized as follows: In § 2, the cohomology groups $H^{2n-2}(M^*; Z[v])$ and $\tilde{\rho}_2 H^{2n-3}(M^*; Z[v])$ are calculated for an odd dimensional manifold M satisfying the condition (1.1) above. The proofs of Theorems A, B and C are given in §§ 3-5, respectively.

§ 2. Cohomology of M*

We begin this section by explaining notations.

$Z_r\langle a\rangle$ denotes the cyclic group of order r, Z_r, generated by a ($r \leqq \infty$).

$Z_r[x]$ for $x \in H_1(X; Z_2)$ denotes the sheaf of coefficients over X, locally isomorphic to Z_r, twisted by x ($r \leqq \infty$), and

$$\tilde{\rho}_r : H^i(X; Z_s[x]) \longrightarrow H^i(X; Z_r[x]) \quad (s \leqq \infty,\ s \equiv 0(r)),$$
$$\tilde{\beta}_r : H^{i-1}(X; Z_r[x]) \longrightarrow H^i(X; Z[x]) \quad (r < \infty)$$

denote the reduction mod r and the Bockstein operator, respectively, twisted by x. Then $\tilde{\rho}_r$ and $\tilde{\beta}_r$ for $x = 0$ are the ordinary ones ρ_r and β_r. Moreover the following relations are well-known (e.g.[2] and [10]);

$$(2.1) \qquad \tilde{\rho}_2\tilde{\beta}_2 = Sq^1 + x, \qquad \rho_2\beta_2 = Sq^1.$$

Let M be an unorientable n-manifold and assume that

$$(2.2) \quad \begin{aligned} &H^n(M; Z) = Z_2\langle\tilde{\beta}_2 M'\rangle \quad (Sq^1 M' = M \textit{ is the generator of } H^n(M; Z_2)),\\ &H^m(M; Z) = \sum_{1 \leqq i \leqq \gamma(m)} Z_{r(m,i)}\langle x_{m,i}\rangle \quad (\textit{direct sum}) \textit{ for } m \leqq n-1,\\ &x_{m,i} = \beta_{r(m,i)} y_{m,i} \quad (y_{m,i} \in H^{m-1}(M; Z_{r(m,i)})) \textit{ for } \alpha(m) < i \leqq \gamma(m), \end{aligned}$$

where the order $r(m,i)$ is infinite for $1 \leqq i \leqq \alpha(m)$, a power of 2 for $\alpha(m) < i \leqq \beta(m)$ and a power of odd prime for $\beta(m) < i \leqq \gamma(m)$, and if $\alpha(m) < i < j \leqq \gamma(m)$ then either $(r(m,i), r(m,j)) = 1$ or $r(m,i) \mid r(m,j)$ holds.

For brevity,

(2.2)′ denote $\alpha(m)$, $\beta(m)$, $\gamma(m)$, $r(m,i)$, $x_{m,i}$ and $y_{m,i}$ in (2.2), respectively, by

α, β, γ, $r(i)$, x_i and y_i	when $m = n-1$,
α', β', γ', $r'(i)$, x_i' and y_i'	when $m = n-2$,
α'', β'', γ'', $r''(i)$, x_i'' and y_i''	when $m = n-3$.

If an n-manifold M satisfies the condition (1.1) above, then so does $H^{n-2}(M;$

$Z[w_1(M)])$ by Poincaré duality, and it is expressed in the form

$$(2.3) \qquad H^{n-2}(M\,;Z[w_1(M)]) = \sum_{1\leq i\leq a} Z_2<\tilde{\beta}_2 z_i> \quad (z_i \in H^{n-3}(M\,;Z_2)).$$

Theorem 2.4. *Let n be odd and $n > 4$. If M is an unorientable n-manifold satisfying the condition* (2.3), *then*

(1) *there exists a short exact sequence*

$$0 \longrightarrow [H^{n-2}(M\,;Z_2) + \sigma(H^{n-1}(M\,;Z) \otimes H^{n-1}(M\,;Z))] \longrightarrow H^{2n-2}(M^*\,;Z[v]) \longrightarrow H_1(M\,;Z) \longrightarrow 0,$$

and

(2) $$\tilde{\rho}_2 H^{2n-3}(M^*\,;Z[v]) = \sum_{1\leq j\leq 5} H_j' + \sum_{1\leq i\leq a} Z_2 <\rho(u\otimes((Sq^1+w_1(M))z_i)^2 + U(1\otimes z_i))>,$$

where $\sigma = 1 - t^*$, t being defined by interchanging factors, and

$$H_1' = \{\rho\sigma(\rho_2 x\otimes M) \mid x\in H^{n-3}(M\,;Z)\},$$

$$H_2' = \{\rho\sigma(\rho_2 x\otimes \rho_2 y) \mid x\in H^{n-2}(M\,;Z),\ y\in H^{n-1}(M\,;Z)\},$$

$$H_3' = \sum_{a<i<j\leq\beta} Z_2<\rho\sigma(\rho_2 x_i\otimes\rho_2 y_j + (r(j)/r(i))\rho_2 y_i\otimes\rho_2 x_j)>,$$

$$H_4' = \sum_{a'<k\leq\beta'} Z_2<\rho\sigma(\rho_2 y_k'\otimes M + Sq^1\rho_2 y_k'\otimes M')>,$$

$$H_5' = \sum_{a<i\leq\beta} Z_2<\rho\sigma(\rho_2 y_i\otimes\rho_2 x_i)>.$$

(Here the description of the elements in $H^*(M^*\,;Z_2)$ is due to [11, § 2].)

Proof. The Z_2-action on $M\times M$, defined via the map t, determines two quotient spaces

$$\Lambda^2 M = (M\times M)/Z_2,\ \Delta M = (\Delta M)/Z_2.$$

Hence $\Lambda^2 M - \Delta M = M^*$ holds and there is an exact sequence (e, g., [15, Lemma 1.3])

$$(2.5) \qquad \cdots \longrightarrow H^{2n-3}(\Lambda^2 M, \Delta M\,;Z[v]) \xrightarrow{i^*} H^{2n-3}(M^*\,;Z[v]) \xrightarrow{j^*} H^{2n-3}(PM\,;Z[v]) \xrightarrow{\delta} H^{2n-2}(\Lambda^2 M,\Delta M\,;Z[v]) \xrightarrow{i^*} H^{2n-2}(M^*\,;Z[v]) \xrightarrow{j^*} H^{2n-2}(PM\,;Z[v]) \longrightarrow \cdots,$$

where PM is the tangent projective bundle over M and $j^*v = v$, j being an embedding $PM\subset M^*$ (see [14, § 1]).

By [9, 9.2 Proposition and its proof], there are isomorphisms

(2.6) $$\theta : H^{n-1}(M ; Z[w_1(M)]) + H^n(M ; Z_2) \xrightarrow{\cong} H^{2n-2}(PM ; Z[v]),$$
$$\theta' : H^{n-2}(M ; Z[w_1(M)]) + H^{n-1}(M ; Z_2) \xrightarrow{\cong} H^{2n-3}(PM ; Z[v]),$$
$$\theta'(x) = \bar{\beta}_2(v^{n-3}x) \qquad \text{for } x \in H^{n-1}(M ; Z_2),$$
$$\bar{\rho}_2\theta'(y) = (\sum_{0\leqq i\leqq 2} v^{n-i-1}w_i(M))\bar{\rho}_2(y)^{*)} \qquad \text{for } y \in H^{n-2}(M ; Z[w_1(M)]).$$

For any $z \in H^{n-3}(M ; Z_2)$, we see easily

$$\bar{\rho}_2 j^*\bar{\beta}_2\rho(u^2 \otimes z^2) = \bar{\rho}_2\bar{\beta}_2 j^*\rho(u^2 \otimes z^2),$$
$$= (Sq^1 + v)(\sum_{0\leqq i\leqq 3} v^{n-i-1}Sq^i z) \qquad \text{by (2.1) and [11, § 2]},$$
$$= (v^{n-1} + v^{n-2}w_1(M) + v^{n-3}w_2(M))(w_1(M) + Sq^1)z \quad \text{by [13, (2.5)]},$$
$$= \bar{\rho}_2\theta'\bar{\beta}_2(z) \qquad \text{by (2.6) and (2.1)}.$$

Since $\bar{\rho}_2$ is a monomorphism on $H^{2n-3}(PM ; Z[v])$ by (2.6) and (2.3), we have

(2.7) $$j^*\bar{\beta}_2\rho(u^2 \otimes z_i^2) = \theta'\bar{\beta}_2(z_i) \qquad \text{for } 1 \leqq i \leqq a.$$

On the other hand, we have

(2.8) $$\delta\theta'(x) = \bar{\beta}_2(v^{n-2}\Lambda x) \qquad \text{for } x \in H^{n-1}(M ; Z_2)$$

by (2.6) and [14, Lemma 1.5]. Therefore it follows from (2.7–8), together with [14, Lemma 3.2(4) and Proposition 5.4], that

(2.9) $$0 \to G_1 + G_3 + G_7 \xrightarrow{i^*} H^{2n-2}(M^* ; Z[v]) \xrightarrow{\theta^{-1}j^*} H^{n-1}(M ; Z[w_1(M)]) \longrightarrow 0$$

is an exact sequence and that $j^* : H^{2n-3}(M^* ; Z[v]) \longrightarrow j^*H^{2n-3}(M^* ; Z[v])$ is a split epimorphism. Hence and from [14, Proposition 5.5] and [15, Lemmas 2.8–9], it follows that

(2.10) $$\bar{\rho}_2 H^{2n-3}(M^* ; Z[v]) = \sum_{1\leqq j\leqq 5} i^*H_j + \sum_{1\leqq i\leqq a} Z_2\langle\rho(u \otimes ((Sq^1 + w_1(M))z_i)^2 + U(1 \otimes z_i))\rangle,$$

Here G_i ($i = 1,3,7$) and H_j ($j = 1,\cdots,5$) are the same as those in [14, Propositions 5.4–5]. By the definitions of G_i and H_j [14] and by [15, Lemma 3.3], we see easily that there are isomorphisms

$$G_1 + G_3 \xrightarrow{\cong} (1 - t^*)(H^{n-1}(M ; Z) \otimes H^{n-1}(M ; Z)),$$
$$G_7 \xrightarrow{\cong} H^{n-2}(M ; Z_2),$$

*) This relation is different from that of Rigdon [9], but his relation can be modified in such a way as was stated in (2.6)

and equalities

$$i^*H_j = H_j' \qquad \text{for } 1 \leqq j \leqq 5.$$

Hence and from the fact that $H^{n-1}(M; Z[w_1(M)])$ is isomorphic to $H_1(M; Z)$ by Poincaré duality, the theorem follows.

§ 3. Proof of Theorem A.

The following proposition plays an important role for proving Theorem A.

Proposition 3.1 (e.g., [13, Proposition 10.1]). *Let n be odd and $n > 6$. Then for an n-manifold M, there is a bijection*

$$[M \subset R^{2n-1}] = H^{2n-2}(M^*; Z[v]) \times H^{2n-1}(M^*; Z_2)/Sq^2\tilde{\rho}_2 H^{2n-3}(M^*; Z[v]).$$

Because for an n-manifold M satisfying the condition (2.3), the group $H^{2n-2}(M^*; Z[v])$ is given in Theorem 2.4(1), it is sufficient to study Coker $Sq^2\tilde{\rho}_2$. We now recall the following fact.

Lemma 3.2 (Thomas [11, Proposition 2.9]). *There is an isomorphism*

$$\rho\sigma(\otimes M) : H^{n-1}(M; Z_2) \xrightarrow{\cong} H^{2n-1}(M^*; Z_2),$$

defined by

$$\rho\sigma(\otimes M)(x) = \rho\sigma(x \otimes M) \qquad \textit{for } x \in H^{n-1}(M; Z_2).$$

By the argument similar to that used in proving [15, Lemma 7.2], we have the following lemma.

Lemma 3.3. *Let n be odd and $n > 6$ and assume that M is an unorientable n-manifold satisfying* (2.3) *above. Then* Im $Sq^2\tilde{\rho}_2$ *is a Z_2-vector space generated by the elements listed below*:

(1) $\rho\sigma(Sq^2\rho_2 x \otimes M)$ *for* $x \in H^{n-3}(M; Z)$,
(2) $\rho\sigma(Sq^2\rho_2 x \otimes \rho_2 y)$ *for* $x \in H^{n-2}(M; Z)$, $y \in H^{n-1}(M; Z)$,
(3) $\rho\sigma(\rho_2 x_i \otimes Sq^2\rho_2 y_j + (r(j)/r(i))Sq^2\rho_2 y_i \otimes \rho_2 x_j)$ *for* $\alpha < i < j \leqq \beta$,
(4) $\rho\sigma(Sq^2\rho_2 y_k' \otimes M + Sq^2 Sq^1 \rho_2 y_k' \otimes M')$ *for* $\alpha' < k \leqq \beta'$,
(5) $\rho\sigma(Sq^2\rho_2 y_i \otimes \rho_2 x_i)$ *for* $\alpha < i \leqq \beta$,
(6) $\rho\sigma(w_1(M) Sq^1 z_i \otimes M)$ *for* $1 \leqq i \leqq a$,

where $\sigma = 1 + t^*$.

First notice that the Z_2-bese of $H^i(M; Z_2)$ and the action of Sq^1 on it are completely determined by the integral cohomology structure (2.2). In particular

$$S=\{\rho_2 x_i'',\ \rho_2 y_j' \mid 1\leqq i\leqq\beta'',\ \alpha'<j\leqq\beta'\}$$

is a Z_2-base of $H^{n-3}(M;Z_2)$. Thus if z_i can be expressed explicitly as a linear combination of the elements in S and if the action of $w_1(M)$ on $H^{n-2}(M;Z_2)$ is given, then $w_1(M)Sq^1z_i$ $(1\leqq i\leqq a)$ in Lemma 3.3 (6) can be expressed explicitly as a linear combination of the elements of the base of $H^{n-1}(M;Z_2)$. Here, by (2.1) and (2.3), we have

$$\begin{aligned}&\operatorname{Ker}\tilde{\beta}_2=\operatorname{Ker}(Sq^1+w_1(M)) \qquad \text{in } H^{n-3}(M;Z_2),\\ &a=\beta''+\beta'-\alpha'-\dim_{Z_2}\operatorname{Ker}(Sq^1+w_1(M)).\end{aligned}$$

If the action of $w_1(M)$ on $H^{n-3}(M;Z_2)$ is given, then we can choose the elements z_i $(1\leqq i\leqq a)$, each of which is expressed explicitly as a linear combination of elements in S above, so as to satisfy (2.3). Therefore if the actions of $w_1(M)$ on $H^i(M;Z_2)$ for $i=n-2,\ n-3$ are given explicitly, then $w_1(M)Sq^1z_i$ $(1\leqq i\leqq a)$ can be determined completely. Hence and by the argument similar to that used in proving [15, Corollary 7.3], we have the following

Corollary 3.4. *Let n be odd and $n\geqq 7$ and let M be an unorientable n-manifold whose integral cohomology groups $H^i(M;Z)$ for $n-3\leqq i\leqq n$ are given as in (2.2). Moreover assume that $H_2(M;Z)$ satisfies the condition (1.1) above. Then the following information is sufficient to determine* Im $Sq^2\tilde{\beta}_2$:

(i) *the actions of Sq^2 on $H^i(M;Z_2)$ for $i=n-3,\ n-2$,*

(ii) *the actions of $w_1(M)$ on $H^i(M;Z_2)$ for $i=n-3,\ n-2$.*

Remark. The actions of Sq^2 on $H^i(M;Z_2)$ for $i=n-3,\ n-2$ are given, e.g., by [1, pp. 273–4] as follows:

$$\begin{aligned}&Sq^2x=(w_2(M)+w_1(M)^2)x &&\text{for } x\in H^{n-2}(M;Z_2),\\ &Sq^2x=(w_2(M)+w_1(M)^2+w_1(M)Sq^1)x &&\text{for } x\in H^{n-3}(M;Z_2).\end{aligned}$$

Hence we can replace the information (i) in Corollary 3.4 by

(i)′ *the actions of $w_2(M)$ on $H^i(M;Z_2)$ for $i=n-3,\ n-2$.*

Theorem A follows from Proposition 3.1, Theorem 2.4 (1) and Corollary 3.4.

§ 4. Proof of Theorem B

The Dold manifold $P(m,n)$ of type (m,n), introduced by Dold [3], is the quotient space obtained from $S^m\times CP^n$ by identifying (x,z) with $(-x,\bar{z})$, S^m and CP^n being the usual m-sphere and the complex projective space of complex dimension

n. In [3], $P(m,n)$ is given a cell-decomposition with k-cell (C_i, D_j) for each pair (i,j), $i, j \geqq 0$, for which $i+2j=k \leqq m+2n$, and the boundary operator satisfies

$$(4.1) \qquad \partial(C_i, D_j) = \begin{cases} (1+(-1)^{i+j})(C_{i-1}, D_j) & \text{for } i>0, \\ 0 & \text{for } i=0. \end{cases}$$

Let C^iD^j denote the cochain which assigns 1 to (C_i, D_j) and 0 to all the other $(i+2j)$-cells or its integral cohomology class if it is a cocycle, and let c^id^j denote the mod 2 cohomology class defined by the cochain C^iD^j. Then it has been shown in [3] that

$$(4.2) \qquad \begin{aligned} &H^*(P(m,n);Z_2) = Z_2[c]/(c^{m+1}) \otimes Z_2[d]/(d^{n+1}), \\ &Sq^1 d = cd, \\ &\sum_{i\geqq 0} w_i(P(m,n)) = (1+c)^m(1+c+d)^{n+1}, \end{aligned}$$

where

$$c = c^1d^0, \qquad d = c^0d^1.$$

In particular we have

$$(4.3) \qquad \begin{aligned} &Sq^1 d^j = jcd^{j+1}, \ Sq^2 d^j = jd^{j+1} + \binom{j}{2} c^2 d^j, \\ &w_1(P(m,n)) = (m+n+1)c. \end{aligned}$$

In the rest of this section, assume that $m+2n>6$, $m \equiv 1\,(2)$ and $n \equiv 1\,(2)$. The last two are equivalent to the condition

$$w_1(P(m,n)) = c \neq 0, \qquad \dim P(m,n) = m+2n \equiv 1\,(2).$$

Under this assumption, the cohomology groups of $P(m,n)$ can easily be calculated by using (4.1−3) and the Bockstein exact sequence.

Lemma 4.4. *Let $n \equiv 1\,(2)$ and $m \equiv 1\,(2)$. Then*

$$\begin{aligned} &H^{m+2n-1}(P(m,n);Z) = 0, \\ &H^{m+2n-1}(P(m,n);Z[c]) = \begin{cases} Z_2 & \text{if } m \geqq 3, \\ Z & \text{if } m=1, \end{cases} \\ &H^{m+2n-1}(P(m,n);Z_2) = Z_2\langle c^{m-1}d^n\rangle, \\ &H^{m+2n-2}(P(m,n);Z) \ni \beta_2(c^{m-3}d^n) \quad \text{if } m \geqq 3, \\ &H^{m+2n-2}(P(m,n);Z[c]) = Z_2, \\ &H^{m+2n-2}(P(m,n);Z_2) = Z_2 + Z_2 \quad \text{if } m \geqq 3. \end{aligned}$$

From Theorem 2.4 and Lemma 4.4, it follows that

$$(4.5) \qquad \# H^{2m+4n-2}(P(m,n)^*;Z[v]) = \begin{cases} 8 & \text{if } n \text{ is odd and } m \geqq 3, \\ \infty & \text{if } n \text{ is odd and } m=1, \end{cases}$$

and that

$$\rho\sigma(c^{m-3}d^n \otimes c^m d^n + c^{m-2}d^n \otimes c^{m-1}d^n) \in \bar{p}_2 H^{2m+4n-3}(P(m,n)^* ; Z[v]) \qquad \text{if } m \geqq 3.$$

Using (4.2–3), we have

$$Sq^2\rho\sigma(c^{m-3}d^n \otimes c^m d^n + c^{m-2}d^n \otimes c^{m-1}d^n) = \rho\sigma(c^{m-1}d^n \otimes c^m d^n).$$

Hence and from Lemma 3.3, it follows that

$$Sq^2\bar{p}_2 H^{2m+4n-3}(P(m,n)^* ; Z[v]) = H^{2m+4n-1}(P(m,n)^* ; Z_2) \qquad \text{if } m \geqq 3,$$

which, together with (4.5), establishes Theorem B.

§ 5. Proof of Theorem C

R. D. Rigdon [9, 11, 24 Theorem] has proved that an unorientable n-manifold M $(n > 4)$ is embedded in Euclidean $(2n-1)$-space if and only if its $(n-1)$-th mod 2 normal Stiefel-Whitney class $\bar{w}_{n-1}(M)$ vanishes. On the other hand, it is known (e.g. [5, Theorem (1.1)] and [8]) that an orientable n-manifold M $(n > 4)$ is always embedded in Euclidean $(2n-1)$-space and $\bar{w}_{n-1}(M) = 0$ for an orientable n-manifold M. Thus we have the following

Theorem 5.1. *For $n > 4$, an n-manifold M is embedded in Euclidean $(2n-1)$-space if and only if its $(n-1)$-th* mod 2 *normal Stiefel-Whitney class vanishes.*

Therefore to prove Theorem C, it is sufficient to calculate $\bar{w}_{m+2n-1}(P(m,n))$. By (4.2), we have a relation

$$\sum_{i\geqq 0} \bar{w}_i(P(m,n)) = (1+c)^{-m}(1+c+d)^{-1-n}$$
$$= \Big(\sum_{i\geqq 0}\binom{m-1+i}{i}c^i\Big)\Big(\sum_{j\geqq 0}\binom{n+j}{j}(c+d)^j\Big),$$

and in particular

$$(5.2) \qquad \bar{w}_{m+2n-1}(P(m,n)) = \sum_{0\leqq i\leqq m-1}\binom{2n+i}{n+i}\binom{n+i}{i}\binom{2m-2-i}{m-1-i}c^{m-1}d^n.$$

If $n=0$, then a simple calculation yields

$$(5.3) \qquad \bar{w}_{m-1}(P(m,0)) = \sum_{0\leqq i\leqq m-1}\binom{2m-2-i}{m-1-i}c^{m-1}$$
$$= \binom{2m-1}{m-1}c^{m-1} = \begin{cases} c^{m-1} & \text{if } m = 2^r, \\ 0 & \text{if } m \neq 2^r. \end{cases}$$

Otherwise let

$$n = 2^r + s \qquad (0 \leqq s < 2^r,\ 0 \leqq r).$$

We can put

$$i = 2^{r+1} j + k \qquad (0 \leqq j,\ 0 \leqq k < 2^{r+1}).$$

If $k + s < 2^r$, then $k + 2s < k + s + 2^r < 2^{r+1}$ and so

$$\binom{2n+i}{n+i} = \binom{2^{r+1}(j+1)+2s+k}{2^{r+1}j+2^r+s+k} \equiv \binom{j+1}{j}\binom{2s+k}{2^r+k+s} \equiv 0\,(2),$$

and if $k + s \geqq 2^r$, then $0 \leqq k + s - 2^r < k < 2^{r+1}$ and so

$$\binom{n+i}{i} = \binom{2^{r+1}(j+1)+s+k-2^r}{2^{r+1}j \;+\; k} \equiv \binom{j+1}{j}\binom{s+k-2^r}{k} \equiv 0\,(2).$$

The above two relations, together with (5.2), imply

(5.4) $$\overline{w}_{m+2n-1}(P(m,n)) = 0 \qquad \text{for } n > 0.$$

Therefore Theorem 5.1 and (5.3–4) imply Theorem C.

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