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The Reduced Symmetric Product of a Complex Projective Space and the Embedding Problem

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§0. Introduction

A given embedding f of a topological space X in the real m-space R^m induces the continuous map F of the space $X\times X-\Delta$ (Δ is the diagonal of $X\times X$) into the unit $(m-1)$-sphere S^{m-1} in R^m, which is defined as follows:

$$F(x,\ y)=\frac{f(x)-f(y)}{\|f(x)-f(y)\|}\ \text{for any distinct points } x,\ y \text{ of } X.$$

Then it is clear that F is equivariant with respect to the symmetry which interchanges the factors in $X\times X-\Delta$ and the antipodal map of S^{m-1}. Also, an isotopy $f_t(t\in[0,1])$ of two embeddings f_0, f_1 of X in R^m induces the equivariant homotopy F_t.

A. Haefliger [3] investigated the embeddings of compact differentiable manifolds in Euclidean spaces using the above equivariant maps and proved

THEOREM (Haefliger). *Let M be an n-dimensional compact differentiable manifold. Consider the correspondence which associates with an isotopy class of a differentiable embedding $f\colon M\longrightarrow R^m$ the equivariant homotopy class of the map F defined as above. Then this correspondence is surjective if $2m\geq 3(n+1)$ and bijective if $2m>3(n+1)$.*

Let the reduced symmetric product space M^* be the quotient space obtained from $M\times M-\Delta$ by identifying $(x, y)\sim(y, x)$. Then the projection $M\times M-\Delta\longrightarrow M^*$ is a double covering, and there exists a sphere bundle $S^{m-1}\longrightarrow(M\times M-\Delta)\times_{Z_2}S^{m-1}\longrightarrow M^*$ associated with this covering. Since there is a one-to-one correspondence between the equivariant homotopy classes of equivariant maps $M\times M-\Delta\longrightarrow S^{m-1}$ and the homotopy classes of cross sections of the above sphere bundle $S^{m-1}\longrightarrow(M\times M-\Delta)\times_{Z_2}S^{m-1}\longrightarrow M^*$, the study of this sphere bundle and so the cohomology of M^* play an important part in studying embeddings of M in R^m. In fact, D. Handel [4] and S. Feder [2] studied the cohomology of $(RP^n)^*$ and applied it to the existence and the classification of embeddings of the real projective spaces RP^n in Euclidean spaces.

In this paper, we try to determine the cohomology of $(CP^n)^*$ and to study the double covering $CP^n\times CP^n-\Delta\longrightarrow(CP^n)^*$ and to apply it to the embedding problem of the complex projective spaces CP^n.

This paper is organized as follows: In §1, we construct the double covering $Z_{n+1,2} \longrightarrow SZ_{n+1,2}$ in (1.3-4) which is homotopy equivalent to the double covering $CP^n \times CP^n - \Delta \longrightarrow (CP^n)^*$ of above. We prepare some results concerning the cohomology of real and complex projective bundles in §2. In §3, we determine the cohomology of $Z_{n+1,2}$ in Theorem 3.1 using the results of §2. In §4, we determine the cohomology of $SZ_{n+1,2}$ and so the reduced symmetric product space $(CP^n)^*$ in Theorems 4.9, 4.10, 4.15. In §5, we consider the isotopy classification of embeddings of CP^n in R^m ($m=4n$, $4n-1$, $4n-2$) and so we have the main theorem:

THEOREM 5.5. *Let* $n \geq 4$.

(1) *There exists a unique isotopy class of embeddings of* CP^n *in* R^{4n}.

(2) *There exist just two isotopy classes of embeddings of* CP^n *in* R^{4n-1}.

(3) *There exist just two isotopy classes of embeddings of* CP^n *in* R^{4n-2} *for* $n \neq 2^r$.

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§1. Construction of the double covering $Z_{n+1,2} \longrightarrow SZ_{n+1,2}$

Let $U(2)$ be the unitary group on the complex 2-space C^2 and $T^2 = S^1 \times S^1$ be the maximal torus of $U(2)$ and let

$$S^1 = \{e^{i\theta} \mid 0 \leq \theta < 2\pi\},$$

$$G = \left\{ \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}, \begin{pmatrix} 0 & \gamma_3 \\ \gamma_4 & 0 \end{pmatrix} \middle| \gamma_i \in S^1, \quad i=1, 2, 3, 4 \right\}.$$

Then we have a sequence of inclusions

$$S^1 \subset T^2 \subset G \subset U(2), \tag{1.1}$$

where S^1 is embedded in T^2 by the diagonal map.

It is clear that $G/T^2 = Z_2$ and we have the following

LEMMA 1.2. *The quotient spaces* $U(2)/T^2$ *and* $U(2)/G$ *are diffeomorphic to* S^2 *and* RP^2 *respectively, and natural projection* $U(2)/T^2 \longrightarrow U(2)/G$ *corresponds to the double covering* $S^2 \longrightarrow RP^2$.

Set $W_{n,2} = U(n)/U(n-2)$. Then $W_{n,2}$ is the complex Stiefel manifold of orthonormal 2-frames in C^n, and $U(2)$ acts freely on $W_{n,2}$ as follows: If $\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}$ is an element of $U(2)$ and $(u_1, u_2) \in W_{n,2}$, then

$$\alpha(u_1, u_2)=(\alpha_1 u_1+\alpha_2 u_2, \alpha_3 u_1+\alpha_4 u_2).$$

We consider the following quotient manifolds:

$$(1.3)\qquad \begin{array}{ll} X_{n,2}=W_{n,2}/S^1, & Z_{n,2}=W_{n,2}/T^2 \\ SZ_{n,2}=W_{n,2}/G, & G_{n,2}(C)=W_{n,2}/U(2). \end{array}$$

Here $X_{n,2}$ is called the complex projective Stiefel manifold [7] and $G_{n,2}(C)$ is the complex Grassmann manifold of complex 2-spaces in C^n.

The sequence (1.1) induces the following commutative diagram of fibrations:

$$(1.4)\qquad \begin{array}{ccccccc} S^1 & \longrightarrow & T^2 & \longrightarrow & G & \longrightarrow & U(2) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ W_{n,2} & = & W_{n,2} & = & W_{n,2} & = & W_{n,2} \\ \downarrow & & \downarrow \pi & & \downarrow & & \downarrow \\ X_{n,2} & \xrightarrow{\pi_1} & Z_{n,2} & \xrightarrow{\pi_2} & SZ_{n,2} & \xrightarrow{\pi_3} & G_{n,2}(C), \end{array}$$

where $\pi_2: Z_{n,2}\longrightarrow SZ_{n,2}$ is a double covering.

Let $f: Z_{n+1,2}\longrightarrow CP^n\times CP^n-\Delta$ be a map defined by

$$f(\pi(u_1, u_2))=([u_1], [u_2]),$$

where $[u_i](i=1, 2)$ is the element of CP^n determined by $u_i \in S^{2n+1}$. Then f is well-defined and is an equivariant map, which induces the map $\bar{f}$: $SZ_{n+1,2} \longrightarrow (CP^n)^*$ and so we obtain the map of double coverings

$$(1.5)\qquad \begin{array}{ccc} Z_{n+1,2} & \xrightarrow{f} & CP^n\times CP^n-\Delta \\ \downarrow \pi_2 & & \downarrow \\ SZ_{n+1,2} & \xrightarrow{\bar{f}} & (CP^n)^* \end{array}$$

Proposition 1.6. *In* (1.5), *the map* f *is a homotopy equivalence and* $\bar{f}$ *is a weak homotopy equivalence.*

Proof. Let (u_1, u_2) be a pair of linearly independent unit vectors in C^{n+1}. Then $\left(u_1, \frac{u_2-<u_2, u_1>u_1}{\|u_2-<u_2, u_1>u_1\|}\right)$ is a pair of orthonormal vectors in C^{n+1} which is obtained from (u_1, u_2) by the Gram-Schmidt process, where $<u_2, u_1>$ stands for the inner product of u_2 and u_1. We define a map g: $CP^n\times CP^n-\Delta\longrightarrow Z_{n+1,2}$ by

$$g([u_1], [u_2]) = \pi\left(u_1, \frac{u_2-<u_2, u_1>u_1}{\|u_2-<u_2, u_1>u_1\|}\right).$$

Then g is a well-defined map such that gf is the identity map. Let f_t:

$CP^n \times CP^n - \Delta \longrightarrow CP^n \times CP^n - \Delta$ be the homotopy defined by

$$f_t([u_1], [u_2]) = \left([u_1], \left[\frac{u_2 - t<u_2, u_1>u_1}{\|u_2 - t<u_2, u_1>u_1\|}\right]\right).$$

Then f_t is a well-defined homotopy between the identity map and fg. Hence f is a homotopy equivalence.

By the exact sequences of homotopy groups of fibrations and the five lemma, $\bar{f}$ induces isomorphisms of all homotopy groups of $SZ_{n+1,2}$ and $(CP^n)^*$ and so $\bar{f}$ is a weak homotopy equivalence. Q. E. D.

Let $V_{n,2}$ be the real Stiefel manifold of orthonormal 2-frames in the real n-space R^n. The orthogonal group $O(2)$ acts on $V_{n,2}$ as follows: If $\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}$ is an element of $O(2)$ and $(v_1, v_2) \in V_{n,2}$, then

$$\alpha(v_1, v_2) = (\alpha_1 v_1 + \alpha_2 v_2, \alpha_3 v_1 + \alpha_4 v_2).$$

Let

$$G' = \left\{\begin{pmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{pmatrix}, \begin{pmatrix} 0 & \varepsilon_3 \\ \varepsilon_4 & 0 \end{pmatrix} \middle| \varepsilon_i = \pm 1, \ i = 1, 2, 3, 4\right\},$$

$$O(1) \times O(1) = \left\{\begin{pmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{pmatrix} \middle| \varepsilon_i = \pm 1, \ i = 1, 2\right\}, \quad D = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right\},$$

and consider the quotient manifolds

$$X'_{n,2} = V_{n,2}/D, \quad Z'_{n,2} = V_{n,2}/O(1) \times O(1), \quad SZ'_{n,2} = V_{n,2}/G',$$

and the double coverings $X'_{n,2} \longrightarrow Z'_{n,2}$, $Z'_{n,2} \longrightarrow SZ'_{n,2}$. Considering the 2-frame in R^n as that in C^n, we have a map $h: V_{n,2} \longrightarrow W_{n,2}$. The map h induces the equivariant map $Z'_{n,2} \longrightarrow Z_{n,2}$ and so the map of double coverings. Also, let $g: X'_{n,2} \longrightarrow Z'_{n,2}$ be the equivariant map defined by

$$g(\pi'(v_1, v_2)) = \pi''\left(\frac{v_1 + v_2}{\sqrt{2}}, \frac{v_1 - v_2}{\sqrt{2}}\right)$$

where $(v_1, v_2) \in V_{n,2}$ and $\pi': V_{n,2} \longrightarrow X'_{n,2}$, $\pi'': V_{n,2} \longrightarrow Z'_{n,2}$ are the projections. Then we obtain the following commutative diagram of double coverings:

$$\text{(1.7)} \qquad \begin{array}{ccccc} X'_{n+1,2} & \xrightarrow{g} & Z'_{n+1,2} & \xrightarrow{h} & Z_{n+1,2} \\ \downarrow & & \downarrow & & \downarrow \pi_2 \\ Z'_{n+1,2} & \xrightarrow{\bar{g}} & SZ'_{n+1,2} & \xrightarrow{\bar{h}} & SZ_{n+1,2}. \end{array}$$

REMARK. D. Handel [4] treated the spaces $Z'_{n,2}$ and $SZ'_{n,2}$ and applied them to embedding problem for real projective spaces. Our notations are due to D. Handel.

§2. Projective bundles

In this section, we prepare some results concerning the cohomology of projective bundles, which will be applied in §§3-4.

For a complex (or real) n-plane bundle $\xi=(E(\xi), p(\xi), B(\xi))$, there determines the associated sphere bundle $S(\xi)=(S(\xi), p_0(\xi), B(\xi))$ with S^{2n-1} (or S^{n-1}) as the fiber. Let $P(\xi)$ be the quotient space of $S(\xi)$ where two unit vectors in the same fiber in $S(\xi)$ are identified by the standard free action of S^1 (or Z_2) on S^{2n-1} (or S^{n-1}), and let $q(\xi)\colon P(\xi)\longrightarrow B(\xi)$ be the factorization of $p_0(\xi)\colon S(\xi)\longrightarrow B(\xi)$ through $P(\xi)$ by the natural projection $q'(\xi)\colon S(\xi)\longrightarrow P(\xi)$. The bundle $P(\xi)=(P(\xi), q(\xi), B(\xi))$ with CP^{n-1} (or RP^{n-1}) as the fiber is the projective bundle associated with ξ.

Let λ_ξ be the complex (or real) line bundle associated with the S^1-bundle (or double covering) $(S(\xi), q'(\xi), P(\xi))$. Then, for the inclusion $i\colon CP^{n-1}\longrightarrow P(\xi)$ (or $i\colon RP^{n-1}\longrightarrow P(\xi)$) in any fiber of $P(\xi)$, $i^*\lambda_\xi$ is the canonical line bundle of CP^{n-1} (or RP^{n-1}).

Under the above situations, we have

THEOREM 2.1. *Let ξ be a complex n-plane bundle and let $a_\xi\in H^2(P(\xi); Z)$ be the first Chern class of λ_ξ^*, the dual of λ_ξ. Then $1, a_\xi,\dots,a_\xi^{n-1}$ form a base of $H^*(B(\xi); Z)$-module $H^*(P(\xi); Z)$. Moreover $q(\xi)^*\colon H^*(B(\xi); Z)\longrightarrow H^*(P(\xi); Z)$ is a monomorphism. The ring structure of $H^*(P(\xi); Z)$ is given by*

$$a_\xi^n=-\sum_{i=1}^{n} c_i(\xi)a_\xi^{n-i}$$

where $c_i(\xi)$ is the i-th Chern class of ξ. If $H^i(B(\xi); Z)=0$ for $i>2n$, then there is the following relation:

$$a_\xi^{n+k}=-\sum_{i=1}^{n-k}\sum_{j=0}^{k}\bar{c}_j(\xi)c_{i+k-j}(\xi)a_\xi^{n-i} \qquad \text{for } k\geq 0, \tag{2.2}$$

where $\bar{c}_j(\xi)$ is the j-th dual Chern class of ξ.

Similarly, we have

THEOREM 2.3. *Let ξ be a real n-plane bundle and let $a_\xi\in H^1(P(\xi); Z_2)$ be the first Stiefel-Whitney class of λ_ξ and let $w_i(\xi)$ (resp. $\bar{w}_i(\xi)$) be the i-th Stiefel-Whitney class (resp. dual Stiefel-Whitney class) of ξ. Then $1, a_\xi,\dots, a_\xi^{n-1}$ form a base of $H^*(B(\xi); Z_2)$-module $H^*(P(\xi); Z_2)$. Moreover $q(\xi)^*\colon H^*(B(\xi); Z_2)\longrightarrow H^*(P(\xi); Z_2)$ is a monomorphism. The ring structure of $H^*(P(\xi); Z_2)$ is given by*

$$a_\xi^n=\sum_{i=1}^{n} w_i(\xi)a_\xi^{n-i}.$$

If $H^i(B(\xi); Z_2)=0$ *for* $i>n$, *then there is the following relation*:

(2.4) $$a_\xi^{n+k}=\sum_{i=1}^{n-k}\sum_{j=0}^{k}\bar{w}_j(\xi)w_{i+k-j}(\xi)a_\xi^{n-i} \qquad \textit{for } k\geq 0.$$

PROOF OF THEOREMS 2.1, 2.3. The first half of each theorem is well-known (e.g. [5]), and the straightforward induction provides the proofs of (2.2) and (2.4) (see [4]). Q. E. D.

§3. Cohomology of $Z_{n+1,2}$

It is easily seen that $X_{n+1,2}$ of (1.3) is the total space of the tangent sphere bundle of CP^n and $Z_{n+1,2}$ of (1.3) is the total space of the complex projective bundle associated with the tangent bundle of CP^n. Also, it is well-known that the i-th Chern class $c_i(CP^n)$ and the i-th dual Chern class $\bar{c}_i(CP^n)$ of the tangent bundle of CP^n are equal to $\binom{n+1}{i}z^i$ and $(-1)^i\binom{n+i}{i}z^i$, respectively, where z is the generator of $H^2(CP^n; Z)$. Therefore the cohomology $H^*(Z_{n+1,2}; Z)$ is determined by Theorem 2.1 as follows:

THEOREM 3.1. *As* $H^*(CP^n; Z)$*-module,* $H^*(Z_{n+1,2}; Z)$ *has* $\{1, a, \cdots, a^{n-1}\}$ *as basis, where* a $(\neq 0)\in H^2(Z_{n+1,2}; Z)$ *is the first Chern class of the dual of the complex line bundle associated with the* S^1*-bundle* $\pi_1: X_{n+1,2}\longrightarrow Z_{n+1,2}$. *The ring structure is given by*

$$a^{n+k}=-\sum_{i=1}^{n-k}\sum_{j=0}^{k}(-1)^j\binom{n+j}{j}\binom{n+1}{i+k-j}z^{i+k}a^{n-i} \qquad \textit{for } k\geq 0,$$

where z *is the generator of* $H^2(CP^n; Z)$.

Similarly, $Z'_{n+1,2}$ is the total space of the real projective bundle associated with the tangent bundle of RP^n. Therefore, by Theorem 2.3 we have

PROPOSITION 3.2 [4, Proposition 3.1]. *In* $H^*(Z'_{n+1,2}; Z_2)$, *the following relation holds*:

$$v'^{n+k}=\sum_{i=1}^{n-k}\sum_{j=0}^{k}\bar{w}_j(RP^n)w_{i+k-j}(RP^n)v'^{n-1} \qquad \textit{for } k\geq 0,$$

where v' $(\neq 0)$ *is the first Stiefel-Whitney class of the double covering* $X'_{n+1,2}\longrightarrow Z'_{n+1,2}$ *and* $w_j(RP^n)$ *and* $\bar{w}_j(RP^n)$ *are the* j*-th Stiefel-Whitney class and the* j*-th dual Stiefel-Whitney class of* RP^n, *respectively.*

COROLLARY 3.3 [4, Corollary 3.2]. *If* $k=\max\left\{i \,\middle|\, \binom{n+i}{i}\not\equiv 0 \bmod 2,\right.$

$0 \leq i \leq n\Big\}$, *then* $v'^{n+k-1} \neq 0$, $v'^{n+k}=0$.

LEMMA 3.4 [4, Lemma 3.3]. *Let* u' *denote the first Stiefel-Whitney class of the double covering* $Z'_{n+1,2} \longrightarrow SZ'_{n+1,2}$, *and* $k=\max\Big\{i \Big| \binom{n+i}{i} \not\equiv 0 \bmod 2, 0 \leq i \leq n\Big\}$. *Then* $u'^{n+k-1} \neq 0$.

PROOF. By the diagram (1.7), it is evident. Q. E. D.

COROLLARY 3.5. *If* $n \geq 4$, *then* $u'^4 \neq 0$.

§4. Cohomology of $(CP^n)^*$

By the mapping cylinder considerations, the diagram (1.4) gives rise to the commutative diagram of fibrations:

$$\begin{array}{ccccccc} W_{n+1,2} & = & W_{n+1,2} & = & W_{n+1,2} & = & W_{n+1,2} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X_{n+1,2} & \xrightarrow{\pi_1} & Z_{n+1,2} & \xrightarrow{\pi_2} & SZ_{n+1,2} & \xrightarrow{\pi_3} & G_{n+1,2}(C) \\ \downarrow p_1 & & \downarrow p_2 & & \downarrow p_3 & & \downarrow p_4 \\ BU(1) & \xrightarrow{i_1} & BT^2 & \xrightarrow{i_2} & BG & \xrightarrow{i_3} & BU(2). \end{array} \tag{4.1}$$

The cohomology structures of $SZ_{n+1,2}$ and BG are unknown. On the other hand, the cohomology of $Z_{n+1,2}$ has been determined in §3 and the cohomology of $X_{n+1,2}$ was determined by C.A. Ruiz [7], and the others are well-known:

(4.2) $H^*(W_{n+1,2}; Z) = \wedge(w_n, w_{n+1})$ where deg $w_i = 2i-1$ $(i=n, n+1)$.

(4.3) $H^*(BU(2); Z) = Z[c_1, c_2]$

where $c_i (i=1, 2)$ is the universal i-th Chern class.

(4.4) $H^*(BT^2; Z) = Z[x_1, x_2]$ where deg $x_i = 2$ $(i=1, 2)$,

and there are the relations

$$i_2^* i_3^* c_1 = x_1 + x_2, \qquad i_2^* i_3^* c_2 = x_1 x_2. \tag{4.5}$$

For $G_{n+1,2}(C)$, it is known that

$$H^*(G_{n+1,2}(C); Z) = S(y_1, y_2) \otimes S(y_3, \cdots, y_{n+1}) / S^+(y_1, \cdots, y_{n+1})$$

where deg $y_i = 2 (i=1, \cdots, n+1)$ and $S(y_1, \cdots, y_k)$ is the ring of symmetric polynomials of k variables $y_1, \cdots, y_k$ with integral coefficients and $S^+(y_1, \cdots, y_k)$

is the ideal generated by the elements of positive degree [1, Proposition 31.1].

Let $\sigma_i (i=1,\cdots, n-1)$ be the i-th elementary symmetric function with respect to $n-1$ variables $y_3,\cdots,y_{n+1}$ and let $c_1=y_1+y_2$, $c_2=y_1y_2$. Then the ideal $S^+(y_1,\cdots,y_{n+1})$ is generated by the elements σ_1+c_1, $\sigma_2+\sigma_1c_1+c_2$, $\sigma_i+\sigma_{i-1}c_1+\sigma_{i-2}c_2 (i>2)$, where $\sigma_i=0$ for $i\geq n$. By a straightforward induction, we obtain

$$\sigma_r=\sum_{i\geq 0}(-1)^{r-i}\binom{r-i}{i}c_1^{r-2i}c_2^i \qquad \text{for } r\geq 1, \tag{4.6}$$

and

$$H^*(G_{n+1,2}(C);Z)=Z[c_1, c_2]/(\sigma_n, \sigma_{n+1}). \tag{4.7}$$

From now on, we shall study the cohomology of $SZ_{n+1,2}$ and BG. Consider the following commutative diagram of fibrations:

$$\begin{array}{ccccc} T^2 & \longrightarrow & U(2) & \longrightarrow & U(2)/T^2=S^2 \\ \downarrow & & \| & & \downarrow \\ G & \longrightarrow & U(2) & \longrightarrow & U(2)/G=RP^2. \end{array}$$

This diagram induces the following two commutative diagrams such that each row is a fibration and each column is a double covering:

$$\begin{array}{ccccc} S^2 & \longrightarrow & Z_{n+1,2} & \xrightarrow{\pi_3\pi_2} & G_{n+1,2}(C) \\ \downarrow & & \downarrow{\scriptstyle \pi_2} & & \| \\ RP^2 & \longrightarrow & SZ_{n+1,2} & \xrightarrow{\pi_3} & G_{n+1,2}(C), \end{array} \qquad \begin{array}{ccccc} S^2 & \longrightarrow & BT^2 & \xrightarrow{i_3i_2} & BU(2) \\ \downarrow & & \downarrow{\scriptstyle i_2} & & \| \\ RP^2 & \longrightarrow & BG & \xrightarrow{i_3} & BU(2). \end{array} \tag{4.8}$$

Therefore $SZ_{n+1,2}$ and BG are the total spaces of the real projective bundles over $G_{n+1,2}(C)$ and $BU(2)$, respectively.

Since $H^*(G_{n+1,2}(C);Z)$ and $H^*(BU(2);Z)$ have no torsion, we adopt the same symbol for each element of $H^*(G_{n+1,2}(C);Z)$ and $H^*(BU(2);Z)$ and its image in $H^*(G_{n+1,2}(C);Z_2)$ and $H^*(BU(2);Z_2)$ by the mod 2 reduction, in the rest of this paper.

THEOREM 4.9. *Let $n\geq 4$ and let $v\in H^1(SZ_{n+1,2};Z_2)$ be the first Stiefel-Whitney class of the double covering $Z_{n+1,2}\xrightarrow{\pi_2}SZ_{n+1,2}$. Then, as $H^*(G_{n+1,2}(C);Z_2)$-module, $H^*(SZ_{n+1,2};Z_2)$ has $\{1, v, v^2\}$ as basis and π_3^*: $H^*(G_{n+1,2}(C);Z_2)\longrightarrow H^*(SZ_{n+1,2};Z_2)$ is a monomorphism. Moreover the ring structure of $H^*(SZ_{n+1,2};Z_2)$ is given by*

$$v^3=c_1v$$

where $c_1\in H^*(G_{n+1,2}(C);Z_2)$ is the mod 2 *reduction of the element of* (4.7).

PROOF. The first half follows from Theorem 2.3. Hence it is sufficient to show that $v^3=c_1v$. By (1.7), we have $\bar{h}^*v=u'$, the first Stiefel-Whitney class of the double covering $Z'_{n+1,2}\longrightarrow SZ'_{n+1,2}$. Since $u'^3\neq 0$ for $n\geq 4$ by Corollary 3.5, we have $v^3\neq 0$. On the other hand, $H^3(SZ_{n+1,2};Z_2)=Z_2$ and its generator is c_1v by the first half of this theorem. Therefore we have $v^3=c_1v$. Q. E. D.

Let $\delta_2: H^*(\ ;Z_2)\longrightarrow H^{*+1}(\ ;Z)$ be the Bockstein homomorphism associated with the exact sequence $0\longrightarrow Z\xrightarrow{\times 2}Z\xrightarrow{\rho_2}Z_2\longrightarrow 0$.

Since $\rho_2\delta_2=Sq^1$ and $Sq^1v=v^2\neq 0$ in $H^*(SZ_{n+1,2};Z_2)$, we have $\delta_2v\neq 0$. Put $\delta_2v=u\in H^2(SZ_{n+1,2};Z)$. Then we have

THEOREM 4.10. *Let $n\geq 4$. Then $H^*(G_{n+1,2}(C);Z)$-module $H^*(SZ_{n+1,2};Z)$ has $\{1, u\}$ as generators and $\pi_3^*: H^*(G_{n+1,2}(C);Z)\longrightarrow H^*(SZ_{n+1,2};Z)$ is a monomorphism. Moreover there are the following relations:*

$$2u=0,\qquad \rho_2u=v^2,\qquad u^2=c_1u.$$

PROOF. The first two relations follow from the fact that $\delta_2v=u$.

In the integral cohomology spectral sequence of the fibration $RP^2\longrightarrow SZ_{n+1,2}\xrightarrow{\pi_3}G_{n+1,2}(C)$, E_2-term is given as follows:

$$E_2^{s,t}=H^s(G_{n+1,2}(C);\ H^t(RP^2;\ Z))=\begin{cases}H^s(G_{n+1,2}(C);Z) & \text{for } t=0\\ H^s(G_{n+1,2}(C);Z_2) & \text{for } t=2\\ 0 & \text{otherwise.}\end{cases}$$

Therefore, each differential is trivial and so we have $E_2=E_\infty$. Hence we obtain the following exact sequence:

$$0\longrightarrow E_\infty^{s,0}\longrightarrow H^s(SZ_{n+1,2};Z)\longrightarrow E_\infty^{s-2,2}\longrightarrow 0.$$

This gives rise to the exact sequence

(4.11) $\quad 0\longrightarrow H^s(G_{n+1,2}(C);Z)\longrightarrow H^s(SZ_{n+1,2};Z)\longrightarrow H^{s-2}(G_{n+1,2}(C);Z_2)\longrightarrow 0.$

(4.11) induces that $H^{2s-1}(SZ_{n+1,2};Z)=0$ for all s and $H^{2s}(SZ_{n+1,2};Z)$ has no p-torsion for odd prime p. Since $H^{2s-1}(SZ_{n+1,2};Z)=0$, the Bockstein cohomology exact sequence associated with the exact sequence of coefficients $0\longrightarrow Z\xrightarrow{\times 2}Z\xrightarrow{\rho_2}Z_2\longrightarrow 0$ induces the exact sequence

$$0\longrightarrow H^{2s-1}(SZ_{n+1,2};Z_2)\xrightarrow{\delta_2}H^{2s}(SZ_{n+1,2};Z)\xrightarrow{\times 2}$$
$$H^{2s}(SZ_{n+1,2};Z)\xrightarrow{\rho_2}H^{2s}(SZ_{n+1,2};Z_2)\longrightarrow 0.$$

This exact sequence implies that the torsion part of $H^{2s}(SZ_{n+1,2};Z)$ is isomorphic to $H^{2s-1}(SZ_{n+1,2};Z_2)$ by δ_2. Since $H^{2s-2}(G_{n+1,2}(C);Z_2)$ is isomorphic to $H^{2s-1}(SZ_{n+1,2};Z_2)$ by the cup product with v, $H^{2s-2}(G_{n+1,2}(C);Z_2)$ is isomorphic to the torsion part of $H^{2s}(SZ_{n+1,2};Z)$, which is given by

$uH^{2s-2}(G_{n+1,2}(C); Z)$. Therefore the exact sequence (4.11) is split. Thus $H^*(G_{n+1,2}(C); Z)$-module $H^*(SZ_{n+1,2}; Z)$ has $\{1, u\}$ as generators and π_3^*: $H^*(G_{n+1,2}(C); Z) \longrightarrow H^*(SZ_{n+1,2}; Z)$ is a monomorphism.

Since $\rho_2 u^2 = v^4$ in $H^*(SZ_{n+1,2}; Z_2)$ and $\bar{h}^* v^4 = u'^4 \neq 0$ by (1.7) and Corollary 3.5, we have $u^2 \neq 0$ in $H^4(SZ_{n+1,2}; Z)$. On the other hand, the torsion part of $H^4(SZ_{n+1,2}; Z)$ is Z_2 and its generator is $c_1 u$. Therefore we have the last relation $u^2 = c_1 u$. Q. E. D.

The integral and the mod 2 cohomology of BG are given by the same way as Theorems 4.9–10 and we omit the details.

THEOREM 4.12. *Let $n \geq 4$ and let $v \in H^1(BG; Z_2)$ be the first Stiefel-Whitney class of the double covering $BT^2 \xrightarrow{i_2} BG$ and let $u = \delta_2 v$. Then $H^*(BU(2); Z_2)$-module $H^*(BG; Z_2)$ has $\{1, v, v^2\}$ as basis and $H^*(BU(2); Z)$-module $H^*(BG; Z)$ has $\{1, u\}$ as generators, and $i_3^*: H(BU(2); Z_2) \longrightarrow H^*(BG; Z_2)$ and $i_3^*: H^*(BU(2); Z) \longrightarrow H^*(BG; Z)$ are both monomorphic. Moreover the following relations hold:*

$$v^3 = c_1 v, \qquad u^2 = c_1 u, \qquad p_3^* v = v, \qquad p_3^* u = u.$$

REMARK. If we notice that the transgression of the fibration $W_{n+1,2} \longrightarrow G_{n+1,2}(C) \longrightarrow BU(2)$ is given by $\tau w_i = \bar{c}_i (i = n, n+1)$, the universal i-th dual Chern class of the complex 2-plane bundle, and that i_3^* is a monomorphism because $i_2^* i_3^*$ is so, we see easily

$$H^*(SZ_{n+1,2}; Z) = H^*(BG; Z)/(i_3^* \bar{c}_n, i_3^* \bar{c}_{n+1}) \qquad \text{for } n \geq 1,$$

$$H^*(SZ_{n+1,2}; Z_2) = H^*(BG; Z_2)/(i_3^* \bar{c}_n, i_3^* \bar{c}_{n+1}) \qquad \text{for } n \geq 1.$$

LEMMA 4.13. *Let $n \geq 4$. Then the homomorphism $\pi_2^*: H^*(SZ_{n+1,2}; Z_2) \longrightarrow H^*(Z_{n+1,2}; Z_2)$ is given by*

$$\pi_2^* c_1 = a, \qquad \pi_2^* c_2 = az + z^2, \qquad \pi_2^* v = 0,$$

where a, z in $H^*(Z_{n+1,2}; Z_2)$ are the images of a, z in $H^*(Z_{n+1,2}; Z)$ respectively, by the mod 2 *reduction.*

PROOF. It is easily seen that $\pi_2^* v = 0$. Since $W_{n+1,2}$ is 6-connected for $n \geq 4$, $p_i^* (i = 1, 2, 3, 4)$ is isomorphic in degree smaller than 7. Therefore there exists a unique element a' in $H^2(BT^2; Z_2)$ such that $p_2^* a' = a$. Since $0 = \pi_1^* a = p_1^* i_1^* a'$ and p_1^* is isomorphic in degree 2, we have $i_1^* a' = 0$. On the other hand, the generator of $H^2(BU(1); Z_2)$ is $i_1^* x_1 = i_1^* x_2$. Therefore the kernel of i_1^* of degree 2 is generated by $x_1 + x_2$. Hence we have $a' = x_1 + x_2 = i_2^* c_1$ by (4.5) and so we have $\pi_2^* c_1 = a$.

By Theorem 3.1, $\pi_2^* c_2$ has the form $\pi_2^* c_2 = \varepsilon_1 a^2 + \varepsilon_2 az + \varepsilon_3 z^2$, where $\varepsilon_i = 0$ or $1 (i = 1, 2, 3)$. Then we have

$$\pi_2^* Sq^2 c_2 = \pi_2^*(c_1 c_2) = \varepsilon_1 a^3 + \varepsilon_2 a^2 z + \varepsilon_3 a z^2.$$

However we have

$$Sq^2 \pi_2^* c_2 = Sq^2(\varepsilon_1 a^2 + \varepsilon_2 a z + \varepsilon_3 z^2) = \varepsilon_2 a^2 z + \varepsilon_2 a z^2.$$

Comparing the coefficients of the corresponding terms of $\pi_2^* Sq^2 c_2$ and $Sq^2 \pi_2^* c_2$, we obtain $\varepsilon_1 = 0$ and $\varepsilon_2 = \varepsilon_3$, since $n \geq 4$. Assume that $\varepsilon_2 = \varepsilon_3 = 0$. Then $0 = \pi_2^* c_2 = p_2^* i_2^* c_2 = p_2^*(x_1 x_2)$. This contradicts the fact that p_2^* is isomorphic in degree 4. Therefore we have $\pi_2^* c_2 = az + z^2$. Q. E. D.

PROPOSITION 4.14. *Let $n \geq 4$ and set $n = 2^r + s$, $0 \leq s \leq 2^r - 1$. The following relations hold in $H^*(SZ_{n+1,2}; Z_2)$:*

$$c_1^{2^{r+1}-1} = 0, \qquad c_1^{2^{r+1}-2} c_2^s v^2 \neq 0.$$

PROOF. By Lemma 4.6, we have $\sigma_r = \sum_{i \geq 0} \binom{r-i}{i} c_1^{r-2i} c_2^i$ for $r \geq 1$ in $H^*(G_{n+1,2}(C); Z_2)$. If $r \geq n$, then $\sigma_r = 0$. Therefore we obtain $c_1^{2^{r+1}-1} = 0$. To prove the second relation, it is sufficient to show that $c_1^{2^{r+1}-2} c_2^s \neq 0$ and so $\pi_2^*(c_1^{2^{r+1}-2} c_2^s) \neq 0$ in $H^*(Z_{n+1,2}; Z_2)$. By Theorem 3.1, we have

$$\pi_2^*(c_1^{2^{r+1}-2} c_2^s) = \sum_{i=0}^{n-1} b_i a^i, \qquad b_i \in H^*(CP^n; Z_2).$$

On the other hand, by Theorem 3.1 and Lemma 4.13, we have

$$\pi_2^*(c_1^{2^{r+1}-2} c_2^s) = a^{2^{r+1}-2}(a+z)^s z^s = \sum_{t=0}^{s} \binom{s}{t} a^{2^{r+1}+s-t-2} z^{s+t}$$

$$= \sum_{t=0}^{s} \binom{s}{t} \sum_{i=1}^{n-(2^r-t-2)} \sum_{j=0}^{2^r-t-2} \bar{c}_j(CP^n) c_{i+2^r-t-2-j}(CP^n) z^{s+t} a^{n-i},$$

where $c_j(CP^n)$, $\bar{c}_j(CP^n)$ are the j-th Chern and dual Chern classes of CP^n. Comparing the coefficients of a^{n-1}, we have

$$b_{n-1} = \sum_{t=0}^{s} \binom{s}{t} \sum_{j=0}^{2^r-t-2} \bar{c}_j(CP^n) c_{2^r-t-1-j}(CP^n) z^{s+t}$$

$$= \sum_{t=0}^{s} \binom{s}{t} \bar{c}_{2^r-t-1}(CP^n) z^{s+t}$$

$$= \sum_{t=0}^{s} \binom{s}{t} \binom{2^{r+1}+s-t-1}{2^r-t-1} z^{2^r+s-1}$$

By a simple calculation, we have $\binom{2^{r+1}+s-t-1}{2^r-t-1} = 0$ or $\neq 0$ according as $t \leq s-1$ or $t = s$, and so we obtain $b_{n-1} = z^{n-1} \neq 0$ in $H^*(CP^n; Z_2)$. Q. E. D.

Using the above proposition, we have

THEOREM 4.15. *Let* $n \geq 4$. *Then* $SZ_{n+1,2}$ *is an unorientable* $(4n\text{-}2)$*-dimensional manifold which is weakly homotopy equivalent to the reduced symmetric product of* CP^n, *and* $H^{4n-2}(SZ_{n+1,2}; Z) = Z_2$ *with the generator* $c_1^{2^{r+1}-2} c_2^s u$ *for* $n = 2^r + s$, $0 \leq s \leq 2^r - 1$.

§5. Classification of embeddings of CP^n in Euclidean spaces

A. Haefliger investigated the embeddings in the stable range [3] and proved the following theorem.

THEOREM 5.1 (Haefliger). *Let* M *be an n-dimensional compact differentiable manifold. The correspondence which associates with a given differentiable embedding* $f\colon M \longrightarrow R^m$ *the equivariant map* $F\colon M \times M - \Delta \longrightarrow S^{m-1}$ *defined by* $F(x, y) = \dfrac{f(x) - f(y)}{\|f(x) - f(y)\|}$ *induces the correspondence which associates with a given isotopy class of* f *the equivariant homotopy class of* F. *This correspondence is surjective if* $2m \geq 3(n+1)$ *and bijective if* $2m > 3(n+1)$.

We now know that there exists a one-to-one correspondence between the equivariant homotopy classes of equivariant maps $M \times M - \Delta \longrightarrow S^{m-1}$ and the homotopy classes of cross sections of the sphere bundle $S^{m-1} \longrightarrow (M \times M - \Delta) \times_{Z_2} S^{m-1} \longrightarrow M^*$ associated with the double covering $M \times M - \Delta \longrightarrow M^*$.

Let λ be the real line bundle over $(CP^n)^*$ associated with the double covering $CP^n \times CP^n - \Delta \longrightarrow (CP^n)^*$. Then the sphere bundle

$$S^{m-1} \longrightarrow (CP^n \times CP^n - \Delta) \times_{Z_2} S^{m-1} \longrightarrow (CP^n)^*$$

is the sphere bundle associated with $m\lambda$, the Whitney sum of m copies of λ.

Therefore we have

PROPOSITION 5.2. (1) *Let* $2m \geq 3(2n+1)$. *If* $m\lambda$ *has a non-zero cross section, then there exists an embedding of* CP^n *in* R^m.

(2) *Let* $2m > 3(2n+1)$. *Then there exists a one-to-one correspondence between the isotopy classes of embeddings of* CP^n *in* R^m *and the homotopy classes of cross sections of the sphere bundle associated with* $m\lambda$ *over* $(CP^n)^*$.

By Propositions 1.6 and 5.2, the obstructions for $m\lambda$ to have a non-zero cross section are the elements of $H^{i+1}(SZ_{n+1,2}; \pi_i(S^{m-1}))$ and its primary obstruction for even m is the Euler class $\chi(m\lambda)$ of $m\lambda$, and the obstructions for two given cross sections to be homotopic are the elements of $H^i(SZ_{n+1,2}; \pi_i(S^{m-1}))$.

LEMMA 5.3. *Let* η *be a real line bundle. Then the Euler class* $\chi(2\eta)$ *is given by*

$$\chi(2\eta) = \delta_2 w_1(\eta),$$

where $w_1(\eta)$ *is the first Stiefel-Whitney class of* η.

PROOF. Let ξ be the canonical line bundle over RP^∞. By the universality of ξ, it is sufficient to show that $\chi(2\xi)=\delta_2 w_1(\xi)$. Consider the Bockstein cohomology exact sequence of RP^∞

$$0\longrightarrow H^1(RP^\infty; Z_2)\xrightarrow{\delta_2} H^2(RP^\infty; Z)\xrightarrow{\times 2} H^2(RP^\infty; Z)\xrightarrow{\rho_2} H^2(RP^\infty; Z_2)\longrightarrow 0,$$

where $H^1(RP^\infty; Z_2)=Z_2$ with the generator $w_1(\xi)$ and $H^2(RP^\infty; Z)=Z_2$ with the generator $\delta_2 w_1(\xi)$. Since $\rho_2\chi(2\xi)=w_2(2\xi)=w_1(\xi)^2\neq 0$, it follows that $\chi(2\xi)\neq 0$ in $H^2(RP^\infty; Z)$ and so we have $\chi(2\xi)=\delta_2 w_1(\xi)$. Q. E. D.

REMARK. The above lemma is generalized as follows: Let η^1 and ζ^n be a real line bundle and a real n-plane bundle over the same space with $w_1(\eta^1)=w_1(\zeta^n)$. Then we have

$$\chi(\eta^1\oplus\zeta^n)=\delta_2 w_n(\zeta^n).$$

By the above considerations, we have the following theorem, which is already known ([6], [8], [9]):

THEOREM 5.4. (1) *CP^n is embeddable in R^{4n-2} for $n\geq 4$, $n\neq 2^r$.*
(2) *CP^{2^r} is embeddable in $R^{2^{r+2}-1}$ but not embeddable in $R^{2^{r+2}-2}$ for $r\geq 2$.*

PROOF. The obstructions for the existence of a non-zero cross section of $(4n-1)\lambda$ are in $H^{i+1}(SZ_{n+1,2};\ \pi_i(S^{4n-2}))$ which is 0, since $SZ_{n+1,2}$ is a $(4n-2)$-dimensional manifold. Hence CP^n is embeddable in R^{4n-1} by Proposition 5.2, (1). The obstructions for the existence of a non-zero cross section of $(4n-2)\lambda$ are in $H^{i+1}(SZ_{n+1,2};\ \pi_i(S^{4n-3}))$ and non-trivial obstruction is the Euler class $\chi((4n-2)\lambda)$ in $H^{4n-2}(SZ_{n+1,2}; Z)$. By Lemma 5.3, we have $\chi(2\lambda)=u=\delta_2 v$ and using Proposition 4.14, we have

$$\chi((4n-2)\lambda)=\chi(2\lambda)^{2n-1}=u^{2n-1}=uc_1^{2n-2}\begin{cases}=0 & \text{for } n\neq 2^r\\ \neq 0 & \text{for } n=2^r.\end{cases}$$

Therefore by Proposition 5.2 (1), it follows that CP^n is embeddable or not embeddable in R^{4n-2} according as $n\neq 2^r$ or $n=2^r$. Q. E. D.

Our main theorem is the following

THEOREM 5.5. *Let $n\geq 4$.*
(1) *There exists a unique isotopy class of embeddings of CP^n in R^{4n}.*
(2) *There exist just two isotopy classes of embeddings of CP^n in R^{4n-1}.*
(3) *There exist just two isotopy classes of embeddings of CP^n in R^{4n-2} for $n\neq 2^r$.*

PROOF. The obstructions for two non-zero cross sections of $4n\lambda$ being homotopic are the elements of $H^i(SZ_{n+1,2};\ \pi_i(S^{4n-1}))$ which is 0 for all i.

This implies (1). The obstructions for two non-zero cross sections of $(4n-1)\lambda$ being homotopic are in $H^i(SZ_{n+1,2};\pi_i(S^{4n-2}))$ and

$$H^i(SZ_{n+1,2};\pi_i(S^{4n-2}))=\begin{cases}0 & \text{for } i\neq 4n-2\\ Z_2 & \text{for } i=4n-2,\end{cases}$$

by Theorem 4.15. Therefore we have (2). By Theorems 4.9-10, 4.15,

$$H^i(SZ_{n+1,2};\pi_i(S^{4n-3}))=\begin{cases}0 & \text{for } i\neq 4n-2\\ Z_2 & \text{for } i=4n-2,\end{cases}$$

and so we have (3). Q. E. D.

REMARK 1. W.-T. Wu [10] proved that any two embeddings of an n-dimensional differentiable manifold in R^{2n+1} are isotopic.

REMARK 2. T. Watabe [9] proved that any two immersions of CP^n in R^{4n-1} are regularly homotopic for even n.

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