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Note on the Enumeration of Embeddings of Real Projective Spaces

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§ 1. Introduction

Recently, Y. Nomura [12] has studied the enumeration problem of liftings of a given map to a fibration and its application to the enumeration problem of immersions of certain manifolds. In this note, using his results we enumerate the non-zero cross sections of certain vector bundles, and then study the embedding problem of the real projective spaces in the euclidean spaces.

Let ξ be an orientable n-plane bundle over a CW-complex X of dimension less than $n+2$, and let $w_2(\xi)$ be the second Stiefel-Whitney class of ξ. Consider the homomorphisms

$$\begin{aligned} &\Theta_\xi^i\colon H^{i-1}(X; Z) \longrightarrow H^{i+1}(X; Z_2),\\ &\Gamma_\xi^i\colon H^i(X; Z_2) \longrightarrow H^{i+2}(X; Z_2), \end{aligned} \tag{1.1}$$

of the cohomology groups, defined by

$$\Theta_\xi^i(a) = Sq^2 \rho_2 a + \rho_2 a \cdot w_2(\xi),$$

$$\Gamma_\xi^i(b) = Sq^2 b + b \cdot w_2(\xi),$$

where ρ_2 is the mod 2 reduction. Then we prove the following theorem in §§ 2–4, using Nomura's theorem [12, § 2] and the Postnikov factorization of the universal orientable $(n-1)$-sphere bundle $BSO(n-1) \to BSO(n)$.

THEOREM A. *Let $n \geq 6$ and let ξ be an orientable n-plane bundle over a CW-complex X of dimension less than $n+2$ with a non-zero cross section. Then, the set cross (ξ) of (free) homotopy classes of non-zero cross sections of ξ is given by*

$$cross(\xi) = \begin{cases} \operatorname{Ker} \Theta_\xi^n \times \operatorname{Coker} \Theta_\xi^{n-1}, & \textit{if } \Gamma_\xi^{n-1} \textit{ is epimorphic,} \\ \operatorname{Ker} \Theta_\xi^n \times \operatorname{Coker} \Theta_\xi^{n-1} \times \operatorname{Coker} \Gamma_\xi^{n-1}, & \textit{if } \Theta_\xi^{n-1} \textit{ is monomorphic,} \end{cases}$$

where Θ_ξ^i, Γ_ξ^i are the homomorphisms of (1.1).

This is a generalization of a part of the theorem of I. M. James [8, Th. 5.1] for the case $\dim X \leq n$.

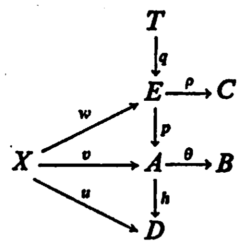
Applying the above theorem, we prove the following theorem in §§ 5–7, using the results of A. Haefliger [6].

THEOREM B. *Let n be an even integer and let* $n \geq 10$, $n \neq 2^r$. *Then, there exists only one isotopy class of embeddings of the real n-dimensional projective space* RP^n *in the real* $(2n-2)$*-space* R^{2n-2}.

§ 2. Nomura's theorem

Let h: $A \to D$ be a principal fibration with fiber F, and let p: $E \to A$ and q: $T \to E$ be the principal fibrations with the classifying maps θ: $A \to B$ and ρ: $E \to C$, respectively. For a given CW-complex X and a map u: $X \to D$, we assume that there are liftings v and w in the following commutative diagram:

(2.1)



and also we assume that w has a lifting to T.

In this section, we consider the set $[X, T; u]$ of homotopy classes of liftings $X \to T$ of u, under the following stability condition (i)–(iii) for the sequence $\{h, p, q\}$ of fibrations:

(i) the spaces B and C are homotopy associative H-spaces,

(ii) there exists a map d: $F \times D \to B$ such that

$$\theta m \simeq d(id_F \times h) + \theta \pi_2 \quad \text{and} \quad d i_2 \simeq 0,$$

(iii) there exists a map c: $\Omega B \times D \to C$ such that

$$\rho \mu \simeq c(id_{\Omega B} \times hp) + \rho \pi_2 \quad \text{and} \quad c i_2 \simeq 0,$$

where m: $F \times A \to A$ and μ: $\Omega B \times E \to E$ are the actions of fibers in the principal fibrations h: $A \to D$ and p: $E \to A$, respectively, π_2 and i_2 denote the projection and the injection to the second factors, and + denotes the multiplication of an H-space.

The maps d and c define the maps d': $\Omega F \times D \to \Omega B$ and c': $\Omega^2 B \times D \to \Omega C$ by $d'(\lambda, x)(t) = d(\lambda(t), x)$ and $c'(v, y)(t) = c(v(t), y)$. These maps induce the maps between homotopy sets;

(2.2) $$\Theta_u\colon [X, F] \longrightarrow [X, B], \qquad \Theta'_u\colon [X, \Omega F] \longrightarrow [X, \Omega B],$$
$$\Gamma_u\colon [X, \Omega B] \longrightarrow [X, C], \qquad \Gamma'_u\colon [X, \Omega^2 B] \longrightarrow [X, \Omega C],$$

by setting

$$\Theta_u(a) = d_*(a, u), \qquad \Theta'_u(a') = d'_*(a', u),$$
$$\Gamma_u(b) = c_*(b, u), \qquad \Gamma'_u(b') = c'_*(b', u),$$

where $u \in [X, D]$ is a given map, and $d_*\colon [X, F] \times [X, D] \to [X, B]$ is the induced map of d and so on. Then it is easy to see that the maps of (2.2) are homomorphisms of groups, by the existence of a lifting of u and the above stability condition (i)–(iii). Further, we define

(2.3) $$\varphi\colon \operatorname{Ker} \Theta_u \longrightarrow \operatorname{Coker} \Gamma_u$$

as follows: For a fixed lifting $v\colon X \to A$ of u, the correspondence $[X, F] \ni \sigma \to m_*(\sigma, v) \in [X, A; u]$ is, as is well-known, a bijection. We see easily that $\sigma \in \operatorname{Ker} \Theta_u$ if and only if $m_*(\sigma, v)$ has a lifting to E. Let $w_\sigma\colon X \to E$ be a lifting of $m_*(\sigma, v)$ and define

$$\varphi(\sigma) = \rho_*(w_\sigma) \mod \operatorname{Im} \Gamma_u.$$

It is easily shown that φ is well-defined.

The following theorem is proved by Y. Nomura [12, Cor. 2.5–6].

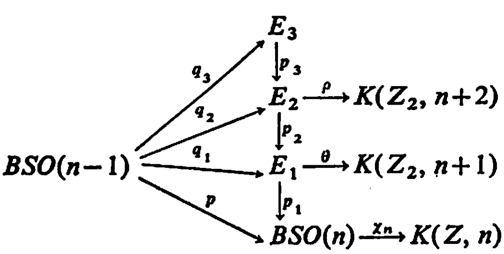
THEOREM. *Under the above assumptions and notations, we obtain, as a set,*

$$[X, T; u] = \begin{cases} \operatorname{Ker} \varphi \times (\operatorname{Ker} \Gamma_u / \operatorname{Im} \Theta'_u) & \text{if } \Gamma'_u \text{ is an epimorphism,} \\ \operatorname{Ker} \varphi \times (\operatorname{Ker} \Gamma_u / \operatorname{Im} \Theta'_u) \times \operatorname{Coker} \Gamma'_u & \text{if } \Theta'_u \text{ is a monomorphism.} \end{cases}$$

§ 3. The Postnikov factorization of the universal orientable S^{n-1}-bundle

Let $n \geq 6$. The Postnikov factorization for the fourth stage of the universal orientable S^{n-1}-bundle $BSO(n-1) \xrightarrow{p} BSO(n)$, induced by the inclusion $SO(n-1) \subset SO(n)$, is given as follows:

(3.1)



where $\chi_n \in H^n(BSO(n); Z)$ represents the Euler class, $p_1: E_1 \to BSO(n)$ is the principal fibration with the classifying map χ_n, and θ and ρ are the second and the third k-invariants, and $p_2: E_2 \to E_1$ and $p_3: E_3 \to E_2$ are the principal fibrations with the classifying maps θ and ρ, respectively. Furthermore $q_3: BSO(n-1) \to E_3$ is an $(n+2)$-equivalence, i.e., $q_{3*}: \pi_i(BSO(n-1)) \to \pi_i(E_3)$ is isomorphic for $i < n+2$ and epimorphic for $i = n+2$.

Let $m_1: K(Z, n-1) \times E_1 \to E_1$ be the action of fiber in $p_1: E_1 \to BSO(n)$ and consider the map $v_1 = m_1(id \times q_1): K(Z, n-1) \times BSO(n-1) \to E_1$. Then, by the results of E. Thomas [14, p. 21], the second k-invariant $\theta \in H^{n+1}(E_1; Z_2)$ is characterized by the equality

$$v_1^* \theta = Sq^2 \rho_2 \iota_1 \times 1 + \rho_2 \iota_1 \times p^* w_2, \tag{3.2}$$

where $v_1^*: H^{n+1}(E_1; Z_2) \to H^{n+1}(K(Z, n-1) \times BSO(n-1); Z_2)$ and $\iota_1 \in H^{n-1}(K(Z, n-1); Z)$ is the fundamental class and w_2 is the second universal Stiefel-Whitney class.

Now, consider the homomorphism

$$m_1^* - \pi_2^*: H^r(E_1; Z_2) \longrightarrow H^r(K(Z, n-1) \times E_1; Z_2),$$

where π_2 is the projection to the second factor. Since $(id \times q_1)^* \pi_2^*(\theta) = 1 \times q_1^*(\theta) = 0$, we have $(id \times q_1)^*(m_1^* - \pi_2^*)(\theta) = (id \times q_1)^* m_1^*(\theta) = v_1^*(\theta)$. On the other hand, $(id \times q_1)^*: \sum_{i=0}^{2} H^{n+1-i}(K(Z, n-1); Z_2) \otimes H^i(E_1; Z_2) \to \sum_{i=0}^{2} H^{n+1-i}(K(Z, n-1); Z_2) \otimes H^i(BSO(n-1); Z_2)$ is monomorphic, because $q_1^*: H^r(E_1; Z_2) \to H^r(BSO(n-1); Z_2)$ is so for $r \leq 2$. Therefore, (3.2) shows that

$$(m_1^* - \pi_2^*)(\theta) = Sq^2 \rho_2 \iota_1 \times 1 + \rho_2 \iota_1 \times p_1^* w_2. \tag{3.3}$$

Similarly, let $m_2: K(Z_2, n) \times E_2 \to E_2$ be the action of fiber in $p_2: E_2 \to E_1$, and consider the map $v_2 = m_2(id \times q_2): K(Z_2, n) \times BSO(n-1) \to E_2$. Then the third k-invariant $\rho \in H^{n+2}(E_2; Z_2)$ is characterized by

$$v_2^* \rho = Sq^2 \iota_2 \times 1 + \iota_2 \times p^* w_2,$$

where $\iota_2 \in H^n(K(Z_2, n); Z_2)$ is the fundamental class (cf. [15, Th. 3.5]). Therefore we have

$$(m_2^* - \pi_2^*)(\rho) = Sq^2 \iota_2 \times 1 + \iota_2 \times p_2^* p_1^* w_2, \tag{3.4}$$

by the same argument as above.

§ 4. Proof of Theorem A

Continuing the previous section, we choose the maps

$$d: (K(Z, n-1)\times BSO(n), BSO(n)) \longrightarrow (K(Z_2, n+1), *),$$

$$c: (K(Z_2, n)\times BSO(n), BSO(n)) \longrightarrow (K(Z_2, n+2), *)$$

such that they represent the elements $d = Sq^2\rho_2\iota_1 \times 1 + \rho_2\iota_1 \times w_2$ and $c = Sq^2\iota_2 \times 1 + \iota_2 \times w_2$, respectively. Then from the equalities (3.3) and (3.4), it is easy to see that the sequence $\{p_1, p_2, p_3\}$ of principal fibrations in the diagram (3.1) satisfies the stability condition (i)–(iii) in § 2. Therefore, for a given map $\xi: X \to BSO(n)$ which has a lifting $X \to E_3$, we can define the homomorphisms

$$\Theta^i_\xi: H^{i-1}(X; Z) \longrightarrow H^{i+1}(X; Z_2) \qquad \text{for } i=n, n-1,$$

$$\Gamma^i_\xi: H^i(X; Z_2) \longrightarrow H^{i+2}(X; Z_2) \qquad \text{for } i=n, n-1,$$

corresponding to Θ_u, Θ'_u, Γ_u and Γ'_u of (2.2) and these are the homomorphisms of (1.1) by definition.

We now prove Theorem A in § 1.

Let ξ be an orientable n-plane bundle over a CW-complex X of dimension less than $n+2$ and suppose that ξ has a non-zero cross section. Then the set cross (ξ) of homotopy classes of non-zero cross sections of ξ is

$$cross(\xi) = [X, BSO(n-1); \xi]$$

by [9, Lemma 2.2], where $\xi: X \to BSO(n)$ denotes the classifying map of ξ. Since $\dim X < n+2$ and $q_3: BSO(n-1) \to E_3$ is an $(n+2)$-equivalence, we obtain

$$[X, BSO(n-1); \xi] = [X, E_3; \xi]$$

by [9, Th. 3.2]. Now we can apply the theorem in § 2. Since $\dim X < n+2$, we have $H^{n+2}(X; Z_2) = 0$ and so $\mathrm{Ker}\,\Gamma^n_\xi = H^n(X; Z_2)$ and $\mathrm{Ker}(\varphi: \mathrm{Ker}\,\Theta^n_\xi \to \mathrm{Coker}\,\Gamma^n_\xi) = \mathrm{Ker}\,\Theta^n_\xi$. This completes the proof.

EXAMPLE. Let ξ be a $(2n-1)$-plane bundle over the real $2n$-dimensional complex projective space CP^n with a non-zero cross section. Then the set cross (ξ) is equal to Z, the set of integers. In fact, $\Theta^{2n-2}_\xi: H^{2n-3}(CP^n; Z) \to H^{2n-1}(CP^n; Z_2)$ is obviously monomorphic and $\mathrm{Coker}\,\Theta^{2n-2}_\xi = 0$. Also $\mathrm{Ker}(\Theta^{2n-1}_\xi: H^{2n-2}(CP^n; Z) \to H^{2n}(CP^n; Z_2))$ is equal to Z and $\mathrm{Coker}(\Gamma^{2n-2}_\xi: H^{2n-2}(CP^n; Z_2) \to H^{2n}(CP^n; Z_2))$ is Z_2 or 0.

§ 5. Enumeration of embeddings

Let M be an n-dimensional differentiable closed manifold, M^* be its reduced symmetric product obtained from $M \times M - \Delta$ (Δ is the diagonal of M) by identifying (x, y) with (y, x) and let η be the real line bundle over M^* associated with the double covering $M\times M-\Delta\to M^*$. Then the set $[M\subset R^{2n-2}]$ of isotopy classes of embeddings of M into R^{2n-2} for $n\geq 8$ is equal to the set of homotopy classes of cross sections of the associated S^{2n-3}-bundle $(M\times M-\Delta)\times_{Z_2}S^{2n-3}\to M^*$ and so equal to *cross* $((2n-2)\eta)$, by the theorem of A. Haefliger [6, § 1].

Since M^* is an open $2n$-manifold, there is a proper Morse function on M^* with no critical points of index $2n$ by [13, Lemma 1.1] and so M^* has the homotopy type of a CW-complex of dimension less than $2n$ by [11, Th. 3.5]. Therefore we obtain the following proposition from Theorem A.

PROPOSITION. *Let $n\geq 8$ and let M be an n-dimensional differentiable closed manifold which is embedded in R^{2n-2}. Then the set $[M\subset R^{2n-2}]$ of isotopy classes of embeddings of M into R^{2n-2} is given by*

$$[M\subset R^{2n-2}]=\begin{cases}\operatorname{Ker}\Theta^{2n-2}\times\operatorname{Coker}\Theta^{2n-3}, & \text{if } \Gamma \text{ is epimorphic,}\\ \operatorname{Ker}\Theta^{2n-2}\times\operatorname{Coker}\Theta^{2n-3}\times\operatorname{Coker}\Gamma, & \text{if } \Theta^{2n-3} \text{ is monomorphic,}\end{cases}$$

where the homomorphisms

$$\Theta^i\colon H^{i-1}(M^*;Z)\longrightarrow H^{i+1}(M^*;Z_2)\qquad \text{for } i=2n-2,\ 2n-3,$$

$$\Gamma\colon H^{2n-3}(M^*;Z_2)\longrightarrow H^{2n-1}(M^*;Z_2),$$

are defined by

$$\Theta^i(a)=Sq^2\rho_2 a+(n-1)\rho_2 a\cdot v^2,$$

$$\Gamma(b)=Sq^2 b+(n-1)b\cdot v^2,$$

and $v\in H^1(M^*;Z_2)$ is the first Stiefel-Whitney class of the double covering $M\times M-\Delta\to M^*$.

COROLLARY. *In addition to the conditions of the above proposition, we assume that $H_1(M;Z_2)=0$. Then we have*

$$[M\subset R^{2n-2}]=H^{2n-3}(M^*;Z)\times\operatorname{Coker}\Theta^{2n-3}.$$

PROOF. Since $H_1(M;Z_2)=0$, we have $H_1(M\times M,\Delta;Z_2)=0$ by the exact sequence of the pair $(M\times M,\Delta)$ and so $H^{2n-1}(M\times M-\Delta;Z_2)=H_1(M\times M,\Delta;Z_2)=0$ by the Poincaré duality. Therefore, the Thom-Gysin exact sequence of the double covering $M\times M-\Delta\to M^*$:

$$\cdots\to H^{2n-1}(M\times M-\Delta;Z_2)\to H^{2n-1}(M^*;Z_2)\to H^{2n}(M^*;Z_2)\quad (=0)$$

shows that $H^{2n-1}(M^*;Z_2)=0$ and we have the desired result by the above proposition.

§ 6. Remarks on the cohomology of $(RP^n)^*$

Let $G_{n+1,2}$ be the Grassmann manifold of 2-planes in R^{n+1}. By [2, Th. 11], the mod 2 cohomology of $G_{n+1,2}$ is given by

$$H^*(G_{n+1,2}; Z_2) = Z_2[x, y]/(a_n, a_{n+1}),$$

where $\deg x=1$, $\deg y=2$ and $a_r = \sum_i \binom{r-i}{i} x^{r-2i} y^i$ $(r=n,\ n+1)$.

S. Feder [4], [5] and D. Handel [7] investigated the mod 2 cohomology of the reduced symmetric product $(RP^n)^*$ of the n-dimensional real projective space RP^n and they showed that

(6.1) *$H^*((RP^n)^*; Z_2)$ has $\{1, v\}$ as basis of $H^*(G_{n+1,2}; Z_2)$*-module, *where $v \in H^1((RP^n)^*; Z_2)$ is the first Stiefel-Whitney class of the double covering $RP^n \times RP^n - \Delta \to (RP^n)^*$ and there are the relations*

$$v^2 = vx,\ Sq^1 y = xy,\ \text{and}\ x^{2^{r+1}-1} = 0 \quad \text{for } n = 2^r + s,\ 0 \leq s < 2^r.$$

We study $H^*((RP^n)^*; Z)$ for even n. According to [7, (3.4)], there exists a fibration

$$V_{n+1,2} \longrightarrow SZ_{n+1,2} \longrightarrow BG,$$

such that $V_{n+1,2}$ is the Stiefel manifold of 2-frames in R^{n+1}, $SZ_{n+1,2}$ is a $(2n-1)$-dimensional closed manifold having the homotopy type of $(RP^n)^*$ and BG is the classifying space of a group G of order 8 (as a matter of fact, G is the dihedral group D_4). Let p be an odd prime. The E_2-term of the mod p cohomology spectral sequence of the above fibration is given by

$$E_2^{s,t} = H^s(BG; \underline{H}^t(V_{n+1,2}; Z_p)),$$

which is the cohomology with local coefficients $\{H^t(V_{n+1,2}; Z_p)\}$. Since $H^*(V_{n+1,2}; Z_p) = H^*(S^{2n-1}; Z_p)$ for even n by [1, (10.5)], we have

$$E_2^{s,t} = \begin{cases} H^s(BG; \underline{H}^0(V_{n+1,2}; Z_p)) & \text{for } t=0 \\ H^s(BG; \underline{H}^{2n-1}(V_{n+1,2}; Z_p)) & \text{for } t=2n-1 \\ 0 & \text{for } t \neq 0,\ 2n-1. \end{cases}$$

Since the action of $\pi_1(BG)$ on $H^0(V_{n+1,2}; Z_p)$ is trivial and $H^i(BG; Z_p)=0$ for $i>0$ by [3, Chap. 12, Cor. 2.7], we have

$$E_2^{s,0} = H^s(BG; Z_p) = \begin{cases} Z_p & s=0 \\ 0 & s \neq 0. \end{cases}$$

These imply that $H^s((RP^n)^*; Z_p)=0$ for $0<s<2n-1$ and so

(6.2) *the orders of elements of* $H^s((RP^n)^*; Z)$ *for* $0<s<2n-1$ *are powers of* 2.

Using the above facts, we determine the groups $H^{2n-3}((RP^n)^*; Z)$ and $\rho_2 H^{2n-4}((RP^n)^*; Z)$. Let $n=2^r+s$, $0<s<2^r$ and s be even. By (6.1) and the Poincaré duality for the manifold $SZ_{n+1,2}$,

(6.3) *the* mod 2 *cohomology groups* $H^t((RP^n)^*; Z_2)$ *for* $2n-4\leq t\leq 2n-1$ *are given as follows*:

t	$H^t((RP^n)^*; Z_2)$	basis
$2n-1$	Z_2	$vx^{2^{r+1}-2}y^s$
$2n-2$	Z_2+Z_2	$vx^{2^{r+1}-3}y^s$, $x^{2^{r+1}-2}y^s$
$2n-3$	$Z_2+Z_2+Z_2$	$vx^{2^{r+1}-4}y^s$, $x^{2^{r+1}-3}y^s$, $vx^{2^{r+1}-2}y^{s-1}$
$2n-4$	$Z_2+Z_2+Z_2+Z_2$	$vx^{2^{r+1}-5}y^s$, $x^{2^{r+1}-4}y^s$, $vx^{2^{r+1}-3}y^{s-1}$, $x^{2^{r+1}-2}y^{s-1}$

Consider the exact seuqence associated with $0\to Z\xrightarrow{\times 2}Z\xrightarrow{\rho_2}Z_2\to 0$:

$$\cdots\to H^{2n-4}((RP^n)^*; Z)\xrightarrow{\rho_2}H^{2n-4}((RP^n)^*; Z_2)\xrightarrow{\beta_2}H^{2n-3}((RP^n)^*; Z)\xrightarrow{\times 2}$$
$$H^{2n-3}((RP^n)^*; Z)\xrightarrow{\rho_2}H^{2n-3}((RP^n)^*; Z_2)\xrightarrow{\beta_2}H^{2n-2}((RP^n)^*; Z)\to\cdots,$$

where β_2 is the Bockstein homomorphism. By simple calculations, we have the following relations for the elements of $H^{2n-3}((RP^n)^*; Z_2)$ by (6.1):

$$Sq^1(vx^{2^{r+1}-4}y^s)=vx^{2^{r+1}-3}y^s,\quad Sq^1(x^{2^{r+1}-3}y^s)=x^{2^{r+1}-2}y^s,$$
$$vx^{2^{r+1}-2}y^{s-1}=Sq^1(vx^{2^{r+1}-3}y^{s-1})=\rho_2\beta_2(vx^{2^{r+1}-3}y^{s-1}).$$

These imply that $\rho_2 H^{2n-3}((RP^n)^*; Z)$ is Z_2 generated by $vx^{2^{r+1}-2}y^{s-1}$. Hence we have

(6.4) $H^{2n-3}((RP^n)^*; Z)=Z_2$ *generated by* $\beta_2(vx^{2^{r+1}-3}y^{s-1})$

by (6.2) and the above exact sequence.

This shows that $\rho_2\colon H^{2n-3}((RP^n)^*; Z)\to H^{2n-3}((RP^n)^*; Z_2)$ is a monomorphism. Furthermore $\rho_2 H^{2n-4}((RP^n)^*; Z)=\operatorname{Ker}\beta_2=\operatorname{Ker}(Sq^1\colon H^{2n-4}((RP^n)^*; Z_2)\to H^{2n-3}((RP^n)^*; Z_2))$, because $Sq^1=\rho_2\beta_2$. On the other hand, we have the relations:

$$Sq^1(vx^{2^{r+1}-5}y^s)=0,\qquad Sq^1(x^{2^{r+1}-4}y^s)=0,$$
$$Sq^1(vx^{2^{r+1}-3}y^{s-1})=vx^{2^{r+1}-2}y^{s-1},\quad Sq^1(x^{2^{r+1}-2}y^{s-1})=0.$$

Therefore, by (6.3), we have
(6.5) $\rho_2 H^{2n-4}((RP^n)^*; Z) = Z_2 + Z_2 + Z_2$ *generated by* $\{vx^{2^{r+1}-5}y^s, x^{2^{r+1}-4}y^s, x^{2^{r+1}-2}y^{s-1}\}$.

§ 7. Proof of Theorem B

We now prove Theorem B in § 1.

The existence of embeddings of RP^n in R^{2n-2} is shown in [7, Th. 4.1] and [10, Th. 7.2.2]. To prove that any two embeddings of RP^n in R^{2n-2} are isotopic, we apply the proposition in § 5 for $M = RP^n$, where the homomorphisms

$$\Theta^i: H^{i-1}((RP^n)^*; Z) \longrightarrow H^{i+1}((RP^n)^*; Z_2) \quad \textit{for } i = 2n-2, 2n-3,$$
$$\Gamma: H^{2n-3}((RP^n)^*; Z_2) \longrightarrow H^{2n-1}((RP^n)^*; Z_2)$$

are defined by $\Theta^i(a) = Sq^2\rho_2 a + \rho_2 av^2$ and $\Gamma(b) = Sq^2 b + bv^2$. We see that Θ^{2n-2} is a monomorphism by (6.4) and the following relations:

$$\Theta^{2n-2}(\beta_2(vx^{2^{r+1}-3}y^{s-1})) = Sq^2(vx^{2^{r+1}-2}y^{s-1}) + vx^{2^{r+1}-2}y^{s-1}v^2$$
$$= vx^{2^{r+1}-2}y^s \neq 0 \ (by\ (6.3)).$$

Also, the equation $\Gamma(vx^{2^{r+1}-2}y^{s-1}) = vx^{2^{r+1}-2}y^s$ and (6.3) imply that Γ is an epimorphism. Consider the homomorphism $\Theta': \rho_2 H^{2n-4}((RP^n)^*; Z) \to H^{2n-2}((RP^n)^*; Z_2)$ defined by $\Theta'(a) = Sq^2 a + av^2$. Then we have the relations

$$\Theta'(x^{2^{r+1}-2}y^{s-1}) = x^{2^{r+1}-2}y^s,$$
$$\Theta'(x^{2^{r+1}-4}y^s) = vx^{2^{r+1}-3}y^s + \binom{s}{2}x^{2^{r+1}-2}y^s.$$

These and (6.3), (6.5) show that Θ' is an epimorphism, and so is $\Theta^{2n-3} = \Theta'\rho_2$. This completes the proof of Theorem B.

References

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