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The Enumeration of Liftings in Fibrations and the Embedding Problem II

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Introduction

The present paper is continued from the first part [27] with the same title. In Part I, we studied the enumeration problem for liftings in fibrations which are the compositions of two twisted principal fibrations, the enumeration problem for cross sections of $(n-1)$-sphere bundles over CW-complexes of dimension less than $n+1$, and the enumeration problem for embeddings of n-dimensional closed differentiable manifolds in the real $(2n-1)$-space R^{2n-1}. As an application, we determined the cardinality of the set of isotopy classes of embeddings of the n-dimensional real projective spaces RP^n in R^{2n-1}.

The organization of the present paper, which is divided into two chapters, is analogous to that of Part I. In Chapter IV, we study the enumeration problem for liftings in fibrations which are the compositions $H \xrightarrow{h} T \xrightarrow{q} E \xrightarrow{p} D$ of three twisted principal fibrations, and with some assumption we obtain in Theorem D of § 16 the formula for determining the set $[X, H]_D$ of homotopy classes of liftings of a map $u: X \to D$. This is a generalization of a theorem of Y. Nomura [23, Theorem 2.4] and on the other hand, an extension of Theorem A in § 2 of Part I, and further a version of a theorem of J. F. McClendon [13, Theorem 5.1], where the stability conditions are woven. Chapter V is as follows. In the beginning, we construct the fourth stage Postnikov factorization of the universal S^{n-1}-bundle $p: BO(n-1) \to BO(n)$, which is continued from § 7. This factorization is a composition of three twisted principal fibrations which satisfies the assumption of Theorem D. Next, applying Theorem D, we enumerate non-zero cross sections of n-plane bundles over CW-complexes of dimension less than $n+2$ in Theorem E of § 18. This is an extension of Theorem B of § 9. Lastly, as an application of Theorem E to the enumeration problem of embeddings, we have the following theorem in § 19.

THEOREM F. *Let $n \geq 5$ and let $n \neq 2^r + 2^s$ $(r \geq s > 0)$. Then the n-dimensional complex projective space CP^n is embedded in the real $(4n-3)$-space R^{4n-3} and there are countably many distinct isotopy classes of embeddings of CP^n in R^{4n-3}.*

Chapter IV. The enumeration of liftings in certain fibrations

§ 13. The situation and the preliminaries

Let B, C and A be H-groups with base point $*$ and let $\phi(G): \pi(G) \to \mathrm{Homeo}(G, *)$ $(G = B, C, A)$ be homomorphisms such that they satisfy the assumption (1.2). Then there are fiber bundles

$$q_G: L(G) = L_{\phi(G)}(G) \longrightarrow K(G) = K(\pi(G), 1) \qquad (G = B, C, A),$$

with fiber G and with canonical cross sections s_G, which are constructed in (1.1).

In this chapter, we consider the following situation:

(13.1)
$$\begin{tikzcd}
& H \arrow[d, "h"] & & \\
& T \arrow[r, "\sigma"] \arrow[d, "q"] & L(A) \arrow[r, "q_A"] & K(A) \\
& E \arrow[r, "\rho"] \arrow[d, "p"] & L(C) \arrow[r, "q_C"] & K(C) \\
X \arrow[uur, "w"] \arrow[ur, "v"] \arrow[r, "u"] & D \arrow[r, "\theta"] & L(B) \arrow[r, "q_B"] & K(B).
\end{tikzcd}$$

Here, $p: E \to D$, $q: T \to E$ and $h: H \to T$ are the twisted principal fibrations with classifying maps $\theta: D \to L(B)$, $\rho: E \to L(C)$ and $\sigma: T \to L(A)$, respectively, and moreover it is assumed that there exist $\bar{\rho}: D \to K(C)$ and $\bar{\sigma}: D \to K(A)$ satisfying

(13.2)
$$q_C \rho = \bar{\rho} p, \qquad q_A \sigma = \bar{\sigma} p q.$$

Let $u: X \to D$ be a given map of a CW-complex X to D. Then, the purpose of this chapter is the investigation of the set of homotopy classes of liftings of u to H, that is, the set $[X, H]_D$ of D-homotopy classes of D-maps of the D-space (X, u) to the D-space (H, pqh).

For the neatness of the description, we assume that the H-groups C and A are topological groups in the rest of this chapter. Further, for the simplicity,

$$n_G: L(G) \times_{K(G)} L(G) \longrightarrow L(G), \qquad ^{-1}: L(G) \longrightarrow L(G) \quad (G = C, A)$$

denote the $K(G)$-maps $\mu_{\phi(G)}$ and $\nu_{\phi(G)}$ of (1.3) induced from the multiplication and the inverse of G, respectively.

Let $\lambda \vee \mu$ denote the join of two paths λ and μ with $\lambda(1) = \mu(0)$ and let

$$m_B: \Omega_{K(B)} L(B) \times_{K(B)} E \longrightarrow E,$$

$$(13.3)\qquad \begin{aligned} m_C&: \Omega_{K(C)}L(C)\times_{K(C)} T \longrightarrow T,\\ m_A&: \Omega_{K(A)}L(A)\times_{K(A)} H \longrightarrow H,\end{aligned}$$

be the maps defined by $m_G(\lambda, (x, \mu))=(x, \lambda\vee\mu)$ for $G=B, C, A$, (cf. (1.7)).

By using the $K(C)$-maps

$$(13.4)\qquad \rho_1: (\Omega_{K(B)}L(B)\times_{K(B)} E, E) \longrightarrow (L(C), K(C))$$

of (2.3) and

$$(13.5)\qquad \rho'_1: (\Omega^2_{K(B)}L(B)\times_{K(B)} E, E) \longrightarrow (\Omega_{K(C)}L(C), K(C))$$

defined by $\rho'_1(\lambda, x)(t)=\rho_1(\lambda(t), x)$, the homomorphism

$$(13.6)\qquad \Delta_p(\rho, [v]): [X, \Omega^2_{K(B)}L(B)]_{K(B)} \longrightarrow [X, \Omega_{K(C)}L(C)]_{K(C)}$$

of (4.1) is defined by

$$\Delta_p(\rho, [v])([a]) = [\rho'_1(a, v)],$$

where $v: X\to E$ is a lifting of a fixed map $u: X\to D$. In the same way, the $K(A)$-maps

$$(13.7)\qquad \begin{aligned} \sigma_1&: (\Omega_{K(C)}L(C)\times_{K(C)} T, T) \longrightarrow (L(A), K(A)),\\ \sigma'_1&: (\Omega^2_{K(C)}L(C)\times_{K(C)} T, T) \longrightarrow (\Omega_{K(A)}L(A), K(A))\end{aligned}$$

are defined by

$$\sigma_1(\lambda, x) = n_A(\sigma m_C(\lambda, x), [\sigma m_C(c_{\lambda(0)}, x)]^{-1}),$$

$$\sigma'_1(\mu, y)(t) = \sigma_1(\mu(t), y),$$

and, for a lifting $w: X\to T$ of v, the homomorphism

$$(13.8)\qquad \Delta_q(\sigma, [w]): [X, \Omega^2_{K(C)}L(C)]_{K(C)} \longrightarrow [X, \Omega_{K(A)}L(A)]_{K(A)}$$

is defined by

$$\Delta_q(\sigma, [w])([b]) = [\sigma'_1(b, w)].$$

In §4, we show that $n'_G: \Omega_{K(G)}L(G)\times_{K(G)} L(G)\to\Omega^*_{K(G)}L(G)$ defined by $n'_G(\lambda, x)(t)=n_G(\lambda(t), x)$ is a $K(G)$-homeomorphism for $G=C, A$ and the map n'_G induces a bijection

$$(13.9)\qquad n'_{G*}: [X, \Omega_{K(G)}L(G)]_{K(G)}\times[X, L(G)]_{K(G)} \xrightarrow{\approx} [X, \Omega^*_{K(G)}L(G)]_{K(G)} \qquad (G = C, A)$$

for any $K(G)$-space X. Let

$$\text{(13.10)} \qquad \rho' : \Omega_D^* E \longrightarrow \Omega_{K(C)}^* L(C), \qquad \sigma' : \Omega_E^* T \longrightarrow \Omega_{K(A)}^* L(A),$$

be the maps defined by

$$\rho'(\lambda)(t) = \rho(\lambda(t)), \qquad \sigma'(\mu)(t) = \sigma(\mu(t)).$$

Then the following diagrams are commutative by (13.2):

$$\text{(13.11)} \qquad \begin{array}{ccc} \Omega_D^* E & \xrightarrow{\rho'} & \Omega_{K(C)}^* L(C) \\ {\scriptstyle pr_E}\downarrow & & \downarrow{\scriptstyle q_C r} \\ D & \xrightarrow{\bar{\rho}} & K(C), \end{array} \qquad \begin{array}{ccc} \Omega_E^* T & \xrightarrow{\sigma'} & \Omega_{K(A)}^* L(A) \\ {\scriptstyle qr_T}\downarrow & & \downarrow{\scriptstyle q_A r} \\ E \xrightarrow{p} D & \xrightarrow{\bar{\sigma}} & K(A). \end{array}$$

where $r_E: \Omega_D^* E \to E$, $r_T: \Omega_E^* T \to T$ and $r: \Omega_{K(G)}^* L(G) \to L(G)$ are the evaluation maps as in Lemma 3.3. Consider the maps

$$m'_B: \Omega_{K(B)}^2 L(B) \times_{K(B)} E \longrightarrow \Omega_D^* E, \qquad m'_C: \Omega_{K(C)}^2 L(C) \times_{K(C)} T \longrightarrow \Omega_E^* T,$$

defined by $m'_G(\lambda, x)(t) = m_G(\lambda(t), x)$ for $G = B, C$. Then, for any CW-complex X, there are two bijections (see §4)

$$\text{(13.12)} \qquad \begin{aligned} m'_{B*}&: [X, \Omega_{K(B)}^2 L(B)]_{K(B)} \times [X, E]_D \xrightarrow{\approx} [X, \Omega_D^* E]_D \\ &\qquad\qquad\qquad\qquad \text{for any} \quad u: X \longrightarrow D, \\ m'_{C*}&: [X, \Omega_{K(C)}^2 L(C)]_{K(C)} \times [X, T]_E \xrightarrow{\approx} [X, \Omega_E^* T]_E \\ &\qquad\qquad\qquad\qquad \text{for any} \quad v: X \longrightarrow E. \end{aligned}$$

Using (13.9–12) and Lemma 4.2. (2), we have the following

LEMMA 13.13. *The primary operations $\Delta_p(\rho, \eta)$ of* (13.6) *for $\eta \in [X, E]_D$ and $\Delta_q(\sigma, \xi)$ of* (13.8) *for $\xi \in [X, T]_E$ are determined uniquely by the relations*

$$\rho'_* m'_{B*}(\beta, \eta) = n'_{C*}(\Delta_p(\rho, \eta)(\beta), \rho_* \eta),$$

$$\sigma'_* m'_{C*}(\gamma, \xi) = n'_{A*}(\Delta_q(\sigma, \xi)(\gamma), \sigma_* \xi).$$

§14. The twisted secondary operations

In this section, we shall define the operations $\Phi([w])$: $\operatorname{Ker} \Delta_p(\rho, [v]) \to \operatorname{Coker} \Delta_q(\sigma, [w])$ for any lifting $w: X \to T$ of u and $v = qw$.

The following three lemmas are easily verified and so we omit the proofs.

LEMMA 14.1. *Let $\phi'(C): \pi(C) \to \operatorname{Homeo}(\Omega^* C, *)$ be the homomorphism defined by $\phi'(C)(\alpha)(\lambda)(t) = \phi(C)(\alpha)(\lambda(t))$ and let $L_{\phi'(C)}(\Omega^* C) \to K(C)$ be the fiber* *bundle constructed in* (1.1). *Then there exists a natural $K(C)$-homeomorphism* $\psi\colon \Omega^*_{K(C)}L(C) \xrightarrow{\approx} L_{\phi'(C)}(\Omega^* C)$.

LEMMA 14.2. *There exist natural $K(C)$-homeomorphisms*

$$\Omega^*_{K(C)}P_{K(C)}L(C) \approx P_{K(C)}\Omega^*_{K(C)}L(C) \approx P_{K(C)}L_{\phi'(C)}(\Omega^* C),$$

$$\Omega^*_{K(C)}\Omega_{K(C)}L(C) \approx \Omega_{K(C)}\Omega^*_{K(C)}L(C).$$

LEMMA 14.3. *Let $q'\colon \Omega^*_D T \to \Omega^*_D E$ be the map which is defined by the relation $q'(\lambda)(t) = q(\lambda(t))$. Then, q' is the twisted principal fibration induced from $P_{K(C)}\Omega^*_{K(C)}L(C) = \Omega^*_{K(C)}P_{K(C)}L(C) \to \Omega^*_{K(C)}L(C)$ with classifying map $p'\colon \Omega^*_D E \to \Omega^*_{K(C)}L(C)$.*

The next two lemmas play an important part in the definition of the twisted secondary operation $\Phi([w])$.

LEMMA 14.4. *Let $w \in [X, T]_D$ and let $\beta \in [X, \Omega^*_D E]_D$. If β lies in the image of $q'_*\colon [X, \Omega^*_D T]_D \to [X, \Omega^*_D E]_D$ and if $q_* w = r_{E*}\beta$ in $[X, E]_D$. Then, there is an element $\chi \in [X, \Omega^*_D T]_D$ such that $r_{T*}\chi = w$ and $q'_*\chi = \beta$, where $r_E\colon \Omega^*_D E \to E$ and $r_T\colon \Omega^*_D T \to T$ are the maps of* Lemma 3.3.

PROOF. By the assumption, there is an element χ_0 in $[X, \Omega^*_D T]_D$ such that $q'_*\chi_0 = \beta$. Since $q_* w = q_* r_{T*}\chi_0$ by the assumption, there is an element ω in $[X, \Omega_{K(C)}L(C)]_{K(C)}$ such that $m_{C*}(\omega, r_{T*}\chi_0) = w$ by Lemma 15.2 below. Now, let $\bar{m}_C\colon \Omega_{K(C)}L(C) \times_{K(C)} \Omega^*_D T \to \Omega^*_D T$ be the map defined by the equation $\bar{m}_C(\lambda, \mu)(t) = m_C(\lambda, \mu(t))$. Then the map $\bar{m}_C$ makes the following diagram commutative:

$$\begin{array}{ccccc}
\Omega_{K(C)}L(C) \times_{K(C)} T & \xleftarrow{1 \times r_T} & \Omega_{K(C)}L(C) \times_{K(C)} \Omega^*_D T & \xrightarrow{\pi_2} & \Omega^*_D T \\
\downarrow{\scriptstyle m_C} & & \downarrow{\scriptstyle \bar{m}_C} & & \downarrow{\scriptstyle q'} \\
T & \xleftarrow{r_T} & \Omega^*_D T & \xrightarrow{q'} & \Omega^*_D E.
\end{array}$$

Therefore, the following commutative diagram holds:

$$\begin{array}{ccccc}
[X, \Omega_{K(C)}L(C)]_{K(C)} \times [X, T]_D & \xleftarrow{1 \times r_{T*}} & [X, \Omega_{K(C)}L(C)]_{K(C)} \times [X, \Omega^*_D T]_D & \xrightarrow{\pi_{2*}} & [X, \Omega^*_D T]_D \\
\downarrow{\scriptstyle m_{C*}} & & \downarrow{\scriptstyle \bar{m}_{C*}} & & \downarrow{\scriptstyle q'_*} \\
[X, T]_D & \xleftarrow{r_{T*}} & [X, \Omega^*_D T]_D & \xrightarrow{q'_*} & [X, \Omega^*_D E]_D.
\end{array}$$

Set $\chi = \bar{m}_{C*}(\omega, \chi_0)$. Then, the relations

$$r_{T*}\chi = w, \qquad q'_*\chi = q'_*\bar{m}_{C*}(\omega, \chi_0) = q'_*\chi_0 = \beta$$

follow from the above diagram.

LEMMA 14.5. *Let v: $X\to E$ be a lifting of u and let s_E: $E\to\Omega_D^*E$ be the canonical cross section of the fibration r_E: $\Omega_D^*E\to E$ defined by $s_E(x)=c_x$ for any $x\in E$. If $\chi\in[X, \Omega_D^*T]_D$ satisfies $q'_*\chi=[s_Ev]$ in $[X, \Omega_D^*E]_D$, then χ is contained in the image of i_*: $[X, \Omega_E^*T]_D\to[X, \Omega_D^*T]_D$, where i: $\Omega_E^*T\to\Omega_D^*T$ is the natural inclusion.*

PROOF. This is a simple application of the homotopy lifting property.

Using the preparation made above, we now construct the twisted secondary operation

$$(14.6) \qquad \Phi([w])\colon \operatorname{Ker}\Delta_p(\rho, [v]) \longrightarrow \operatorname{Coker}\Delta_q(\sigma, [w]).$$

Here w: $X\to T$ is a lifting of u: $X\to D$ and $v=qw$ and $\Delta_p(\rho, [v])$, $\Delta_q(\sigma, [w])$ are the homomorphisms of (13.6), (13.8).

Let $\tau\in\operatorname{Ker}\Delta_p(\rho, [v])$. Since v has a lifting w, the relation $\rho_*[v]=0$ holds. Hence the relation $\rho'_*m'_{B*}(\tau, [v])=0$ follows from Lemma 13.13. This relation and Lemma 14.3 state that $m'_{B*}(\tau, [v])$ lies in the image of q'_*: $[X, \Omega_D^*T]_D\to[X, \Omega_D^*E]_D$. The equation $r_{E*}m'_{B*}(\tau, [v])=[v]=q_*[w]$ follows immediately. Therefore by Lemma 14.4, there is an element χ in $[X, \Omega_D^*T]_D$ such that

$$q'_*\chi = m'_{B*}(\tau, [v])\in[X, \Omega_D^*E]_D, \qquad r_{T*}\chi = [w]\in[X, T]_D.$$

By (13.9), the element $\sigma'_*\chi$ in $[X, \Omega_{K(A)}^*L(A)]_{K(A)}$ is described uniquely in the form $\sigma'_*\chi=n'_{A*}(\alpha, \beta)$ for some $\alpha\in[X, \Omega_{K(A)}L(A)]_{K(A)}$ and $\beta\in[X, L(A)]_{K(A)}$. Since $\beta=r_*n'_{A*}(\alpha, \beta)=r_*\sigma'_*\chi=\sigma_*r_{T*}\chi=\sigma_*[w]$, we have

$$\sigma'_*\chi = n'_{A*}(\alpha, \sigma_*[w]) \qquad \text{for some} \quad \alpha\in[X, \Omega_{K(A)}L(A)]_{K(A)}.$$

By the use of the element α of this equation, we define $\Phi([w])\tau$ by

$$(14.7) \qquad \Phi([w])\tau = \alpha+\operatorname{Im}\Delta_q(\sigma, [w])\in[X, \Omega_{K(A)}L(A)]_{K(A)}/\operatorname{Im}\Delta_q(\sigma, [w]).$$

Suppose that another element χ' in $[X, \Omega_D^*T]_D$ satisfies the relations

$$q'_*\chi' = m'_{B*}(\tau, [v])\in[X, \Omega_D^*E]_D, \qquad r_{T*}\chi' = [w]\in[X, T]_D,$$

and suppose that

$$\sigma'_*\chi' = n'_{A*}(\alpha', \sigma_*[w]) \qquad \text{for some} \quad \alpha'\in[X, \Omega_{K(A)}L(A)]_{K(A)}.$$

We may choose χ and χ' such that $r_T\chi=r_T\chi'=w$ as maps because r_T: $\Omega_D^*T\to T$ is a fibration by Lemma 3.3. Then $\chi'\vee\chi^{-1}$ makes sense and $q'_*(\chi'\vee\chi^{-1})=[s_Ev]$ in $[X, \Omega_D^*E]_D$. Therefore, $\chi'\vee\chi^{-1}\in[X, \Omega_E^*T]_D$ by Lemma 14.5 and this is considered as an element in $[X, \Omega_E^*T]_E$, where X is considered as an E-space with the map v: $X\to E$. Hence (13.9) implies that $\chi'\vee\chi^{-1}=m'_{C*}(\gamma, [w])$ for $[w]\in[X,$

$T]_E$ and some $\gamma \in [X, \Omega^2_{K(C)}L(C)]_{K(C)}$. Further, Lemma 13.13 states

$$\sigma'_* m'_{C\bullet}(\gamma, [w]) = n'_{A\bullet}(\Delta_q(\sigma, [w])\gamma, \sigma_*[w]).$$

Also, it follows that

$$\begin{aligned}\sigma'_* m'_{C\bullet}(\gamma, [w]) &= \sigma'_*(\chi' \vee \chi^{-1}) = \sigma'_*\chi' \vee \sigma'_*\chi^{-1} \\ &= n'_{A\bullet}(\alpha', \sigma_*[w]) \vee n'_{A\bullet}(\alpha^{-1}, \sigma_*[w]) = n'_{A\bullet}(\alpha' \vee \alpha^{-1}, \sigma_*[w]).\end{aligned}$$

Therefore, $\alpha' \vee \alpha^{-1} = \Delta_q(\sigma, [w])\gamma$ by (13.9) and so $\alpha + \mathrm{Im}\,\Delta_q(\sigma, [w]) = \alpha' + \mathrm{Im}\,\Delta_q(\sigma, [w])$. This shows that the definition (14.7) of $\Phi([w])$ is well-defined. By a similar calculation, we see easily that $\Phi([w])$ is a homomorphism.

§15. The actions of groups $[X, \Omega_{K(G)}L(G)]_{K(G)}$ $(G = C, A)$

The maps m_C and m_A in (13.3) determine the actions of groups

$$\begin{aligned} &m_{C\bullet}\colon [X, \Omega_{K(C)}L(C)]_{K(C)} \times [X, T]_D \longrightarrow [X, T]_D \quad \textit{for} \quad u\colon X \longrightarrow D, \\ (15.1)\quad &m_{A\bullet}\colon [X, \Omega_{K(A)}L(A)]_{K(A)} \times [X, H]_D \longrightarrow [X, H]_D \quad \textit{for} \quad u\colon X \longrightarrow D, \\ &m_{A\sharp}\colon [X, \Omega_{K(A)}L(A)]_{K(A)} \times [X, H]_E \longrightarrow [X, H]_E \quad \textit{for} \quad v\colon X \longrightarrow E,\end{aligned}$$

which are given by $m_{C*}([a], [w]) = [m_C(a, w)]$ and so on. For the element w in $[X, T]_D$, $I_u(w)$ denotes the isotropy subgroup of $[X, \Omega_{K(C)}L(C)]_{K(C)}$ at w under the action m_{C*}. Let $\zeta\colon X \to H$ be a lifting of u and let $v = qh\zeta$. Then $I_u([\zeta])$ and $I_v([\zeta])$ denote the isotropy subgroups of $[X, \Omega_{K(A)}L(A)]_{K(A)}$ at $[\zeta] \in [X, H]_D$ and $[\zeta] \in [X, H]_E$ under the actions of $m_{A\bullet}$ and $m_{A\sharp}$, respectively.

From Proposition 1.8, the following lemma holds (see §3).

LEMMA 15.2. *Let $w\colon X \to T$ be a lifting of $u\colon X \to D$ and let $v = qw$. Then, the actions $m_{C\bullet}$, $m_{A\bullet}$ and $m_{A\sharp}$ are transitive on $q_*^{-1}([v])$, $h_*^{-1}([w])$ and $h_\sharp^{-1}([w])$, where $q_*\colon [X, T]_D \to [X, E]_D$, $h_*\colon [X, H]_D \to [X, T]_D$ and $h_\sharp\colon [X, H]_E \to [X, T]_E$.*

The following lemma is proved in Lemmas 3.4–5 and Proposition 4.3.

LEMMA 15.3. *Let $\zeta\colon X \to H$ be a lifting of $u\colon X \to D$ and let $w = h\zeta$ and $v = qw$. Then the following equalities hold:*

$$I_u([w]) = \mathrm{Im}\,\Delta_p(\rho, [v]), \qquad I_v([\zeta]) = \mathrm{Im}\,\Delta_q(\sigma, [w]),$$

where $\Delta_p(\rho, [v])$, $\Delta_q(\sigma, [w])$ are the ones of (13.6), (13.8).

The map $\sigma'\colon \Omega_D^* T \to \Omega^*_{K(A)}L(A)$ defined by $\sigma'(\lambda)(t) = \sigma(\lambda(t))$ and the assumption $q_A\sigma = \bar{\sigma}pq$ of (13.2) give rise to the commutative diagram

$$\begin{array}{ccccc} & & [X, \Omega_D^* T]_D & \xrightarrow{r_{T*}} & [X, T]_D \\ & & \downarrow{\scriptstyle \sigma'_*} & & \downarrow{\scriptstyle \sigma_*} \\ [X, \Omega_{K(A)} L(A)]_{K(A)} & \xrightarrow{i_*} & [X, \Omega_{K(A)}^* L(A)]_{K(A)} & \xrightarrow{r_*} & [X, L(A)]_{K(A)}, \end{array}$$

where $i\colon \Omega_{K(A)} L(A) \to \Omega^*_{K(A)} L(A)$ is the natural inclusion. We say that an element γ in $[X, \Omega_{K(A)} L(A)]_{K(A)}$ is *σ-correlated* to η in $[X, T]_D$ if there exists an element χ in $[X, \Omega_D^* T]_D$ such that $r_{T*}\chi = \eta$ in $[X, T]_D$ and $\sigma'_*\chi = i_*\gamma$. Then, the methods similar to Lemmas 3.4–5 lead to the following

LEMMA 15.4. *Let $\zeta \in [X, H]_D$ and $h_*\zeta = \eta$ in $[X, T]_D$. Then $\gamma \in I_u(\zeta)$ if and only if γ is σ-correlated to η.*

LEMMA 15.5. *Let $\zeta\colon X \to H$ be a lifting of $u\colon X \to D$ and let $w = h\zeta$ and $v = qw$. Then we have*

$$\mathrm{Im}\, \Phi([w]) = I_u([\zeta])/I_v([\zeta]),$$

where $\Phi([w])$ is the operation of (14.6).

PROOF. Let $\gamma \in [X, \Omega_{K(A)} L(A)]_{K(A)}$ lie in the coset $\Phi([w])\tau$. By the definition (14.7) of $\Phi([w])$, there is an element χ in $[X, \Omega_D^* T]_D$ such that

$$q'_*\chi = m'_{B*}(\tau, [v]) \in [X, \Omega_D^* E]_D, \qquad r_{T*}\chi = [w] \in [X, T]_D,$$

$$\sigma'_*\chi = n'_{A*}(\gamma, \sigma_*[w]) \in [X, \Omega^*_{K(A)} L(A)]_{K(A)}.$$

Since w has a lifting to H, it follows that $\sigma_*[w] = 0$ and so $n'_{A*}(\gamma, \sigma_*[w]) = i_*\gamma$. This shows that γ is σ-correlated to $[w]$. Lemma 15.4 implies $\gamma \in I_u([\zeta])$. Conversely, suppose that $\gamma \in [X, \Omega_{K(A)} L(A)]_{K(A)}$ is contained in $I_u([\zeta])$. By Lemma 15.4, γ is σ-correlated to $[w] \in [X, T]_D$, i.e., there is an element χ in $[X, \Omega_D^* T]_D$ such that $\sigma'_*\chi = i_*\gamma$ and $r_{T*}\chi = [w]$ in $[X, T]_D$. From the facts $r_{E*}q'_*\chi = q_*r_{T*}\chi = [v]$ and (13.12), it follows that $q'_*\chi = m'_{B*}(\tau, [v]) \in [X, \Omega_D^* E]_D$ for some τ in $[X, \Omega^2_{K(B)} L(B)]_{K(B)}$. Thus we have $0 = \rho'_* m'_{B*}(\tau, [v])$ by Lemma 14.3 and we have $0 = \rho'_* m'_{B*}(\tau, [v]) = n'_{C*}(\Delta_p(\rho, [v])\tau, \rho_*[v])$ by Lemma 13.13. As a consequence, $\Delta_p(\rho, [v])\tau = 0$ follows from (13.9) and so $\gamma \in \Phi([w])\tau$.

§16. The main theorem of Chapter IV

We say that the composition of fibrations $H \xrightarrow{h} T \xrightarrow{q} E \xrightarrow{p} D$ in the diagram (13.1) is *stable* if there exist two maps

$$d\colon (\Omega_{K(B)} L(B) \times_{K(B)} D, D) \longrightarrow (L(C), K(C)),$$

$$c\colon (\Omega_{K(C)} L(C) \times_{K(C)} D, D) \longrightarrow (L(A), K(A)),$$

such that the diagram

$$\text{(16.1)}\quad\text{(a)}\qquad \begin{array}{ccc} (\Omega_{K(B)}L(B)\times_{K(B)}E,\ E) & \xrightarrow{\rho_1} & (L(C),\ K(C)) \\ \downarrow{\scriptstyle 1\times p} & & \Vert \\ (\Omega_{K(B)}L(B)\times_{K(B)}D,\ D) & \xrightarrow{d} & (L(C),\ K(C)) \end{array}$$

is $K(C)$-homotopy commutative and the diagram

$$\text{(16.1)}\quad\text{(b)}\qquad \begin{array}{ccc} (\Omega_{K(C)}L(C)\times_{K(C)}T,\ T) & \xrightarrow{\sigma_1} & (L(A),\ K(A)) \\ \downarrow{\scriptstyle 1\times pq} & & \Vert \\ (\Omega_{K(C)}L(C)\times_{K(C)}D,\ D) & \xrightarrow{c} & (L(A),\ K(A)) \end{array}$$

is $K(A)$-homotopy commutative, where ρ_1 and σ_1 are the maps in (13.4) and (13.7), (cf. § 2).

Let

$$d'\colon (\Omega^2_{K(B)}L(B)\times_{K(B)}D,\ D) \longrightarrow (\Omega_{K(C)}L(C),\ K(C)),$$

$$c'\colon (\Omega^2_{K(C)}L(C)\times_{K(C)}D,\ D) \longrightarrow (\Omega_{K(A)}L(A),\ K(A)),$$

be the maps defined by the relations (cf. (2.4))

$$d'(\lambda,\ x)(t) = d(\lambda(t),\ x), \qquad c'(\mu,\ y)(t) = c(\mu(t),\ y).$$

Then the diagrams below are $K(G)$-homotopy commutative ($G=C,\ A$):

$$\begin{array}{ccc} (\Omega^2_{K(B)}L(B)\times_{K(B)}E,\ E) & \xrightarrow{\rho'_1} & (\Omega_{K(C)}L(C),\ K(C)) \\ \downarrow{\scriptstyle 1\times p} & & \Vert \\ (\Omega^2_{K(B)}L(B)\times_{K(B)}D,\ D) & \xrightarrow{d'} & (\Omega_{K(C)}L(C),\ K(C)), \end{array}$$

$$\begin{array}{ccc} (\Omega^2_{K(C)}L(C)\times_{K(C)}T,\ T) & \xrightarrow{\sigma'_1} & (\Omega_{K(A)}L(A),\ K(A)) \\ \downarrow{\scriptstyle 1\times pq} & & \Vert \\ (\Omega^2_{K(C)}L(C)\times_{K(C)}D,\ D) & \xrightarrow{c'} & (\Omega_{K(A)}L(A),\ K(A)), \end{array}$$

where ρ'_1 and σ'_1 are the maps defined in (13.5) and (13.7).

Now, for a given map $u\colon X\to D$, we have four functions

$$\text{(16.2)}\qquad \begin{aligned} \Theta &\colon [X,\ \Omega_{K(B)}L(B)]_{K(B)} \longrightarrow [X,\ L(C)]_{K(C)}, \\ \Theta' &\colon [X,\ \Omega^2_{K(B)}L(B)]_{K(B)} \longrightarrow [X,\ \Omega_{K(C)}L(C)]_{K(C)}, \\ \Gamma &\colon [X,\ \Omega_{K(C)}L(C)]_{K(C)} \longrightarrow [X,\ L(A)]_{K(A)}, \\ \Gamma' &\colon [X,\ \Omega^2_{K(C)}L(C)]_{K(C)} \longrightarrow [X,\ \Omega_{K(A)}L(A)]_{K(A)}, \end{aligned}$$

by setting

$$\Theta([a]) = [d(a, u)], \qquad \Theta'([b]) = [d'(b, u)],$$

$$\Gamma([x]) = [c(x, u)], \qquad \Gamma'([y]) = [c'(y, u)],$$

(cf. (2.5)). Then, we have the following results by the consideration given in §5: The functions Θ' and Γ' are homomorphisms and moreover

$$(16.3) \qquad \begin{aligned} \Theta' &= \Delta_p(\rho, [v]) && \text{for any lifting} \quad v\colon X \longrightarrow E \quad of \quad u, \\ \Gamma' &= \Delta_q(\sigma, [w]) && \text{for any lifting} \quad w\colon X \longrightarrow T \quad of \quad v, \end{aligned}$$

where $\Delta_p(\rho, [v])$ and $\Delta_q(\sigma, [w])$ are the ones of (13.6) and (13.8). Further, if u has a lifting $\zeta\colon X \to H$, then Θ and Γ are homomorphisms and

$$(16.4) \qquad \Theta = \rho_* m_{B^*}(\ , [qh\zeta]), \qquad \Gamma = \sigma_* m_{C^*}(\ , [h\zeta]).$$

We, now, turn to the study of the twisted secondary operation

$$\Phi([w])\colon \operatorname{Ker}\Theta' = \operatorname{Ker}\Delta_p(\rho, [v]) \longrightarrow \operatorname{Coker}\Gamma' = \operatorname{Coker}\Delta_q(\sigma, [w])$$

of (14.6) on the assumption that the composition pqh is stable.

LEMMA 16.5. *Let $w, w'\colon X \to T$ be liftings of $v\colon X \to E$ such that both w and w' admit liftings to H. Then $\Phi([w]) = \Phi([w'])$.*

PROOF. Using the diagram (16.1) (b) and the relations

$$\Omega^*_{K(C)}\Omega_{K(C)}L(C) \times_{K(C)} \Omega^*_D T = \Omega^*_D(\Omega_{K(C)}L(C) \times_{K(C)} T),$$

$$\Omega^*_{K(C)}\Omega_{K(C)}L(C) \times_{K(C)} D = \Omega^*_D(\Omega_{K(C)} L(C) \times_{K(C)} D),$$

we have a $K(A)$-homotopy commutative diagram

$$\begin{array}{ccc} (\Omega^*_{K(C)}\Omega_{K(C)}L(C) \times_{K(C)} \Omega^*_D T,\ \Omega^*_D T) & \xrightarrow{\Omega^*(\sigma_1)} & (\Omega^*_{K(A)}L(A),\ K(A)) \\ \downarrow{\scriptstyle 1\times p'q'} & & \| \\ (\Omega^*_{K(C)}\Omega_{K(C)}L(C) \times_{K(C)} D,\ D) & \xrightarrow{\quad\Omega^*(c)\quad} & (\Omega^*_{K(A)}L(A),\ K(A)). \end{array}$$

Since $q'\colon \Omega^*_D T \to \Omega^*_D E$ is the twisted principal fibration with classifying map ρ': $\Omega^*_D E \to \Omega^*_{K(A)}L(C)$ by Lemma 14.3, there exists a map

$$\tilde{m}_C\colon \Omega_{K(C)}\Omega^*_{K(C)}L(C) \times_{K(C)} \Omega^*_D T \longrightarrow \Omega^*_D T,$$

which is defined by $\tilde{m}_C(\lambda, (x, \mu)) = (x, \lambda \vee \mu)$ (cf. (13.3)). In the same way as (13.7), this map determines the map

$$(\sigma')_1\colon (\Omega_{K(C)}\Omega^*_{K(C)}L(C) \times_{K(C)} \Omega^*_D T,\ \Omega^*_D T) \longrightarrow (\Omega^*_{K(A)}L(A),\ K(A)).$$

From the second relation in Lemma 14.2, the relation

$$\Omega^*(\sigma_1) = (\sigma')_1$$

follows immediately, and moreover the following diagram is $K(A)$-homotopy commutative by the commutativity (16.1):

$$\begin{array}{ccc} (\Omega^2_{K(C)}L(C) \times_{K(C)} \Omega_{K(C)}L(C)) \times_{K(C)} D & \xrightarrow{(c',c)} & \Omega_{K(A)}L(A) \times_{K(A)} L(A) \\ \downarrow n'_{\Omega C} \times 1 & & \downarrow n'_A \\ \Omega^*_{K(C)}(\Omega_{K(C)}L(C)) \times_{K(C)} D & \xrightarrow{\Omega^* c} & \Omega^*_{K(A)}L(A), \end{array}$$

where $(c', c)(\lambda, \mu, x) = (c'(\lambda, x), c(\mu, x))$ and $n'_{\Omega C}\colon \Omega_{K(C)}(\Omega_{K(C)}L(C)) \times_{K(C)} \Omega_{K(C)}L(C) \to \Omega^*_{K(C)}(\Omega_{K(C)}L(C))$ is the product induced from the product $n_{\Omega C}\colon \Omega_{K(C)}L(C) \times_{K(C)} \Omega_{K(C)}L(C) \to \Omega_{K(C)}L(C)$.

Suppose that

$$\Phi([w])\tau = \alpha + \operatorname{Im}\Gamma', \quad \Phi([w'])\tau = \alpha' + \operatorname{Im}\Gamma' \qquad \text{for} \quad \tau \in \operatorname{Ker}\Theta'.$$

By the definition, there are two elements $\chi, \chi' \in [X, \Omega^*_D T]_D$ such that

$$q'_*\chi = q'_*\chi' = m'_{B*}(\tau, [v]), \quad r_{T*}\chi = [w], \quad r_{T*}\chi' = [w'],$$

$$\sigma'_*\chi = n'_{A*}(\alpha, \sigma_*[w]) = i_*\alpha, \quad \sigma'_*\chi' = n'_{A*}(\alpha', \sigma_*[w']) = i_*\alpha'.$$

The map $\tilde{m}_C$ mentioned above induces the action $\tilde{m}_{C*}\colon [X, \Omega_{K(C)}\Omega^*_{K(C)}L(C)]_{K(C)} \times [X, \Omega^*_D T]_D \to [X, \Omega^*_D T]_D$ and this action is transitive on $q'^{-1}_*(m'_{B*}(\tau, [v]))$, where $q'_*\colon [X, \Omega^*_D T]_D \to [X, \Omega^*_D E]_D$ (cf. Lemma 15.2 for the action m_{C*}). Therefore we can choose an element $\lambda \in [X, \Omega_{K(C)}\Omega^*_{K(C)}L(C)]_{K(C)}$ such that $\tilde{m}_{C*}(\lambda, \chi) = \chi' \in [X, \Omega^*_D T]_D$. Hence we have

$$\begin{aligned} ((\sigma')_1)_*(\lambda, \chi) &= \sigma'_*(\tilde{m}_{C*}(\lambda, \chi)) \vee [\sigma'_*(\tilde{m}_{C*}(*, \chi))]^{-1} \quad \text{by the definition,} \\ &= \sigma'_*\chi' \vee (\sigma'_*\chi)^{-1} = n'_{A*}(\alpha' \vee \alpha^{-1}, *). \end{aligned}$$

On the other hand,

$$((\sigma')_1)_*(\lambda, \chi) = (\Omega^*(\sigma_1))_*(\lambda, \chi) = (\Omega^* c)_*(1 \times p'q')_*(\lambda, \chi) = (\Omega^* c)(\lambda, u).$$

Since $n'_{\Omega C}\colon \Omega^2_{K(C)}L(C) \times_{K(C)} \Omega_{K(C)}L(C) \to \Omega^*_{K(C)}\Omega_{K(C)}L(C)$ is a weak $K(C)$-homotopy equivalence by [10, Theorem 2.7], there exist two elements $\mu \in [X, \Omega^2_{K(C)}L(C)]_{K(C)}$ and $\nu \in [X, \Omega_{K(C)}L(C)]_{K(C)}$ such that $\lambda = n'_{\Omega C*}(\mu, \nu)$. Hence we have

$$\begin{aligned} ((\sigma')_1)_*(\lambda, \chi) &= (\Omega^* c)_*(\lambda, u) = (\Omega^* c)_*(n'_{\Omega C*}(\mu, \nu), u) \\ &= n'_{A*}(c'_*(\mu, u), c_*(\nu, u)) = n'_{A*}(\Gamma'(\mu), \Gamma(\nu)). \end{aligned}$$

Therefore $\alpha' \vee \alpha^{-1} = \Gamma'(\mu)$ because n'_{A*} is a bijection by (13.9), and we have Lemma 16.5 completely.

Next, we consider the another twisted secondary operation

(16.6) $$\phi\colon \operatorname{Ker}\Theta \longrightarrow \operatorname{Coker}\Gamma \qquad \text{for a fixed} \quad \zeta \in [X, H]_D,$$

where Θ, Γ are the homomorphisms of (16.4). Let $a \in \operatorname{Ker}\Theta$. Then $0=\Theta(a) = p_* m_{B^\bullet}(a, v)$, $v=q_*h_*\zeta$, by (16.4), i.e., $m_{B^\bullet}(a, v) \in [X, E]_D$ lies in the image of $q_*\colon [X, T]_D \to [X, E]_D$. Let $w_a \in [X, T]_D$ be the element such that $q_*w_a = m_{B^\bullet}(a, v)$ and set

$$\phi(a) = \sigma_* w_a + \operatorname{Im}\Gamma \in [X, L(A)]_{K(A)}/\operatorname{Im}\Gamma = \operatorname{Coker}\Gamma.$$

In the same way as [23, § 2], we see that ϕ is well-defined and moreover we have the following

LEMMA 16.7. [23, Lemma 2.3]. *Let $a \in \operatorname{Ker}\Theta$. Then $m_{B^\bullet}(a, v) \in [X, E]_D$ is contained in $q_*h_*[X, H]_D$ if and only if $\phi(a)=0$.*

By Lemma 16.5, it is easily seen that if w runs through the elements of $h_*[X, H]_D \cap [X, T]_D$ then the twisted secondary operation $\Phi(w)$ of (14.6) depends only on $q_*w \in [X, E]_D$. Therefore we set

(16.8) $$\Phi_a = \Phi(w) \qquad \text{for} \quad a \in \operatorname{Ker}\phi,$$

where $q_*w = m_{B^\bullet}(a, v)$, $v=q_*h_*\zeta$.

The following theorem is the main theorem in this chapter.

THEOREM D. *Suppose that the composition of fibrations $H \xrightarrow{h} T \xrightarrow{q} E \xrightarrow{p} D$ in the diagram* (13.1) *is stable by the maps c and d in* (16.1). *Let X be a CW-complex and let $u\colon X \to D$ admit a lifting $\zeta\colon X \to H$. Then the set $[X, H]_D$ of homotopy classes of liftings of u to H is given by*

$$[X, H]_D = \bigcup_{a \in \operatorname{Ker}\phi} (\operatorname{Ker}\Gamma/\operatorname{Im}\Theta') \times \operatorname{Coker}\Phi_a,$$

where Γ, Θ', ϕ and Φ_a are the ones of (16.2), (16.6) *and* (16.8).

PROOF. By Proposition 1.8, there is a bijection

$$m_{B^\bullet}(\ , v)\colon [X, \Omega_{K(B)}L(B)]_{K(B)} \xrightarrow{\approx} [X, E]_D, \qquad (v = q_*h_*\zeta).$$

From Lemma 16.7, it follows that

(16.9) $$[X, H]_D = \bigcup_{a \in \operatorname{Ker}\phi} h_*^{-1} q_*^{-1} m_{B^\bullet}(a, v).$$

Let w_a be a lifting of $m_{B^\bullet}(a, v)$ to T such that w_a has a lifting to H. The correspondence which associates with an element $\tau \in [X, \Omega_{K(C)}L(C)]_{K(C)}$ the element $m_{C^\bullet}(\tau, w_a)$ induces a bijection

$$[X, \Omega_{K(C)}L(C)]_{K(C)}/I_u(w_a) = \operatorname{Coker}\Theta' \xrightarrow{\approx} q_*^{-1}(m_{B^\bullet}(a, v)),$$

by Lemmas 15.2–3 and (16.3). Since $m_{C^\bullet}(\tau, w_a)$ has a lifting to H if and only if $\tau \in \operatorname{Ker}\Gamma$ by (16.4), we have a bijection

$$\operatorname{Ker}\Gamma/\operatorname{Im}\Theta' \xrightarrow{\approx} q_*^{-1}(m_{B^\bullet}(a, v)) \cap h_*[X, H]_D.$$

For any $\tau \in \operatorname{Ker}\Gamma$, let $\zeta_{a,\tau}$ be a lifting of $m_{C^\bullet}(\tau, w_a)$ to H. Then, by Lemma 15.2, the correspondence which associates with $\gamma \in [X, \Omega_{K(A)}L(A)]_{K(A)}$ the element $m_{A^\bullet}(\gamma, \zeta_{a,\tau})$ induces a bijection

$$[X, \Omega_{K(A)}L(A)]_{K(A)}/I_u(\zeta_{a,\tau}) \xrightarrow{\approx} h_*^{-1}(m_{C^\bullet}(\tau, w_a)).$$

Therefore

$$\begin{aligned} h_*^{-1}(m_{C^\bullet}(\tau, w_a)) &= ([X, \Omega_{K(A)}L(A)]_{K(A)}/I_{v'}(\zeta_{a,\tau}))/(I_u(\zeta_{a,\tau})/I_{v'}(\zeta_{a,\tau})), \\ &= \operatorname{Coker}\Phi(m_{C^\bullet}(\tau, w_a)) \quad \text{by Lemma 15.3 and Lemma 15.5}, \\ &= \operatorname{Coker}\Phi_a \quad \text{by (16.8)}, \end{aligned}$$

where $[v'] = q_* h_* \zeta_{a,\tau} = m_{B^\bullet}(a, v)$. This equation and (16.9–10) complete the proof of the theorem.

Chapter V. The enumeration of cross sections of sphere bundles and the enumeration of embeddings of complex projective spaces

§17. The fourth stage Postnikov factorization of $p: BO(n-1) \longrightarrow BO(n)$

The third stage Postnikov factorization of the universal S^{n-1}-bundle p: $BO(n-1) \to BO(n)$ is constructed in §7 (cf. also [22, §6]) and is given as follows:

$$\begin{tikzcd} BO(n-1) \arrow[r, "q_2"] \arrow[dr, "q_1"'] \arrow[ddr, "p"'] & T \arrow[d, "p_2"] & & \\ & E \arrow[r, "\rho"] \arrow[d, "p_1"] & K(Z_2, n+1) & \\ & BO(n) \arrow[r, "W"] & L_\phi(Z, n) \arrow[r] & K(Z_2, 1) = K. \end{tikzcd}$$

Here $\phi: \pi_1(K(Z_2, 1)) = Z_2 \to \operatorname{Aut}(Z)$ is a non-trivial homomorphism, $L_\phi(Z, n) \to K(Z_2, 1) = K$ is the fiber bundle constructed in (1.1), $p_1: E \to BO(n)$ is the twisted principal fibration with classifying map W, $p_2: T \to E$ is the principal fibration with classifying map ρ, and q_2 is an $(n+1)$-equivalence.

This section is continued from §7. Hence, we use λ instead of $\lambda_{BO(n-1)}$ and the other notations in §7 will be used freely if no confusion can arise.

We can assume that q_2: $BO(n-1)\to T$ is a fibration. Then its fiber F' satisfies

$$\pi_i(F') = \begin{cases} 0 & i \leq n \\ \pi_i(S^{n-1}) & i \geq n+1. \end{cases}$$

Now, $\pi_{n+1}(F')=\pi_{n+1}(S^{n-1})=Z_2$, and the generator of $H^{n+1}(F'; Z_2)=Z_2$ is transgressive in the fibration q_2: $BO(n-1)\to T$. Let $\sigma\in H^{n+2}(T; Z_2)$ denote the transgression image and let

$$p_3: H \longrightarrow T$$

be the principal fibration with classifying map σ: $T\to K(Z_2, n+2)$. Then q_2: $BO(n-1)\to T$ admits a lifting q_3: $BO(n-1)\to H$. Moreover, q_3 becomes an $(n+2)$-equivalence.

To characterize the map σ: $T\to K(Z_2, n+2)$, we prepare two lemmas.

LEMMA 17.1. (cf. [28, Lemma 3.3]). *Let* $n\geq 5$. *For the homomorphisms*

$$H^*(BO(n-1); Z_2) \xleftarrow{q_1^*} H^*(E; Z_2) \xrightarrow{p_2^*} H^*(T; Z_2),$$

the following two conditions hold:

(a) $\mathrm{Ker}\, p_2^* \supset \mathrm{Ker}\, q_1^*$ *in dimension* $n+2$.

(b) q_1^* *is surjective in dimension* $n+2$.

PROOF. By (7.4), there is an isomorphism

$$\mu^*: H^{n+2}(E; Z_2)\cap \mathrm{Ker}\, q_1^* \xrightarrow{\cong} H^{n+2}(\Omega_K L_\phi(Z, n)\times_K BO(n-1); Z_2)\cap \mathrm{Ker}\, s^* \cap \mathrm{Ker}\, \tau_1.$$

By Lemma 6.2, any element $x\in H^{n+2}(\Omega_K L_\phi(Z, n)\times_K BO(n-1); Z_2)$ is described uniquely in the form

$$x = \pi_2^* b + \lambda(\varepsilon_1\pi_2^* w_3 + \varepsilon_2\pi_2^* w_2 w_1 + \varepsilon_3\pi_2^* w_1^3) + \varepsilon_4 Sq^2\lambda\pi_2^* w_1 + \varepsilon_5 Sq^3\lambda.$$

If $x\in \mathrm{Ker}\, s^*$, then $b=0$. Since $\tau_1 Sq^i = Sq^i\tau_1$ and τ_1 is an $H^*(BO(n); Z_2)$-homomorphism by [19, §3], there are relations

$$\tau_1(\lambda\pi_2^* w_3) = w_n w_3, \qquad \tau_1(\lambda\pi_2^*(w_2 w_1)) = w_n w_2 w_1,$$

$$\tau_1(\lambda\pi_2^* w_1^3) = w_n w_1^3, \qquad \tau_1(Sq^3\lambda) = Sq^3 w_n = w_n w_3,$$

$$\tau_1((Sq^2\lambda)(\pi_2^* w_1)) = (Sq^2 w_n)w_1 = w_n w_2 w_1,$$

by the equality $\tau_1(\lambda)=w_n$ in § 7 and the formula of *Wu*. Hence we have

$$H^{n+2}(\Omega_K L_\phi(Z, n)\times_K BO(n-1); Z_2) \cap \mathrm{Ker}\, s^* \cap \mathrm{Ker}\,\tau_1 = Z_2+Z_2,$$

generated by $\{\lambda\pi_2^* w_3+Sq^3\lambda,\ \lambda\pi_2^* w_2 w_1+(Sq^2\lambda)(\pi_2^* w_1)\}$. Since λ lies in the image of the mod 2 reduction ρ_2 of $H^{n-1}(\Omega_K L_\phi(Z, n)\times_K BO(n-1); \underline{Z})$ by the definition in § 6, it follows that $0=\rho_2\tilde{\beta}_2\lambda$, which is equal to $Sq^1\lambda+\lambda\pi_2^* w_1$ from [4] and [16], (cf. (12.3)). Therefore we have a relation

$$Sq^1\lambda = \lambda\pi_2^* w_1. \tag{17.2}$$

The relation (17.2) and (7.5) yield two relations

$$\mu^* Sq^1\rho = Sq^1\mu^*\rho = Sq^1(\lambda\pi_2^* w_2+Sq^2\lambda) = \lambda\pi_2^* w_3+Sq^3\lambda,$$

$$\mu^*(\rho p_1^* w_1) = (\lambda\pi_2^* w_2+Sq^2\lambda)\pi_2^* w_1 = \lambda\pi_2^* w_2 w_1+(Sq^2\lambda)(\pi_2^* w_1).$$

Therefore we have

$$H^{n+2}(E; Z_2) \cap \mathrm{Ker}\, q_1^* = Z_2+Z_2 \text{ generated by } \{Sq^1\rho,\ \rho p_1^* w_1\}.$$

This and the equation $p_2^*\rho=0$ imply the statement (a). The statement (b) follows from the fact that p^* is surjective and $p=p_1 q_1$.

Let

$$m_2\colon \Omega K(Z_2, n+1)\times T \longrightarrow T$$

be the action of the fiber $\Omega K(Z_2, n+1)$ of the principal fibration $p_2\colon T\to E$ and let

$$\nu = m_2(1\times q_2)\colon \Omega K(Z_2, n+1)\times BO(n-1) \longrightarrow T.$$

Then this map ν gives rise to the commutative diagram of fibrations

$$\begin{array}{ccc} \Omega K(Z_2, n+1)\times BO(n-1) & \xrightarrow{\nu} & T \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle p_2} \\ BO(n-1) & \xrightarrow{q_1} & E. \end{array} \tag{17.3}$$

Let $s'\colon BO(n-1)\to\Omega K(Z_2, n+1)\times BO(n-1)$ be the canonical cross section. Then there is a relation

$$\nu s' \simeq q_2. \tag{17.4}$$

By the method similar to [28, Corollary 3.4], we have the following

LEMMA 17.5. *The following sequence is exact for $n \geq 5$:*

$$0 \longrightarrow H^{n+2}(T; Z_2) \xrightarrow{\nu^*} H^{n+2}(\Omega K(Z_2, n+1) \times BO(n-1); Z_2)$$
$$\xrightarrow{\tau'_1} H^{n+3}(E; Z_2),$$

where τ'_1 is the transgression associated with (17.3) (*see* [19, § 3]).

THEOREM 17.6. *Let $n \geq 6$. The transgression image $\sigma: T \to K(Z_2, n+2)$ is characterized uniquely by the relation*

$$\nu^*\sigma = \iota' \times w_2 + \iota' \times w_1^2 + Sq^1\iota' \times w_1 + Sq^2\iota' \times 1,$$

where $\iota' \in H^n(K(Z_2, n); Z_2)$ is the fundamental class of $K(Z_2, n)$.

PROOF. The element σ belongs to $\operatorname{Ker} q_2^* \cap H^{n+2}(T; Z_2)$. By Lemma 17.5 and (17.4), ν^* gives an isomorphism

(17.7) $$\nu^*: H^{n+2}(T; Z_2) \cap \operatorname{Ker} q_2^* \xrightarrow{\approx} \operatorname{Ker} s'^* \cap \operatorname{Ker} \tau'_1.$$

By the definition of τ'_1, $\tau'_1(\iota' \times 1) = \tau(\iota')$ holds, where τ is the transgression of $p_2: T \to E$, and $\tau(\iota') = \rho$ because q_2 is the principal fibration with classifying map ρ. Hence it follows

(17.8) $$\tau'_1(\iota' \times 1) = \rho.$$

Any element $x \in H^{n+2}(\Omega K(Z_2, n+1) \times BO(n-1); Z_2)$ is described in the form $x = 1 \times a + \varepsilon_1\iota' \times w_2 + \varepsilon_2\iota' \times w_1^2 + \varepsilon_3 Sq^1\iota' \times w_1 + \varepsilon_4 Sq^2\iota' \times 1$, where $a \in H^{n+2}(BO(n-1); Z_2)$ and $\varepsilon_i = 0$ or 1 for $i = 1, 2, 3, 4$. If $x \in \operatorname{Ker} s'^*$, then $a = 0$. Since μ^* is a monomorphism in dimension $n+3$ on the assumption $n \geq 6$ by (7.4), it follows that $x \in \operatorname{Ker} \tau'_1$ if and only if $\mu^*\tau'_1 x = 0$. Now, τ'_1 is an $H^*(E; Z_2)$-homomorphism and τ'_1 satisfies the relation $Sq^i\tau'_1 = \tau'_1 Sq^i$ by [19, § 3]. Therefore, using the above facts and the relations (7.5), (17.2), (17.8) and the formula of *Wu*, we have

$$\mu^*\tau'_1(\iota' \times w_2) = (\mu^*\rho)w_2 = (Sq^2\lambda)w_2 + \lambda w_2^2,$$

$$\mu^*\tau'_1(\iota' \times w_1^2) = (Sq^2\lambda)w_1^2 + \lambda w_2 w_1^2,$$

$$\mu^*\tau'_1(Sq^1\iota' \times w_1) = \mu^*((Sq^1\rho)w_1) = (Sq^1(Sq^2\lambda + \lambda w_2))w_1$$
$$= (Sq^3\lambda)w_1 + \lambda w_3 w_1,$$

$$\mu^*\tau'_1(Sq^2\iota' \times 1) = (Sq^3\lambda)w_1 + (Sq^2\lambda)(w_2 + w_1^2) + \lambda(w_3 w_1 + w_2 w_1^2 + w_2^2).$$

These relations imply that $H^{n+2}(\Omega K(Z_2, n+1) \times BO(n-1); Z_2) \cap \operatorname{Ker} s'^* \cap \operatorname{Ker} \tau'_1 = Z_2$ generated by $\iota' \times w_2 + \iota' \times w_1^2 + Sq^1\iota' \times w_1 + Sq^2\iota' \times 1$. This result and

(17.7) complete the proof of Theorem 17.6.

Summing up the above arguments, we have

THEOREM 17.9. *The fourth stage Postnikov factorization of the universal* S^{n-1}*-bundle* $p\colon BO(n-1)\to BO(n)$ *for* $n\geq 6$ *is given as follows*:

(17.10)
$$\begin{tikzcd}
BO(n-1) \arrow[r, "q_3"] \arrow[dr, "q_2"] \arrow[ddr, dashed, "q_1"'] \arrow[dddr, "p"'] & H \arrow[d, "p_3"] & \\
& T \arrow[r, "\sigma"] \arrow[d, "p_2"] & K(Z_2, n+2) \\
& E \arrow[r, "\rho"] \arrow[d, "p_1"] & K(Z_2, n+1) \\
& BO(n) \arrow[r, "W"] & L_\phi(Z, n),
\end{tikzcd}$$

where $p_3\colon H\to T$ *is the principal fibration with classifying map* σ *which is characterized by the relation in* Theorem 17.6, $q_3\colon BO(n-1)\to H$ *is an* $(n+2)$*-equivalence and the others give the third stage Postnikov factorization of* $p\colon BO(n-1)\to BO(n)$ *constructed in* Theorem 7.6.

We close this section by verifying the fact that the fourth stage Postnikov factorization (17.10) of $p\colon BO(n-1)\to BO(n)$ is stable in the sense of § 16.

Choose a map

(17.11) $$c\colon (K(Z_2, n)\times BO(n), BO(n)) \longrightarrow (K(Z_2, n+2), *)$$

so as to represent

$$c = \iota'\times(w_2+w_1^2)+Sq^1\iota'\times w_1+Sq^2\iota'\times 1\in H^{n+2}(K(Z_2, n)\times BO(n), BO(n); Z_2)$$

and let

(17.12) $$d\colon (\Omega_K L_\phi(Z, n)\times_K BO(n), BO(n)) \longrightarrow (K(Z_2, n+1), *)$$

be the map of (8.1) which represents the element

$$d = \lambda\pi_2^* w_2+Sq^2\lambda\in H^{n+1}(\Omega_K L_\phi(Z, n)\times_K BO(n), BO(n); Z_2).$$

In Proposition 8.5, we show that the map d satisfies the condition (16.1)(a). In the same way, it is easily seen that the map c of (17.11) satisfies the condition (16.1)(b) and so we have the following

LEMMA 17.13. *The composition of fibrations* $H\xrightarrow{p_3} T\xrightarrow{p_2} E\xrightarrow{p_1} BO(n)$ *of the fourth stage Postnikov factorization* (17.10) *is stable by the maps* c *and* d *of* (17.11–12) *in the sense* (16.1).

§ 18. The enumeration of cross sections of sphere bundles

Let $n \geq 6$ and let ξ be a real n-plane bundle over a CW-complex X of dimension less than $n+2$. If ξ has a non-zero cross section, $cross(\xi)$ denotes the set of (free) homotopy classes of non-zero cross sections of ξ. The space X is considered as a $BO(n)$-space with classifying map $\xi: X \to BO(n)$. Then the relation

$$(18.1) \qquad cross(\xi) = [X, H]_{BO(n)}$$

results from the argument similar to that of § 9.

By Lemma 17.13, there are the four homomorphisms

$$\Theta: H^{n-1}(X; \underline{Z}) = [X, \Omega_K L_\phi(Z, n)]_K \longrightarrow H^{n+1}(X; Z_2) = [X, K(Z_2, n+1)],$$

$$\Theta': H^{n-2}(X; \underline{Z}) \longrightarrow H^n(X; Z_2),$$

$$\Gamma: H^n(X; Z_2) \longrightarrow H^{n+2}(X; Z_2) = 0,$$

$$\Gamma': H^{n-1}(X; Z_2) \longrightarrow H^{n+1}(X; Z_2),$$

of (16.2) by taking the classifying map $\xi: X \to BO(n)$ for u, where $\underline{Z}$ is the local system on X associated with ξ. Then, we have the following relations in the same way as the proofs in § 9 by using (17.11–12):

$$\Theta(a) = Sq^2 \rho_2 a + \rho_2 a w_2(\xi), \quad \Theta'(b) = Sq^2 \rho_2 b + \rho_2 b w_2(\xi),$$

$$\Gamma(x) = x(w_2(\xi) + w_1(\xi)^2) + (Sq^1 x) w_1(\xi) + Sq^2 x,$$

where ρ_2 is the mod 2 reduction and $w_i(\xi)$ is the i-th Stiefel-Whitney class of ξ.

Apply, now, Theorem D to the fourth stage Postnikov factorization (17.10) of the universal S^{n-1}-bundle $p: BO(n-1) \to BO(n)$. Then we have the following theorem, which is an extension of Theorem B in § 9.

THEOREM E. *Let $n \geq 6$ and let ξ be a real n-plane bundle over a CW-complex X of dimension less than $n+2$. If ξ admits a non-zero cross section, then the set $cross(\xi)$ of homotopy classes of non-zero cross sections of ξ is given by*

$$cross(\xi) = \bigcup_{a \in \operatorname{Ker} \Theta^n} \operatorname{Coker} \Theta^{n-1} \times \operatorname{Coker} \Phi_a.$$

Here

$$\Theta^i: H^{i-1}(X; \underline{Z}) \longrightarrow H^{i+1}(X; Z_2), \qquad i = n-1, n,$$

$$\Gamma: H^{n-1}(X; Z_2) \longrightarrow H^{n+1}(X; Z_2),$$

are the homomorphisms defined by

$$\Theta^i(a) = Sq^2\rho_2 a + \rho_2 a w_2(\xi),$$

$$\Gamma(b) = Sq^2 b + Sq^1 b w_1(\xi) + b(w_2(\xi) + w_1(\xi)^2),$$

where $\underline{Z}$ is the local system on X associated with ξ, ρ_2 is the mod 2 *reduction and $w_i(\xi)$ is the i-th Stiefel-Whitney class of ξ, and Φ_a:* $\operatorname{Ker}\Theta^{n-1}\to\operatorname{Coker}\Gamma$ *is the twisted secondary operation of* (16.8).

§ 19. The enumeration of embeddings of complex projective spaces

$[CP^n \subset R^m]$ denotes the set of isotopy classes of embeddings of the n-dimensional complex projective space CP^n in the real m-space R^m. Let $(CP^n)^*$ denote the reduced symmetric product of CP^n and let η denote the real line bundle associated with the double covering $CP^n \times CP^n - \Delta \to (CP^n)^*$. By A. Haefliger's theorem [5, Théorème 1], the set $[CP^n \subset R^{4n-3}]$ is equivalent to the set *cross* $((4n-3)\eta)$ for $n \geq 5$. Hence we determine the cardinality of the set $[CP^n \subset R^{4n-3}]$ by studying the set *cross*$((4n-3)\eta)$ and we have the following

THEOREM F. *Let $n \geq 5$ and let $n \neq 2^r + 2^s$ $(r \geq s > 0)$. Then, the n-dimensional complex projective space CP^n is embedded in the real* $(4n-3)$*-space R^{4n-3} and there are countably many distinct isotopy classes of embeddings of CP^n in R^{4n-3}.*

PROOF. The first half is shown for odd n in [9, Theorem 1.2] and for even n in [25, Theorem 4.1.(2)], and so we concentrate ourselves on the investigation of the cardinality of the set $[CP^n \subset R^{4n-3}] = cross((4n-3)\eta)$.

Since the space $(CP^n)^*$ has the homotopy type of a $(4n-2)$-dimensional manifold by [3, § 2] and [26, Proposition 1.6], we have

$$[CP^n \subset R^{4n-3}] = \bigcup_{a \in \operatorname{Ker}\Theta^{4n-3}} \operatorname{Coker}\Theta^{4n-4} \times \operatorname{Coker}\Phi_a,$$

by Theorem E, where

$$\Theta^{4n-3}\colon H^{4n-4}((CP^n)^*; \underline{Z}) \longrightarrow H^{4n-2}((CP^n)^*; Z_2) = Z_2.$$

Thus in order to prove the theorem, it is sufficient to show that $H^{4n-4}((CP^n)^*; \underline{Z})$ is countable.

Now, $(CP^n)^*$ has the homotopy type of an unorientable $(4n-2)$-dimensional manifold with

$$\pi_1((CP^n)^*) = Z_2, \quad H^3((CP^n)^*; Z) = 0, \quad H^2((CP^n)^*; Z) = Z + Z_2,$$

by [26, Proposition 1.6, Theorem 4.10 and Theorem 4.15]. By Poincaré duality (cf. [17, p. 357]), we have

$$H^{4n-4}((CP^n)^*; \underline{Z}) = H_2((CP^n)^*; Z).$$

Further there is the universal coefficient theorem

$$0 \longrightarrow \mathrm{Ext}(H^3((CP^n)^*; Z), Z) \longrightarrow H_2((CP^n)^*; Z) \longrightarrow \mathrm{Hom}(H^2((CP^n)^*; Z), Z) \longrightarrow 0.$$

These relations imply

$$H^{4n-4}((CP^n)^*; \underline{Z}) = Z,$$

and we have Theorem F completely.

REMARK. By calculating more precisely, one can show that Coker $\Theta^{4n-4}=0$ or Z_2 according as n is even or odd, and Coker $\Gamma=$ Coker $\Phi_a=0$.

For completeness' sake, we mention the cardinality $\#[CP^n \subset R^m]$ for $m \geq 4n-2$, as follows:

(19.1) $\#[CP^n \subset R^{4n+1}] = 1$, [29],

(19.2) $\#[CP^n \subset R^{4n}] = 1$, $\#[CP^n \subset R^{4n-1}] = \aleph_0$, [7, Theorem 2.4],

(19.3) $\#[CP^n \subset R^{4n-2}] = 1$, [26, Theorem 5.5(3)].

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(continued from [27])

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