On relations between Bessel potential spaces and Riesz potential spaces

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Abstract. We present a relation between the Bessel potential spaces and the Riesz potential spaces. The ideas of the proof are to characterize each potential spaces and to give a correspondence between individual Bessel potentials and Riesz potentials.

Mathematics Subject Classification(1991). 31B99, 46E35.

Key words: Bessel potential spaces, Riesz potential spaces.

1. Introduction and preliminaries

The purposes of this paper are to improve a characterization of Riesz potential spaces and to give relations between Bessel potential spaces and Riesz potential spaces. Let R^n be the n-dimensional Euclidean space. Throughout this paper, let $0 < \alpha < \infty$ and $1 < p < \infty$. For a real number r, we define the spaces $L^{p,r}$ and $L^{p,r,\log}$ as follows:

$$L^{p,r} = \{f \in L^1_{loc} : ||f||_{p,r} = (\int_{R^n} |f(x)|^p (1+|x|)^{rp} dx)^{1/p} < \infty\},$$

$$L^{p,r,\log} = \{f \in L^1_{loc} : ||f||_{p,r,\log} = (\int_{R^n} |f(x)|^p (1+|x|)^{rp} (\log(e+|x|))^{-p} dx)^{1/p} < \infty\}$$

where L^1_{loc} is the set of all locally integrable functions and e is the base of the natural logarithm. We simply write $L^{p,0} = L^p$ and $||f||_{p,0} = ||f||_p$. Further, L^1 denotes the space consisting of all integrable functions.

We define the Bessel and Riesz potential spaces. The Bessel kernel $G_\alpha(x)$ of order α is given by

$$G_\alpha(x) = \frac{1}{(4\pi)^{\alpha/2}\Gamma(\alpha/2)} \int_0^\infty e^{-\pi|x|^2/s} e^{-s/4\pi} s^{(\alpha-n)/2} \frac{ds}{s}.$$

Since the Bessel kernel $G_\alpha(x)$ is integrable ([St2: Proposition 2 in Chap.V]), for $f \in L^p$ the Bessel potential of order α

$$G^f_\alpha(x) = \int G_\alpha(x-y) f(y) dy$$

is well-defined. The Bessel potential space B^p_α is defined by

$$B^p_\alpha = \{G^f_\alpha : f \in L^p\}.$$

The norm $||u||_{B^p_\alpha}$ of $u = G^f_\alpha$ is defined to be the L^p-norm of f, i.e.,

$$||u||_{B^p_\alpha} = ||f||_p \quad \text{if} \quad u = G^f_\alpha.$$

Let N denote the set of natural numbers including zero, and let $2N$ stand for the set of nonnegative even numbers.

The Riesz kernel $\kappa_\alpha(x)$ of order α is given by

$$\kappa_\alpha(x) = \frac{1}{\gamma_{\alpha,n}} \begin{cases} |x|^{\alpha-n}, & \alpha - n \notin 2N \\ (\delta_{\alpha,n} - \log|x|)|x|^{\alpha-n}, & \alpha - n \in 2N \end{cases}$$

with

$$\gamma_{\alpha,n} = \begin{cases} \pi^{n/2}2^\alpha\Gamma(\alpha/2)/\Gamma((n-\alpha)/2), & \alpha - n \notin 2N \\ (-1)^{(\alpha-n)/2}2^{\alpha-1}\pi^{n/2}\Gamma(\alpha/2)((\alpha-n)/2)!, & \alpha - n \in 2N \end{cases}$$

and

$$\delta_{\alpha,n} = \frac{\Gamma'(\alpha/2)}{2\Gamma(\alpha)} + \frac{1}{2}(1 + \frac{1}{2} + \cdots + \frac{1}{(\alpha-n)/2} - C) - \log \pi$$

where C is Euler's constant. In order to define the Riesz potential of order α of an L^p-function, we introduce the modified Riesz kernel. For an integer $k < \alpha$, we set

$$\kappa_{\alpha,k}(x,y) = \begin{cases} \kappa_\alpha(x-y) - \sum_{|\gamma|\le k} \frac{D^\gamma\kappa_\alpha(-y)}{\gamma!}x^\gamma, & 0 \le k < \alpha, \\ \kappa_\alpha(x-y), & k < 0 \end{cases}$$

where $\gamma = (\gamma_1, \cdots, \gamma_n)$ is a multi-index, $x^\gamma = x_1^{\gamma_1}\cdots x_n^{\gamma_n}(x = (x_1, \cdots, x_n)), D^\gamma = D_1^{\gamma_1}\cdots D_n^{\gamma_n}(D_j = \partial/\partial x_j), \gamma! = \gamma_1!\cdots\gamma_n!$ and $|\gamma| = \gamma_1 + \cdots + \gamma_n$.

For a function $f \in L^p$, we define the Riesz potential U^f_α of order α of f as follows:

$$U^f_\alpha(x) = \begin{cases} \int \kappa_{\alpha,k}(x,y)f(y)dy, & \alpha - (n/p) \notin N, \\ \int \kappa_{\alpha,k-1}(x,y)f_1(y)dy + \int \kappa_{\alpha,k}(x,y)f_2(y)dy, & \alpha - (n/p) \in N \end{cases}$$

where $k = [\alpha - (n/p)]$ is the integral part of $\alpha - (n/p)$, f_1 is the restriction of f to $\{x : |x| < 1\}$ and $f_2 = f - f_1$. The existence of U^f_α is guaranteed by the following proposition.

Proposition 1.1 (cf. [Ku1: Corollary 5.9 and Proposition 5.15]). *Let $f \in L^p$ and $k = [\alpha - (n/p)]$. Then U^f_α exists and satisfies the following estimates:*

$$(\int |U_\alpha^f(x)|^p|x|^{-\alpha p}dx)^{1/p} \leq C||f||_p, \qquad \alpha - (n/p) \notin N,$$

$$(\int |U_\alpha^f(x)|^p|x|^{-\alpha p}(1 + |\log|x||)^{-p}dx)^{1/p} \leq C||f||_p, \qquad \alpha - (n/p) \in N.$$

The Riesz potential space R_α^p is given by

$$R_\alpha^p = \{U_\alpha^f : f \in L^p\}.$$

By Proposition 1.1 we have

$$R_\alpha^p \subset \begin{cases} L^{p,-\alpha}, & \alpha - (n/p) \notin N \\ L^{p,-\alpha,\log}, & \alpha - (n/p) \in N. \end{cases}$$

We denote by $\mathcal{P}$ the set of all polynomials, and let $\mathcal{P}_k = \{P \in \mathcal{P} : \text{degree of } P \leq k\}$. If $k = [\alpha - (n/p)]$, then

$$\mathcal{P}_k \subset \begin{cases} L^{p,-\alpha}, & \alpha - (n/p) \notin N \\ L^{p,-\alpha,\log}, & \alpha - (n/p) \in N. \end{cases}$$

Consequently, for $k = [\alpha - (n/p)]$ we have

$$R_\alpha^p + \mathcal{P}_k \subset \begin{cases} L^{p,-\alpha}, & \alpha - (n/p) \notin N \\ L^{p,-\alpha,\log}, & \alpha - (n/p) \in N. \end{cases}$$

Let $k = [\alpha - (n/p)]$. Since $U_\alpha^f + P = U_\alpha^g + Q$ ($f, g \in L^p$ and $P, Q \in \mathcal{P}_k$) implies $f = g, P = Q$ (see Lemma 2.15 below), the sum $R_\alpha^p + \mathcal{P}_k$ is a direct sum $R_\alpha^p \oplus \mathcal{P}_k$, and we can define the norm in the direct sum space $R_\alpha^p \oplus \mathcal{P}_k$ as follows:

$$||u||_{R_\alpha^p \oplus \mathcal{P}_k} = \begin{cases} ||f||_p + ||P||_{p,-\alpha}, & \alpha - (n/p) \notin N \\ ||f||_p + ||P||_{p,-\alpha,\log}, & \alpha - (n/p) \in N \end{cases}$$

if $u = U_\alpha^f + P \in R_\alpha^p \oplus \mathcal{P}_k$.

In section 2 we give an improvement of a characterization of the Riesz potential spaces (Proposition 2.17), which is necessary for discussions in section 3. In section 3 we present our main result (Theorem 3.7) concerning a relation between Bessel potential spaces and Riesz potential spaces. Theorem 3.7 below gives the following relation: Let $k = [\alpha - (n/p)]$. Then

$$B_\alpha^p = (R_\alpha^p \oplus \mathcal{P}_k) \cap L^p$$

and

$$||u||_{B_\alpha^p} \approx ||u||_{R_\alpha^p \oplus \mathcal{P}_k} + ||u||_p$$

where the notation $\approx$ stands for equivalent norms.

2. Characterizations of potential spaces

Let ℓ be a positive integer. For a function u on R^n, the difference $\Delta_t^\ell u$ of order ℓ with increment $t \in R^n$ is defined by

$$\Delta_t^\ell u(x) = \sum_{j=0}^{\ell}(-1)^j C_j^\ell u(x + (\ell - j)t)$$

with $C_j^\ell = \frac{\ell!}{j!(\ell-j)!}$. We define the space $\mathcal{L}_{\alpha,\ell}^p$ as follows:

$$\mathcal{L}_{\alpha,\ell}^p = \{u \in L_{loc}^1 : \lim_{\epsilon\to 0}\int_{|t|\ge\epsilon} \frac{\Delta_t^\ell u(x)}{|t|^{n+\alpha}}dt \quad \text{exists in } L^p\}.$$

We write

$$D_\epsilon^{\alpha,\ell}u(x) = \int_{|t|\ge\epsilon} \frac{\Delta_t^\ell u(x)}{|t|^{n+\alpha}}dt$$

and

$$D^{\alpha,\ell}u = \lim_{\epsilon\to 0} D_\epsilon^{\alpha,\ell}u.$$

E.M.Stein [St1: Theorem 2], and S.G.Samko, A.A.Kilbas and O.I.Marichev [SKM: Theorem 27.3] give the following characterization of the Bessel potential spaces.

PROPOSITION 2.1. *Let $2[(\ell+1)/2] > \alpha$ if α is not an odd number, and $\ell = \alpha$ if α is an odd number. Then*

$$B_\alpha^p = \mathcal{L}_{\alpha,\ell}^p \cap L^p$$

and

$$||u||_{B_\alpha^p} \approx ||D^{\alpha,\ell}u||_p + ||u||_p.$$

Characterizations of the Riesz potential spaces are treated in S.G.Samko [Sa: Theorem 10] and the author [Ku2: Remark 4.10]. However, the condition of ℓ in the characterizations can be improved. In this section we give the improvement. We put

$$\rho_\epsilon^{\alpha,\ell}(x) = \int_{|t|\ge\epsilon} \frac{\Delta_t^\ell \kappa_\alpha(x)}{|t|^{n+\alpha}}dt, \qquad \epsilon > 0.$$

We note that $\rho_\epsilon^{\alpha,\ell}(x)$ is finite for $x \neq 0$. In fact

$$(2.1) \qquad |\rho_\epsilon^{\alpha,\ell}(x)| \le \sum_{j=0}^{\ell-1} C_j^\ell \int_{|t|\ge\epsilon} \frac{|\kappa_\alpha(x + (\ell-j)t)|}{|t|^{n+\alpha}}dt + |\kappa_\alpha(x)| \int_{|t|\ge\epsilon} \frac{1}{|t|^{n+\alpha}}dt < \infty$$

for $x \neq 0$. We simply write $\rho_1^{\alpha,\ell}(x) = \rho^{\alpha,\ell}(x)$. We use the symbol C for a generic positive constant whose value may be different at each occurrence.

LEMMA 2.2. (i) ([Ku2: Lemma 3.1]) *Let* ℓ *be a positive integer, and moreover suppose* $\ell > \alpha - n$ *if* $\alpha - n \in 2N$. *Then*

$$\rho_\epsilon^{\alpha,\ell}(x) = \frac{1}{\epsilon^n}\rho^{\alpha,\ell}(\frac{x}{\epsilon}).$$

(ii) ([Ku2: Corollary 3.6]) *Let* $2[(\ell+1)/2] > \alpha$. *Then for* $|x| \geq 1$

$$|\rho^{\alpha,\ell}(x)| \leq C|x|^{\alpha-2[(\ell+1)/2]-n},$$

and for $|x| < 1$

$$|\rho^{\alpha,\ell}(x)| \leq C \begin{cases} |x|^{\alpha-n}, & \alpha < n, \\ (1-\log|x|), & \alpha = n, \\ 1, & \alpha > n. \end{cases}$$

REMARK 2.3. By Lemma 2.2 (ii), if $2[(\ell+1)/2] > \alpha$, then $\rho^{\alpha,\ell}$ is integrable. We set

$$d_{\alpha,\ell} = \int \rho^{\alpha,\ell}(x)dx.$$

By [Sa: Lemma 1] or [SKM: Theorem 26.1], $d_{\alpha,\ell} = 0$ if and only if α is an odd number and $\ell > \alpha$.

LEMMA 2.4 ([Ku2: Corollary 2.3]). *For a positive integer* $m > \alpha - n$ *and* $|x| \geq 2m|h|$,

$$|\Delta_h^m \kappa_\alpha(x)| \leq C|h|^m|x|^{\alpha-m-n}.$$

LEMMA 2.5. (i) (cf. [Ku2: Lemma 4.1]) *If* $m > 2\alpha - (n/p)$ *and* $u \in L^{p,-\alpha,\log}$, *then*

$$\int |\Delta_h^m \kappa_\alpha(x-y)u(y)|dy < \infty$$

for almost every x *in case of* $\alpha \leq n$, *and for all* x *in case of* $\alpha > n$.

(ii) (cf. [Ku2: Lemma 4.3]) *If* $m > 2\alpha - (n/p), \delta > 0$ *and* $u \in L^{p,-\alpha,\log}$, *then*

$$\int |u(z)| \int_{|z-y|\geq\delta} \frac{|\Delta_h^m \kappa_\alpha(x-y)|}{|z-y|^{n+\alpha}} dydz < \infty$$

for all x.

LEMMA 2.6. *If $m > 2\alpha - (n/p)$ and $u \in L^{p,-\alpha,\log}$, then*

$$\Delta_h^m \kappa_\alpha * D_\epsilon^{\alpha,\ell} u(x) = u * (\Delta_h^m \rho_\epsilon^{\alpha,\ell})(x)$$

for almost every x in case of $\alpha \le n$, and for all x in case of $\alpha > n$ where the symbol $*$ means convolution.

PROOF. We put

$$\begin{aligned} K(x) &= \Delta_h^m \kappa_\alpha * D_\epsilon^{\alpha,\ell} u(x) \\ &= \int \Delta_h^m \kappa_\alpha(x-y) \int_{|t|\ge\epsilon} \frac{\sum_{j=0}^{\ell-1}(-1)^j C_j^\ell u(y+(\ell-j)t) + (-1)^\ell u(y)}{|t|^{n+\alpha}} dt dy. \end{aligned}$$

Since

$$\int_{|t|\ge\epsilon} \frac{|u(y+(\ell-j)t)|}{|t|^{n+\alpha}} dt < \infty, \qquad j = 0,1,\cdots,\ell-1$$

by the condition $u \in L^{p,-\alpha,\log}$, we have

$$\begin{aligned} K(x) = & \int \{ \sum_{j=0}^{\ell-1} (-1)^j C_j^\ell \Delta_h^m \kappa_\alpha(x-y) \int_{|t|\ge\epsilon} \frac{u(y+(\ell-j)t)}{|t|^{n+\alpha}} dt \\ & + (-1)^\ell \Delta_h^m \kappa_\alpha(x-y) \int_{|t|\ge\epsilon} \frac{u(y)}{|t|^{n+\alpha}} dt \} dy. \end{aligned}$$

We can use Lemma 2.5 (i) and (ii) because of the conditions $m > 2\alpha - (n/p)$ and $u \in L^{p,-\alpha,\log}$. So, by the changes of variables $y + (\ell - j)t = z$ $(j = 0, 1, \cdots, \ell - 1)$ and Fubini's Theorem, we obtain

$$\begin{aligned} K(x) &= \int \{ \sum_{j=0}^{\ell-1} (-1)^j C_j^\ell (\ell-j)^\alpha \Delta_h^m \kappa_\alpha(x-y) \int_{|z-y|\ge(\ell-j)\epsilon} \frac{u(z)}{|z-y|^{n+\alpha}} dz \\ &\quad + (-1)^\ell \Delta_h^m \kappa_\alpha(x-y) u(y) \int_{|t|\ge\epsilon} \frac{1}{|t|^{n+\alpha}} dt \} dy \\ &= \sum_{j=0}^{\ell-1} (-1)^j C_j^\ell (\ell-j)^\alpha \int u(z) \int_{|z-y|\ge(\ell-j)\epsilon} \frac{\Delta_h^m \kappa_\alpha(x-y)}{|z-y|^{n+\alpha}} dy dz \\ &\quad + (-1)^\ell \int u(z) \int_{|t|\ge\epsilon} \frac{\Delta_h^m \kappa_\alpha(x-z)}{|t|^{n+\alpha}} dt dz \end{aligned}$$

for almost every x in case of $\alpha \le n$, and for all x in case of $\alpha > n$. Further, the changes of variables $(z-y)/(\ell-j) = t$ $(j = 0, 1, \cdots, \ell - 1)$ give

$$K(x) = \sum_{j=0}^{\ell-1} (-1)^j C_j^\ell \int u(z) \int_{|t|\ge\epsilon} \frac{\Delta_h^m \kappa_\alpha(x-z+(\ell-j)t)}{|t|^{n+\alpha}} dt dz$$

$$
\begin{aligned}
&+(-1)^{\ell}\int u(z)\int_{|t|\geq\epsilon}\frac{\Delta_h^m\kappa_\alpha(x-z)}{|t|^{n+\alpha}}dtdz\\
=&\int u(z)\int_{|t|\geq\epsilon}\frac{\sum_{j=0}^{\ell}(-1)^jC_j^{\ell}\Delta_h^m\kappa_\alpha(x-z+(\ell-j)t)}{|t|^{n+\alpha}}dtdz\\
=&\int u(z)\int_{|t|\geq\epsilon}\frac{\sum_{\nu=0}^{m}(-1)^{\nu}C_\nu^m\Delta_t^{\ell}\kappa_\alpha(x-z+(m-\nu)h)}{|t|^{n+\alpha}}dtdz
\end{aligned}
$$

for almost every x in case of $\alpha \leq n$, and for all x in case of $\alpha > n$. Finally, by (2.1) we obtain

$$
\begin{aligned}
K(x) &= \int u(z)\sum_{\nu=0}^{m}(-1)^{\nu}C_\nu^m\int_{|t|\geq\epsilon}\frac{\Delta_t^{\ell}\kappa_\alpha(x-z+(m-\nu)h)}{|t|^{n+\alpha}}dtdz\\
&= \int u(z)\sum_{\nu=0}^{m}(-1)^{\nu}C_\nu^m\rho_\epsilon^{\alpha,\ell}(x-z+(m-\nu)h)dz\\
&= \int u(z)(\Delta_h^m\rho_\epsilon^{\alpha,\ell})(x-z)dz\\
&= u*(\Delta_h^m\rho_\epsilon^{\alpha,\ell})(x)
\end{aligned}
$$

for almost every x in case of $\alpha \leq n$, and for all x in case of $\alpha > n$. This completes the proof of the lemma.

LEMMA 2.7 ([Ku3: Lemma 3.12]). *If $m > \alpha - (n/p)$, then $\Delta_h^m\kappa_\alpha \in \cup_{1<s<p'}L^s$ with $(1/p)+(1/p')=1$.*

LEMMA 2.8. *If $m \geq 2\alpha$ and $|z| \geq 4m|h|$, then*

$$
|\Delta_h^m\rho^{\alpha,\ell}(z)| \leq C\begin{cases}(|h|^\alpha+|h|^{2\alpha})|z|^{-\alpha-n}, & \alpha-n\notin 2N\\ (|h|^\alpha(1+|\log|h||)+|h|^{2\alpha})|z|^{-\alpha-n}, & \alpha-n\in 2N.\end{cases}
$$

PROOF. We have

$$
\begin{aligned}
\Delta_h^m\rho^{\alpha,\ell}(z) &= \int_{|t|\geq 1}\frac{\Delta_t^{\ell}\Delta_h^m\kappa_\alpha(z)}{|t|^{n+\alpha}}dt\\
&= \sum_{j=0}^{\ell-1}(-1)^jC_j^{\ell}\int_{|t|\geq 1}\frac{\Delta_h^m\kappa_\alpha(z+(\ell-j)t)}{|t|^{n+\alpha}}dt+(-1)^{\ell}\Delta_h^m\kappa_\alpha(z)\int_{|t|\geq 1}\frac{1}{|t|^{n+\alpha}}dt.
\end{aligned}
$$

The changes of variables $z+(\ell-j)t = y (j=0,1,\cdots,\ell-1)$ give

$$
\begin{aligned}
|\Delta_h^m\rho^{\alpha,\ell}(z)| &\leq \sum_{j=0}^{\ell-1}C_j^{\ell}(\ell-j)^\alpha\int_{|y-z|\geq\ell-j}\frac{|\Delta_h^m\kappa_\alpha(y)|}{|y-z|^{n+\alpha}}dt+|\Delta_h^m\kappa_\alpha(z)|\int_{|t|\geq 1}\frac{1}{|t|^{n+\alpha}}dt\\
&\leq C(\alpha,\ell)\int_{|y-z|\geq 1}\frac{|\Delta_h^m\kappa_\alpha(y)|}{|y-z|^{n+\alpha}}dy+|\Delta_h^m\kappa_\alpha(z)|\int_{|t|\geq 1}\frac{1}{|t|^{n+\alpha}}dt.
\end{aligned}
$$

Since $|\Delta_h^m \kappa_\alpha(z)| \le C|h|^m|z|^{\alpha-m-n} \le C|h|^{2\alpha}|z|^{-\alpha-n}$ for $|z| \ge 4m|h|$ and $m \ge 2\alpha$ by Lemma 2.4, it is sufficient to show

$$K(z,h) = \int_{|y-z|\ge 1} \frac{|\Delta_h^m \kappa_\alpha(y)|}{|y-z|^{n+\alpha}} dy \le C \begin{cases} (|h|^\alpha + |h|^{2\alpha})|z|^{-\alpha-n}, & \alpha - n \notin 2N \\ (|h|^\alpha(1+|\log|h||) + |h|^{2\alpha})|z|^{-\alpha-n}, & \alpha - n \in 2N \end{cases}$$

for $|z| \ge 4m|h|$ and $m \ge 2\alpha$. We devide $K(z,h)$ as follows:

$$\begin{aligned} K(z,h) &= \int_{|y-z|\ge 1, |y|<2m|h|} \frac{|\Delta_h^m \kappa_\alpha(y)|}{|y-z|^{n+\alpha}} dy + \int_{|y-z|\ge 1, |y|\ge 2m|h|} \frac{|\Delta_h^m \kappa_\alpha(y)|}{|y-z|^{n+\alpha}} dy \\ &= K_1(z,h) + K_2(z,h). \end{aligned}$$

Since $|z| \ge 4m|h|$ and $|y| < 2m|h|$ imply $|y-z| \ge |z|/2$, we see that

$$\begin{aligned} K_1(z,h) &\le \sum_{i=0}^m C_i^m \left(\frac{|z|}{2}\right)^{-\alpha-n} \int_{|y|<2m|h|} |\kappa_\alpha(y+(m-i)h)|dy \\ &\le C|z|^{-\alpha-n} \int_{|y|<3m|h|} |\kappa_\alpha(y)|dy \\ &\le C|z|^{-\alpha-n} \begin{cases} |h|^\alpha, & \alpha - n \notin 2N \\ |h|^\alpha(1+|\log|h||), & \alpha - n \in 2N \end{cases} \end{aligned}$$

for $|z| \ge 4m|h|$ For $K_2(z,h)$, by Lemma 2.4 we have

$$\begin{aligned} K_2(z,h) &\le C \int_{|y-z|\ge 1, |y|\ge 2m|h|} \frac{|h|^m |y|^{\alpha-m-n}}{|y-z|^{n+\alpha}} dy \\ &= C\left(\int_{D_1} + \int_{D_2} + \int_{D_3} + \int_{D_4} \frac{|h|^m|y|^{\alpha-m-n}}{|y-z|^{n+\alpha}} dy\right) \\ &= C(K_{21}(z,h) + K_{22}(z,h) + K_{23}(z,h) + K_{24}(z,h)) \end{aligned}$$

where $D_1 = \{y : |y-z| \ge 1, |y| \ge 2m|h|, |y| < |z|/2\}$, $D_2 = \{y : |y-z| \ge 1, |y| \ge 2m|h|, |y| < |y-z|, |y| \ge |z|/2\}$, $D_3 = \{y : |y-z| \ge 1, |y| \ge 2m|h|, |y| \ge |y-z|, |y-z| \ge |z|/2\}$ and $D_4 = \{y : |y-z| \ge 1, |y| \ge 2m|h|, |y-z| < |z|/2\}$. For $K_{21}(z,h)$, since $|y| < |z|/2$ implies $|z|/2 < |y-z| < 3|z|/2$, we obtain

$$\begin{aligned} K_{21}(z,h) &\le |h|^m \left(\frac{|z|}{2}\right)^{-\alpha-n} \int_{2m|h|\le|y|\le|z|/2} |y|^{\alpha-m-n} dy \\ &\le C|h|^m|z|^{-\alpha-n}|h|^{\alpha-m} = C|h|^\alpha|z|^{-\alpha-n} \end{aligned}$$

for $|z| \ge 4m|h|$. For $K_{22}(z,h)$, the condition $|y| \le |y-z|$ gives

$$K_{22}(z,h) \le |h|^m \int_{|y|\ge|z|/2} |y|^{-m-2n} dy = C|h|^m|z|^{-m-n} \le C|h|^\alpha|z|^{-\alpha-n}$$

for $|z| \geq 4m|h|$ and $m \geq 2\alpha \geq \alpha$. For $K_{23}(z,h)$, the condition $|y| \geq |y-z|$ also implies

$$K_{23}(z,h) \leq C|h|^{\alpha}|z|^{-\alpha-n}$$

for $|z| \geq 4m|h|$ and $m \geq 2\alpha \geq \alpha$. For $K_{24}(z,h)$, since $|y-z| < |z|/2$ implies $|z|/2 < |y| < 3|z|/2$, we see that

$$\begin{aligned} K_{24}(z,h) &\leq |h|^m\left(\frac{|z|}{2}\right)^{\alpha-m-n}\int_{|y-z|\geq 1}\frac{1}{|y-z|^{n+\alpha}}dy \\ &\leq C|h|^m|z|^{\alpha-m-n} \leq C|h|^{2\alpha}|z|^{-\alpha-n} \end{aligned}$$

for $|z| \geq 4m|h|$ and $m \geq 2\alpha$. Thus we obtain the required conclusion.

Corollary 2.9. *If $m \geq 2\alpha, |z| \geq 4m|h|$ and $0 < \epsilon \leq 1$, then*

$$|(\Delta_h^m \rho_\epsilon^{\alpha,\ell})(z)| \leq C\begin{cases} (|h|^{\alpha} + |h|^{2\alpha})\epsilon^{-\alpha}|z|^{-\alpha-n}, & \alpha - n \notin 2N \\ (|h|^{\alpha}(1 + |\log|h||) + |h|^{2\alpha})\epsilon^{-\alpha}|z|^{-\alpha-n}, & \alpha - n \in 2N. \end{cases}$$

Proof. We note that

$$(\Delta_h^m \rho_\epsilon^{\alpha,\ell})(z) = \frac{1}{\epsilon^n}(\Delta_{h/\epsilon}^m \rho^{\alpha,\ell})(\frac{z}{\epsilon}).$$

Hence, in case $\alpha - n \notin 2N$, since $|\frac{z}{\epsilon}| \geq 4m|\frac{h}{\epsilon}|$, by Lemma 2.8 we have

$$\begin{aligned} (\Delta_h^m \rho_\epsilon^{\alpha,\ell})(z) &\leq \frac{C}{\epsilon^n}\left(\left(\frac{|h|}{\epsilon}\right)^{\alpha}\left(\frac{|z|}{\epsilon}\right)^{-\alpha-n} + \left(\frac{|h|}{\epsilon}\right)^{2\alpha}\left(\frac{|z|}{\epsilon}\right)^{-\alpha-n}\right) \\ &= C(|h|^{\alpha}|z|^{-\alpha-n} + \epsilon^{-\alpha}|h|^{2\alpha}|z|^{-\alpha-n}) \\ &\leq C(|h|^{\alpha} + |h|^{2\alpha})\epsilon^{-\alpha}|z|^{-\alpha-n} \end{aligned}$$

for $0 < \epsilon \leq 1$. The proof of the case $\alpha - n \in 2N$ is the same. Hence we obtain the corollary.

The next lemma is the key lemma for an improvement of the characterization of the Riesz potential spaces. The idea of the proof is due partly to [SW: Theorem 1.25].

Lemma 2.10. *If $m \geq 2\alpha, 2[(\ell+1)/2] > \alpha, ((n/\alpha p)+1)[(\ell+1]/2] > \alpha$ and $u \in L^{p,-\alpha,\log}$, then $u * (\Delta_h^m \rho_{1/j}^{\alpha,\ell})(x)$ converges to $d_{\alpha,\ell}\Delta_h^m u(x)$ as $j \to \infty$ for almost every x.*

PROOF. We denote by u_j the restriction of u to $\{x : |x| < j^{\alpha p/n}(\log j)^{(p-1)/n}\}$ for $j = 2, 3, \cdots$. We have

$$\begin{aligned} I_j(x) &= u * (\Delta_h^m \rho_{1/j}^{\alpha,\ell})(x) - d_{\alpha,\ell}\Delta_h^m u(x) \\ &= (u * (\Delta_h^m \rho_{1/j}^{\alpha,\ell})(x) - u_j * (\Delta_h^m \rho_{1/j}^{\alpha,\ell})(x)) \\ &\quad +(u_j * (\Delta_h^m \rho_{1/j}^{\alpha,\ell})(x) - d_{\alpha,\ell}\Delta_h^m u_j(x)) \\ &\quad +(d_{\alpha,\ell}\Delta_h^m u_j(x) - d_{\alpha,\ell}\Delta_h^m u(x)) \\ &= I_j^1(x) + I_j^2(x) + I_j^3(x). \end{aligned}$$

First, it is clear that

$$\lim_{j\to\infty} I_j^3(x) = 0 \quad \text{for all } x. \tag{2.2}$$

Next, we shall prove

$$|I_j^1(x)| \leq C(\int_{|t|\geq j^{\alpha p/n}(\log j)^{(p-1)/n}} |u(t)|^p(1+|t|)^{-\alpha p}(\log(e+|t|))^{-p}dt)^{1/p} \tag{2.3}$$

for j such that $j^{\alpha p/n}(\log j)^{(p-1)/n} \geq \max(2|x|+1, 4(|x|+2m|h|))$. We have

$$\begin{aligned} &|I_j^1(x)| = |\int (u(x-t) - u_j(x-t))(\Delta_h^m \rho_{1/j}^{\alpha,\ell})(t)dt| \\ &= |\int_{|x-t|\geq j^{\alpha p/n}(\log j)^{(p-1)/n}} u(x-t)(1+|x-t|)^{-\alpha}(\log(e+|x-t|))^{-1} \\ &\quad \times(1+|x-t|)^{\alpha}(\log(e+|x-t|))(\Delta_h^m \rho_{1/j}^{\alpha,\ell})(t)dt| \\ &\leq (\int_{|x-t|\geq j^{\alpha p/n}(\log j)^{(p-1)/n}} |u(x-t)|^p(1+|x-t|)^{-\alpha p}(\log(e+|x-t|))^{-p}dt)^{1/p} \\ &\quad \times(\int_{|x-t|\geq j^{\alpha p/n}(\log j)^{(p-1)/n}} (1+|x-t|)^{\alpha p'}(\log(e+|x-t|))^{p'}|(\Delta_h^m \rho_{1/j}^{\alpha,\ell})(t)|^{p'}dt)^{1/p'}. \end{aligned}$$

For $j^{\alpha p/n}(\log j)^{(p-1)/n} \geq \max(2|x|+1, 4(|x|+2m|h|))$, it is easy to check that $|x-t| \geq j^{\alpha p/n}(\log j)^{(p-1)/n}$ implies

$$\begin{aligned} &\text{(i)}\quad |t| \geq |x|+1, \quad \text{(ii)}\quad |t| \geq j^{\alpha p/n}(\log j)^{(p-1)/n}/2 \geq 4m|h|, \\ &\text{(iii)}\quad 1+|x-t| \leq 2|t|, \quad \text{(iv)}\quad e+|x-t| \leq e+2|t|. \end{aligned} \tag{2.4}$$

Hence , for such j by Corollary 2.9 and (2.4) we have

$$\begin{aligned} &(\int_{|x-t|\geq j^{\alpha p/n}(\log j)^{(p-1)/n}} (1+|x-t|)^{\alpha p'}(\log(e+|x-t|))^{p'}|(\Delta_h^m \rho_{1/j}^{\alpha,\ell})(t)|^{p'}dt)^{1/p'} \\ &\leq C(h)j^{\alpha}(\int_{|t|\geq j^{\alpha p/n}(\log j)^{(p-1)/n}/2} |t|^{\alpha p'}(\log(e+2|t|))^{p'}|t|^{(-\alpha-n)p'}dt)^{1/p'} \\ &\leq C(h) \end{aligned}$$

because of

$$(\int_{|t|\geq j^{\alpha p/n}(\log j)^{(p-1)/n}/2} |t|^{-np'}(\log(e+2|t|))^{p'}dt)^{1/p'} \leq Cj^{-\alpha}.$$

Therefore, we obtain (2.3), and hence

$$I_j^1(x) \to 0 \qquad (j \to \infty) \tag{2.5}$$

on account of $u \in L^{p,-\alpha,\log}$. Finally we consider $I_j^2(x)$. Since $d_{\alpha,\ell} = \int \rho_{1/j}^{\alpha,\ell}(t)dt$, we have

$$\begin{aligned} I_j^2(x) &= (\Delta_h^m u_j) * \rho_{1/j}^{\alpha,\ell}(x) - d_{\alpha,\ell}\Delta_h^m u_j(x) \\ &= \sum_{i=0}^{m}(-1)^i C_i^m \int (u_j(x+(m-i)h-t) - u_j(x+(m-i)h))\rho_{1/j}^{\alpha,\ell}(t)dt. \end{aligned}$$

We denote the Lebesgue set of u by B_u. It follows from $u \in L_{loc}^1$ that $m(B_u^c) = 0$ where m denotes the Lebesgue measure and B_u^c is the complement of B_u. We put $E_i = B_u^c - (m-i)h (i = 0, 1, \cdots, m)$ and $E = \cup_{i=0}^m E_i$ Then $m(E) = 0$ and for $x \in E^c, x + (m-i)h$ is a Lebesgue point of u for $i = 0, 1, \cdots, m$. We shall show that for a fixed $x \in E^c$ and $i = 0, 1, \cdots, m$,

$$a_j = \int (u_j(x+(m-i)h-t) - u_j(x+(m-i)h))\rho_{1/j}^{\alpha,\ell}(t)dt \to 0 \quad (j \to \infty).$$

We choose $\delta > 0$. Since $x + (m-i)h$ is a Lebesgue point of u, we can find an $\eta > 0$ such that

$$\frac{1}{r^n}\int_{|t|<r} |u(x+(m-i)h-t) - u(x+(m-i)h)|dt < \delta \tag{2.6}$$

provided $r \leq \eta$. We put $y = x + (m-i)h$. We have

$$\begin{aligned} |a_j| &= |\int (u_j(y-t) - u_j(y))\rho_{1/j}^{\alpha,\ell}(t)dt| \\ &\leq \int_{|t|<\eta} |u_j(y-t) - u_j(y)||\rho_{1/j}^{\alpha,\ell}(t)|dt \\ &\qquad + \int_{|t|\geq\eta} |u_j(y-t) - u_j(y)||\rho_{1/j}^{\alpha,\ell}(t)|dt \\ &= b_j + c_j. \end{aligned}$$

We let

$$w(r) = \int_{S_{n-1}} |u(y - rt') - u(y)| dS(t')$$

and

$$W(s) = \int_0^s r^{n-1} w(r) dr$$

where S_{n-1} denotes the surface of the unit sphere and dS is the element of surface area on S_{n-1}. Then, since $W(r) = \int_{|t|<r} |u(y-t) - u(y)| dt$, by (2.6) we see that

$$W(r) < \delta r^n \tag{2.7}$$

provided $r \leq \eta$. Moreover we define for $r \geq 1$

$$h(r) = r^{\alpha - 2[(\ell+1)/2] - n} \tag{2.8}$$

and for $r < 1$

$$h(r) = \begin{cases} r^{\alpha-n}, & \alpha < n \\ 1 - \log r, & \alpha = n \\ 1, & \alpha > n. \end{cases} \tag{2.9}$$

By Lemma 2.2(i) and (ii), for j such that $j^{\alpha p/n}(\log j)^{(p-1)/n} > |x| + m|h| + \eta$ we have

$$\begin{aligned} b_j &= \int_{|t|<\eta} |u_j(y-t) - u_j(y)| |\rho_{1/j}^{\alpha,\ell}(t)| dt \\ &\leq Cj^n \int_{|t|<\eta} |u(y-t) - u(y)| h(j|t|) dt \\ &= Cj^n \int_0^\eta w(r) r^{n-1} h(jr) dr \\ &= Cj^n [W(r) h(jr)]_0^\eta - Cj^n \int_0^\eta W(r) dh(jr). \end{aligned}$$

It follows from (2.7) and (2.9) that $\lim_{r\to 0} W(r)h(jr) = 0$. Hence

$$\begin{aligned} b_j &\leq Cj^n W(\eta) h(j\eta) + Cj^n \int_0^\eta W(r)(-dh(jr)) \\ &= b_j^1 + b_j^2. \end{aligned}$$

By the change of variables $jr = s$ and (2.7), we obtain

$$\begin{aligned} b_j^2 &= Cj^n \int_0^{j\eta} W(\frac{s}{j})(-dh(s)) \leq Cj^n \int_0^{j\eta} \delta(\frac{s}{j})^n(-dh(s)) \\ &\leq C\delta \int_0^\infty s^n(-dh(s)) = C\delta \frac{n}{\sigma_{n-1}} \int_{R^n} h(|x|) dx \end{aligned}$$

where σ_{n-1} is the surface area of the unit sphere. We note that $h(|x|) \in L^1$ because of (2.8), (2.9) and $\alpha < 2[(\ell+1)/2]$. We may assume that $j\eta > 1$. Then by (2.7) and (2.8) we see that

$$b_j^1 \le Cj^n \delta \eta^n (j\eta)^{\alpha - 2[(\ell+1)/2] - n} = C\delta (j\eta)^{\alpha - 2[(\ell+1)/2]} \le C\delta$$

on account of $\alpha < 2[(\ell+1)/2]$. Thus for j such that $j\eta > 1$ we have

$$b_j \le C\delta \Big(1 + \frac{n}{\sigma_{n-1}} \int_{R^n} h(|x|)dx\Big).$$

Since δ is arbitrary, this implies $\lim_{j\to\infty} b_j = 0$. Furthermore, we have

$$\begin{aligned} c_j &\le \int_{|t|\ge\eta} |u_j(y-t)||\rho^{\alpha,\ell}_{1/j}(t)|dt + |u(y)| \int_{|t|\ge\eta} |\rho^{\alpha,\ell}_{1/j}(t)|dt \\ &= c_j^1 + c_j^2. \end{aligned}$$

It follows from $\rho^{\alpha,\ell} \in L^1$ that $\lim_{j\to\infty} c_j^2 = 0$ in view of $|u(y)| < \infty$. We also may assume that $j\eta > 1$. Then by Lemma 2.2 (i) and (ii) we have

$$\begin{aligned} c_j^1 &\le C \int_{|t|\ge\eta} |u_j(y-t)| j^n |jt|^{\alpha - 2[(\ell+1)/2] - n} dt \\ &= Cj^{\alpha - 2[(\ell+1)/2]} \int_{|t|\ge\eta, |y-t| \le j^{\alpha p/n} (\log j)^{(p-1)/n}} |u(y-t)|(1+|t|)^{-\alpha}(\log(e+|t|))^{-1} \\ &\quad \times (1+|t|)^{\alpha} (\log(e+|t|)) |t|^{\alpha - 2[(\ell+1)/2] - n} dt \\ &\le Cj^{\alpha - 2[(\ell+1)/2]} \Big(\int |u(y-t)|^p (1+|t|)^{-\alpha p} (\log(e+|t|))^{-p} dt\Big)^{1/p} \\ &\quad \times \Big(\int_{|t|\ge\eta, |y-t| \le j^{\alpha p/n}(\log j)^{(p-1)/n}} (1+|t|)^{\alpha p'} (\log(e+|t|))^{p'} |t|^{(\alpha - 2[(\ell+1)/2] - n)p'} dt\Big)^{1/p'} \end{aligned}$$

We note that the condition $\alpha < 2[(\ell+1)/2]$ implies $(1+\frac{n}{\alpha p})[(\ell+1)/2] > [(\ell+1)/2] + \frac{n}{2p}$. We also notice that $\alpha < [(\ell+1)/2] + \frac{n}{2p}$ gives $(2\alpha - 2[(\ell+1)/2] - n)p' < -n$ and the assumption $\alpha < (1+\frac{n}{\alpha p})[(\ell+1)/2]$ implies $\frac{2\alpha^2 p}{n} - 2(1+\frac{\alpha p}{n})[(\ell+1)/2] < 0$. Therefore

$$c_j^1 \le C\|u\|_{p,-\alpha,\log} \begin{cases} j^{\alpha - 2[(\ell+1)/2]}, & \alpha < [\frac{\ell+1}{2}] + \frac{n}{2p} \\ j^{\alpha - 2[(\ell+1)/2]} (\log j)^{p - \frac{1}{p}}, & \alpha = [\frac{\ell+1}{2}] + \frac{n}{2p} \\ j^{\frac{2\alpha^2 p}{n} - 2(1+\frac{\alpha p}{n})[(\ell+1)/2]} (\log j)^c, & [\frac{\ell+1}{2}] + \frac{n}{2p} < \alpha < (1+\frac{n}{\alpha p})[\frac{\ell+1}{2}] \end{cases}$$

with $c = \frac{2(p-1)}{n}(\alpha - [(\ell+1)/2]) + \frac{1}{p}$, and hence $\lim_{j\to\infty} c_j = 0$. Thus

$$\lim_{j\to\infty} I_j^2(x) = 0 \quad \text{for} \quad x \in E^c. \tag{2.10}$$

Taking (2.2), (2.5), and (2.10) into account we can conclude that $\lim_{j\to\infty} I_j(x) = 0$ for $x \in E^c$ with $m(E) = 0$. This proves the lemma.

LEMMA 2.11 ([Ku2: Lemma 4.7]). *Let $f \in L^p$ and $m > \alpha - (n/p)$. Then*

$$\Delta_h^m U_\alpha^f = \Delta_h^m \kappa_\alpha * f.$$

LEMMA 2.12 ([Ku2: Lemma 4.8]). *Let $f \in L^p$ and $\ell > \alpha - (n/p)$. Then*

$$D_\epsilon^{\alpha,\ell} U_\alpha^f = \rho_\epsilon^{\alpha,\ell} * f.$$

COROLLARY 2.13. *Let $f \in L^p, 2[(\ell+1)/2] > \alpha$ and $\ell > \alpha - (n/p)$. Then*

$$D^{\alpha,\ell} U_\alpha^f = d_{\alpha,\ell} f.$$

PROOF. This corollary follows from Lemmas 2.2, 2.11 and Remark 2.3.

LEMMA 2.14 ([Sa: section 3]). *Let u be a locally integrable function. Then $\Delta_t^\ell u(x) = 0$ almost everywhere for all $t \in R^n$ if and only if u is a polynomial of degree at most $\ell - 1$.*

LEMMA 2.15. *Let $f, g \in L^p, k = [\alpha - (n/p)]$ and $P, Q \in \mathcal{P}_k$. Then*

$$U_\alpha^f + P = U_\alpha^g + Q$$

if and only if $f = g$ and $P = Q$.

PROOF. It suffices to show "only if" part. We take a positive integer ℓ such that $2[(\ell+1)/2] > \alpha, \ell > \alpha - (n/p)$ if α is not odd, and $\ell = \alpha$ if α is odd. Since $Q - P \in \mathcal{P}_k$ and $\ell \geq k+1$, by Lemma 2.14 we see

$$\Delta_t^\ell (U_\alpha^f - U_\alpha^g) = \Delta_t^\ell (Q - P) = 0.$$

Hence Corollary 2.13 gives

$$d_{\alpha,\ell} f = D^{\alpha,\ell} U_\alpha^f = D^{\alpha,\ell} U_\alpha^g = d_{\alpha,\ell} g.$$

Therefore, since $d_{\alpha,\ell} \neq 0$ by Remark 2.3, we have $f = g$, and hence $P = Q$.

REMARK 2.16. Let $2[(\ell+1)/2] > \alpha, \ell > \alpha - (n/p)$ if α is not an odd number, and $\ell = \alpha$ if α is an odd number. By Lemma 2.15, the Riesz potential operator U_α^f of order α is a one-to-one mapping from L^p to R_α^p. Hence it follows from Corollary 2.13 and Remark 2.3 that the operator $D^{\alpha,\ell}/d_{\alpha,\ell}$ which maps R_α^p to L^p is the inverse operator of the Riesz potential operator of order α (cf. [Ba: Theorem 4.4], [Sa: Theorem 2] and [SKM: Theorem 26.3]).

Now we give an improvement of S.G.Samko [Sa: Theorem 10] and the author [Ku2: Remark 4.10].

PROPOSITION 2.17. (i) *We assume that* $2[(\ell+1)/2] > \alpha, \ell > \alpha - (n/p)$. *Then*

$$R_\alpha^p \oplus \mathcal{P}_k \subset \mathcal{L}_{\alpha,\ell}^p \cap \begin{cases} L^{p,-\alpha}, & \alpha - (n/p) \notin N \\ L^{p,-\alpha,\log}, & \alpha - (n/p) \in N \end{cases}$$

with $k = [\alpha - (n/p)]$, *and*

$$\begin{cases} \|D^{\alpha,\ell}u\|_p + \|u\|_{p,-\alpha} \leq C\|u\|_{R_\alpha^p \oplus \mathcal{P}_k}, & \alpha - (n/p) \notin N \\ \|D^{\alpha,\ell}u\|_p + \|u\|_{p,-\alpha,\log} \leq C\|u\|_{R_\alpha^p \oplus \mathcal{P}_k}, & \alpha - (n/p) \in N. \end{cases}$$

(ii) *We assume that* $\min(2[(\ell+1)/2], (1+\frac{n}{\alpha p})[(\ell+1)/2] > \alpha$ *and* $\ell > \alpha - (n/p)$ *if* $\alpha \neq$ *odd, and* $\ell = \alpha$ *if* $\alpha =$ *odd and* $\frac{\alpha(\alpha-1)}{\alpha+1} < \frac{n}{p}$. *Then*

$$R_\alpha^p \oplus \mathcal{P}_k \supset \mathcal{L}_{\alpha,\ell}^p \cap \begin{cases} L^{p,-\alpha}, & \alpha - (n/p) \notin N \\ L^{p,-\alpha,\log}, & \alpha - (n/p) \in N \end{cases}$$

and

$$\|u\|_{R_\alpha^p \oplus \mathcal{P}_k} \leq C \begin{cases} \|D^{\alpha,\ell}u\|_p + \|u\|_{p,-\alpha}, & \alpha - (n/p) \notin N \\ \|D^{\alpha,\ell}u\|_p + \|u\|_{p,-\alpha,\log}, & \alpha - (n/p) \in N. \end{cases}$$

PROOF. (i) let $u \in R_\alpha^p \oplus \mathcal{P}_k$. Then $u = U_\alpha^f + P$ with $f \in L^p$ and $P \in \mathcal{P}_k$. Proposition 1.1 and $k = [\alpha - (n/p)]$ imply $u \in L^{p,-\alpha}$ in case $\alpha - (n/p) \notin N$ and $u \in L^{p,-\alpha,\log}$ in case of $\alpha - (n/p) \in N$. Moreover, since $\ell > \alpha - (n/p)$, by Lemmas 2.12 and 2.14 we see

$$D_\epsilon^{\alpha,\ell}u = D_\epsilon^{\alpha,\ell}U_\alpha^f = \rho_\epsilon^{\alpha,\ell} * f.$$

On account of $2[(\ell+1)/2] > \alpha$, it follows from Lemma 2.2 that $D_\epsilon^{\alpha,\ell}u = \rho_\epsilon^{\alpha,\ell} * f$ converges in L^p as $\epsilon \to 0$. Thus we have $u \in \mathcal{L}_{\alpha,\ell}^p$.

(ii) let $u \in \mathcal{L}_{\alpha,\ell}^p \cap L^{p,-\alpha}$ in case of $\alpha - (n/p) \notin N$ and $u \in \mathcal{L}_{\alpha,\ell}^p \cap L^{p,-\alpha,\log}$ in case of $\alpha - (n/p) \in N$. We take an integer m such that $m \geq 2\alpha$. Since $u \in L^{p,-\alpha,\log}$ and $m > 2\alpha - (n/p)$, by Lemma 2.6 we see that

$$\Delta_h^m \kappa_\alpha * D_{1/j}^{\alpha,\ell}u(x) = u * (\Delta_h^m \rho_{1/j}^{\alpha,\ell})(x)$$

for almost every x. Since $u \in \mathcal{L}_{\alpha,\ell}^p$, there exists an $f \in L^p$ such that $D_\epsilon^{\alpha,\ell}u \to f$ in L^p as $\epsilon \to 0$. Since $m > \alpha - (n/p)$, by Lemma 2.7 there exists a number r such that $1 < r < p'$ and $\Delta_h^m \kappa_\alpha \in L^r$. Hence, if we put $1/q = (1/p) + (1/r) - 1$, then

Young's inequality implies $\Delta_h^m \kappa_\alpha * D_{1/j}^{\alpha,\ell} u \to \Delta_h^m \kappa_\alpha * f$ in L^q as $j \to \infty$. On the other hand, since the assumptions about ℓ satisfy those of Lemma 2.10, we have

$$u * (\Delta_h^m \rho_{1/j}^{\alpha,\ell})(x) \to d_{\alpha,\ell} \Delta_h^m u(x) \qquad \text{as } j \to \infty$$

for almost every x. Hence

$$\Delta_h^m \kappa_\alpha * f(x) = d_{\alpha,\ell} \Delta_h^m u(x)$$

for almost every x. Moreover, by Lemma 2.11 we see that

$$\Delta_h^m U_\alpha^f = \Delta_h^m \kappa_\alpha * f.$$

Therefore, $d_{\alpha,\ell} \Delta_h^m u = \Delta_h^m U_\alpha^f$, and hence by Remark 2.3 and Lemma 2.14 there exists a polynomial $P \in \mathcal{P}_{m-1}$ such that

$$u = U_\alpha^{f/d_{\alpha,\ell}} + P.$$

By Proposition 1.1 and the condition

$$u \in \begin{cases} L^{p,-\alpha}, & \alpha - (n/p) \notin N \\ L^{p,-\alpha,\log}, & \alpha - (n/p) \in N, \end{cases}$$

we have

$$P \in \begin{cases} L^{p,-\alpha}, & \alpha - (n/p) \notin N \\ L^{p,-\alpha,\log}, & \alpha - (n/p) \in N. \end{cases}$$

This implies $P \in \mathcal{P}_k$, and hence $u \in R_\alpha^p + \mathcal{P}_k$.

The estimates of the norms in (i) and (ii) follow from Proposition 1.1 and Corollary 2.13. This completes the proof of the proposition.

3. Relations between Bessel and Riesz potential spaces

In this section we are concerned with relations between Bessel and Riesz potential spaces. At first, we treat a correspondence between individual Bessel potentials and Riesz potentials. For a tempered distribution u, we denote by $\mathcal{F}u$ the Fourier transform of u. The Fourier transform of $u \in L^1$ is defined by

$$\mathcal{F}u(x) = \int e^{-ix\cdot y} u(y) dy$$

where $x \cdot y = x_1y_1 + \cdots + x_ny_n$ denotes the usual inner product. The Fourier transforms of the Bessel kernel and the Riesz kernel are given by

(3.1) $$\mathcal{F}G_\alpha(x) = (1+|x|^2)^{-\alpha/2}$$ ([St2: Proposition 2 in Chap.V])

and

(3.2) $$\mathcal{F}\kappa_\alpha(x) = \mathrm{Pf.}|x|^{-\alpha}$$ ([Sc: section 7 in Chap.VII])

where Pf. stands for the pseudo function [Sc: section 3 in Chap.II].

The following lemma is due to E.M.Stein [St2: Lemma 2 in Chap.V].

LEMMA 3.1. There exists an integrable function h_α so that its Fourier transform $\mathcal{F}h_\alpha$ is given by

$$1 + \mathcal{F}h_\alpha(x) = \frac{|x|^\alpha}{(1+|x|^2)^{\alpha/2}}.$$

LEMMA 3.2. *If ℓ is a nonnegative integer such that $\ell > \alpha - (n/2)$, then*

$$\Delta_t^\ell G_\alpha = \Delta_t^\ell \kappa_\alpha + \Delta_t^\ell \kappa_\alpha * h_\alpha$$

PROOF. By (3.1) we have

(3.3) $$\mathcal{F}(\Delta_t^\ell G_\alpha)(x) = (-1)^\ell (1-e^{ix\cdot t})^\ell (1+|x|^2)^{-\alpha/2}.$$

Further, by (3.2) and $\ell - \alpha > -n/2 > -n$, we obtain

(3.4) $$\mathcal{F}(\Delta_t^\ell \kappa_\alpha)(x) = (-1)^\ell (1-e^{ix\cdot t})^\ell \mathrm{Pf.}|x|^{-\alpha} = (-1)^\ell (1-e^{ix\cdot t})^\ell |x|^{-\alpha}.$$

We denote by $g_1(x)$ the restriction of $\Delta_t^\ell \kappa_\alpha(x)$ to $\{x : |x| < 2\ell|t|\}$, and let $g_2(x) = \Delta_t^\ell \kappa_\alpha(x) - g_1(x)$. It is clear that $g_1 \in L^1$, and it follows from Lemma 2.4 and $\ell > \alpha - (n/2)$ that $g_2 \in L^2$. Therefore we have

$$\begin{aligned}\mathcal{F}(\Delta_t^\ell \kappa_\alpha * h_\alpha)(x) &= \mathcal{F}(g_1 * h_\alpha)(x) + \mathcal{F}(g_2 * h_\alpha)(x)\\ &= \mathcal{F}g_1(x)\mathcal{F}h_\alpha(x) + \mathcal{F}g_2(x)\mathcal{F}h_\alpha(x)\\ &= (\mathcal{F}g_1(x) + \mathcal{F}g_2(x))\mathcal{F}h_\alpha(x)\\ &= \mathcal{F}(\Delta_t^\ell \kappa_\alpha)(x)\mathcal{F}h_\alpha(x).\end{aligned}$$

Hence by (3.3), (3.4) and Lemma 3.2 we have

$$\begin{aligned}\mathcal{F}(\Delta_t^\ell \kappa_\alpha &+ \Delta_t^\ell \kappa_\alpha * h_\alpha)(x)\\ &= (-1)^\ell (1-e^{ix\cdot t})^\ell |x|^{-\alpha}(1+\mathcal{F}h_\alpha(x))\\ &= (-1)^\ell (1-e^{ix\cdot t})^\ell |x|^{-\alpha}\frac{|x|^\alpha}{(1+|x|^2)^{\alpha/2}}\\ &= (-1)^\ell (1-e^{ix\cdot t})^\ell (1+|x|^2)^{-\alpha/2}\\ &= \mathcal{F}(\Delta_t^\ell G_\alpha)(x).\end{aligned}$$

Thus we obtain the lemma.

REMARK 3.3. By lemma 2.4, $\Delta_t^\ell \kappa_\alpha * h_\alpha(x)$ exists almost everywhere for $\ell > \alpha - n$.

LEMMA 3.4 ([Ku2: Lemma 4.4 (ii)]). *If $\ell > \alpha - (n/p)$ and $f \in L^p$, then*

$$\int |f(x-y)| \int_{|t|\geq\epsilon} \frac{|\Delta_t^\ell \kappa_\alpha(y)|}{|t|^{n+\alpha}} dt dy < \infty$$

for almost every x in case of $\alpha - (n/p) \leq 0$, and for all x in case of $\alpha - (n/p) > 0$.

LEMMA 3.5. *Let $f \in L^p, 2[(\ell+1)/2] > \alpha$ and $\ell > \max(\alpha - (n/p), \alpha - (n/2))$. Then*

$$D^{\alpha,\ell} G_\alpha^f = d_{\alpha,\ell} f + d_{\alpha,\ell} f * h_\alpha.$$

PROOF. Since $\ell > \max(\alpha - (n/p), \alpha - (n/2))$ and $f, f * h_\alpha \in L^p$, by Lemmas 3.2, 3.4 and Fubini's Theorem we have

$$\begin{aligned}
D_\epsilon^{\alpha,\ell} G_\alpha^f(x) &= \int_{|t|\geq\epsilon} \frac{\Delta_t^\ell G_\alpha * f(x)}{|t|^{n+\alpha}} dt \\
&= \int_{|t|\geq\epsilon} \frac{(\Delta_t^\ell \kappa_\alpha + \Delta_t^\ell \kappa_\alpha * h_\alpha) * f(x)}{|t|^{n+\alpha}} dt \\
&= \int_{|t|\geq\epsilon} \frac{1}{|t|^{n+\alpha}} \int \Delta_t^\ell \kappa_\alpha(y) f(x-y) dy dt \\
&\quad + \int_{|t|\geq\epsilon} \frac{1}{|t|^{n+\alpha}} \int \Delta_t^\ell \kappa_\alpha(y) h_\alpha * f(x-y) dy dt \\
&= \int f(x-y) \int_{|t|\geq\epsilon} \frac{\Delta_t^\ell \kappa_\alpha(y)}{|t|^{n+\alpha}} dt dy \\
&\quad + \int h_\alpha * f(x-y) \int_{|t|\geq\epsilon} \frac{\Delta_t^\ell \kappa_\alpha(y)}{|t|^{n+\alpha}} dt dy \\
&= \rho_\epsilon^{\alpha,\ell} * f(x) + \rho_\epsilon^{\alpha,\ell} * h_\alpha * f(x).
\end{aligned}$$

Since $2[(\ell+1)/2] > \alpha$, by Lemma 2.2 and Remark 2.3 we obtain

$$D^{\alpha,\ell} G_\alpha^f = \lim_{\epsilon\to 0} D_\epsilon^{\alpha,\ell} G_\alpha^f = d_{\alpha,\ell} f + d_{\alpha,\ell} h_\alpha * f$$

in L^p. This completes the proof of the lemma.

We define the bounded operator T^α on L^p as follows:

$$T^\alpha f = f + f * h_\alpha.$$

We set

$$T^{\alpha}(L^p) = \{T^{\alpha} f : f \in L^p\}.$$

PRPPOSITION 3.6. (i) *For a function* $f \in L^p$, *we have*

$$G_{\alpha}^{f} = U_{\alpha}^{T^{\alpha} f} + P$$

where P *is a polynomial of at most degree* k.
(ii) *The operator* T^{α} *is one-to-one on* L^p, *and if* $g \in T^{\alpha}(L^p)$, *then*

$$U_{\alpha}^{g} + P = G_{\alpha}^{(T^{\alpha})^{-1} g}$$

where P *is a polynomial of at most degree* k *and* $(T^{\alpha})^{-1}$ *is the inverse operator of* T^{α}.

PROOF. (i) Let $f \in L^p$. We take a positive integer ℓ such that $\ell > \max(\alpha - (n/p), \alpha - (n/2))$. By Lemmas 3.2 and 2.11 we have

$$\begin{aligned} \Delta_t^{\ell} G_{\alpha}^{f} &= \Delta_t^{\ell} G_{\alpha} * f \\ &= (\Delta_t^{\ell} \kappa_{\alpha} + \Delta_t^{\ell} \kappa_{\alpha} * h_{\alpha}) * f \\ &= \Delta_t^{\ell} \kappa_{\alpha} * (f + h_{\alpha} * f) \\ &= \Delta_t^{\ell} U_{\alpha}^{T^{\alpha} f}. \end{aligned} \tag{3.5}$$

So, it follows from Lemma 2.14 that there exists a polynomial P of degree $\ell - 1$ such that

$$G_{\alpha}^{f} = U_{\alpha}^{T^{\alpha} f} + P.$$

This also implies $P \in L^{p,-\alpha}$ in case of $\alpha - (n/p) \notin N$ and $P \in L^{p,-\alpha,\log}$ in case of $\alpha - (n/p) \in N$, and hence $P \in \mathcal{P}_k$.

(ii) Let $f_1, f_2 \in L^p$ and $T^{\alpha} f_1 = T^{\alpha} f_2 = g$. By (i) there exist polynomials $P_1, P_2 \in \mathcal{P}_k$ such that

$$G_{\alpha}^{f_1} = U_{\alpha}^{g} + P_1, \qquad G_{\alpha}^{f_2} = U_{\alpha}^{g} + P_2.$$

Therefore, $P_1 - P_2 = G_{\alpha}^{f_1} - G_{\alpha}^{f_2} \in L^p$, and hence $P_1 = P_2$. So, $G_{\alpha}^{f_1} = G_{\alpha}^{f_2}$. This implies $f_1 = f_2$. Thus the operator T^{α} is one-to-one on L^p. Next, let $g \in T^{\alpha}(L^p)$. If we put $f = (T^{\alpha})^{-1} g$, then in the same way as (3.5) we have

$$\Delta_t^{\ell} G_{\alpha}^{f} = \Delta_t^{\ell} U_{\alpha}^{g}.$$

Hence there exists a polynomial $P \in \mathcal{P}_k$ such that $U_{\alpha}^{g} + P = G_{\alpha}^{f}$. We have thus proved the proposition.

By Proposition 3.6(ii), for $g \in T^\alpha(L^p)$, we have $G_\alpha^{(T^\alpha)^{-1}g} - U_\alpha^g \in \mathcal{P}_k$. We define the operator $P^{\alpha,p}$ as follows:

$$P^{\alpha,p}(g) = G_\alpha^{(T^\alpha)^{-1}g} - U_\alpha^g, \quad g \in T^\alpha(L^p).$$

The operator $P^{\alpha,p}$ maps $T^\alpha(L^p)$ to $\mathcal{P}_k$. Further, we define the space S_α^p as follows:

$$S_\alpha^p = \{U_\alpha^g + P^{\alpha,p}(g) : g \in T^\alpha(L^p)\}.$$

Let $u \in S_\alpha^p$. Then $u = U_\alpha^g + P^{\alpha,p}(g), g \in T^\alpha(L^p)$. Since the operator T^α is one-to-one on L^p, there exists an unique L^p–function f such that $g = T^\alpha f$. We define the norm $||u||_{S_\alpha^p}$ to be the L^p–norm of f.

Now we are in a position to prove our main theorem.

THEOREM 3.7. *Let $k = [\alpha - (n/p)]$. Then*

$$B_\alpha^p = S_\alpha^p = (R_\alpha^p \oplus \mathcal{P}_k) \cap L^p$$

and

$$||u||_{B_\alpha^p} = ||u||_{S_\alpha^p} \approx ||u||_{R_\alpha^p \oplus \mathcal{P}_k} + ||u||_p.$$

PROOF. First, $B_\alpha^p = S_\alpha^p$ and $||u||_{B_\alpha^p} = ||u||_{S_\alpha^p}$ follow from $U_\alpha^g + P^{\alpha,p}(g) = G_\alpha^{(T^\alpha)^{-1}g}, g \in T^\alpha(L^p)$. Next, $S_\alpha^p \subset (R_\alpha^p \oplus \mathcal{P}_k) \cap L^p$ is clear from the definition. We show

$$||u||_{R_\alpha^p \oplus \mathcal{P}_k} + ||u||_p \leq C||u||_{S_\alpha^p}. \tag{3.6}$$

Let $u \in S_\alpha^p$. Then $u = U_\alpha^g + P^{\alpha,p}(g)$ and $g = T^\alpha f, f \in L^p$. Since

$$||u||_{R_\alpha^p \oplus \mathcal{P}_k} = ||g||_p + \begin{cases} ||P^{\alpha,p}(g)||_{p,-\alpha}, & \alpha - (n/p) \notin N \\ ||P^{\alpha,p}(g)||_{p,-\alpha,\log}, & \alpha - (n/p) \in N, \end{cases}$$

We must estimate $||g||_p, ||P^{\alpha,p}(g)||_{p,-\alpha}, ||P^{\alpha,p}(g)||_{p,-\alpha,\log}$ and $||u||_p$. Since T^α is a bounded operator on L^p, we see that

$$||g||_p = ||T^\alpha f||_p \leq C||f||_p = C||u||_{S_\alpha^p}.$$

Further, in case $\alpha - (n/p) \notin N$, by Proposition 1.1 we have

$$\begin{aligned} ||P^{\alpha,p}(g)||_{p,-\alpha} &= ||G_\alpha^{(T^\alpha)^{-1}g} - U_\alpha^g||_{p,-\alpha} \leq ||G_\alpha^f||_p + ||U_\alpha^g||_{p,-\alpha} \\ &\leq C(||f||_p + ||g||_p) \leq C||f||_p = C||u||_{S_\alpha^p}. \end{aligned}$$

The estimate of $||P^{\alpha,p}(g)||_{p,-\alpha,\log}$ in case of $\alpha - (n/p) \in N$ is the same. Moreover, we have

$$\begin{aligned} ||u||_p &= ||U_\alpha^g + P^{\alpha,p}(g)||_p = ||G_\alpha^{(T^\alpha)^{-1}g}||_p \\ &= ||G_\alpha^f||_p \leq C||f||_p = C||u||_{S_\alpha^p}. \end{aligned}$$

Thus we obtain (3.6). Finally we show that $(R_\alpha^p \oplus \mathcal{P}_k) \cap L^p \subset B_\alpha^p$ and $||u||_{B_\alpha^p} \leq C(||u||_{R_\alpha^p \oplus \mathcal{P}_k} + ||u||_p)$. In case $\alpha - (n/p) \notin N$, we take a positive integer ℓ such that $2[(\ell+1)/2] > \alpha$ and $\ell > \alpha - (n/p)$ Then by Proposition 2.1 and 2.17(i), we have

$$(R_\alpha^p \oplus \mathcal{P}_k) \cap L^p \subset \mathcal{L}_{\alpha,\ell}^p \cap L^{p,-\alpha} \cap L^p = \mathcal{L}_{\alpha\ell}^p \cap L^p = B_\alpha^p$$

and

$$||u||_{B_\alpha^p} \approx ||D^{\alpha,\ell}u||_p + ||u||_p \leq ||D^{\alpha,\ell}||_p + ||u||_{p,-\alpha} + ||u||_p \leq C(||u||_{R_\alpha^p \oplus \mathcal{P}_k} + ||u||_p).$$

The proof of the case $\alpha - (n/p) \in N$ is the same. We have thus completed the proof of the theorem.

REMARK 3.8. If $\alpha \neq$ odd, or $\alpha =$ odd and $\frac{\alpha(\alpha-1)}{\alpha+1} < \frac{n}{p}$, then $B_\alpha^p = (R_\alpha^p \oplus \mathcal{P}_k) \cap L^p$ follows from Propositions 2.1 and 2.17.

COROLLARY 3.9. *Let $g \in L^p$ and $k = [\alpha - (n/p)]$. Then the following three conditions are equivalent:*
(I) $g \in T^\alpha(L^p)$,
(II) *there exists a polynomial P such that $U_\alpha^g + P \in L^p$,*
(III) *there exists an unique polynomial P of degree k such that $U_\alpha^g + P \in L^p$.*

PROOF. First, we show (II) $\Longleftrightarrow$ (III). Since (III) $\Longrightarrow$ (II) is trivial, it suffices to show (II) $\Longrightarrow$ (III). We suppose that there exists a polynomial P such that

$$U_\alpha^g + P \in L^p. \tag{3.7}$$

The condition (3.7) implies $P \in L^{p,-\alpha}$ in case of $\alpha - (n/p) \notin N$ and $P \in L^{p,-\alpha,\log}$ in case of $\alpha - (n/p) \in N$, and hence $P \in \mathcal{P}_k$. In order to show the uniqueness of P, we assume that $P, Q \in \mathcal{P}$ and $U_\alpha^g + P, U_\alpha^g + Q \in L^p$. Then we have

$$P - Q = U_\alpha^g + P - (U_\alpha^g + Q) \in L^p.$$

This gives $P = Q$. Next, we show (III) $\Longrightarrow$ (I). We suppose that there exists a polynomial $P \in \mathcal{P}_k$ such that $U_\alpha^g + P \in L^p$. Then $U_\alpha^g + P \in (R_\alpha^p \oplus \mathcal{P}_k) \cap L^p$, and hence $U_\alpha^g + P \in B_\alpha^p$ by Theorem 3.7. Therefore there exists an $f \in L^p$ such that $U_\alpha^g + P = G_\alpha^f$. We take a positive integer ℓ such that $2[(\ell+1)/2] > \alpha, \ell > \max(\alpha - (n/p), \alpha - (n/2))$ if α is not an odd number, and $\ell = \alpha$ if α is an odd number. Then by Lemma 3.5 we have

$$D^{\alpha,\ell} G_\alpha^f = d_{\alpha,\ell} f + d_{\alpha,\ell} f * h_\alpha.$$

On the other hand, by virtue of Corollary 2.13 and Lemma 2.14 we see

$$D^{\alpha,\ell}(U_\alpha^g + P) = d_{\alpha,\ell} g.$$

Consequently we obtain $g = f + f * h_\alpha$ by Remark 2.3, and hence $g \in T^\alpha(L^p)$. Finally, we show (I) $\Longrightarrow$ (II). Let $g \in T^\alpha(L^p)$. By Proposition 3.6 (ii) there exists a polynomial $P \in \mathcal{P}_k$ such that $U_\alpha^g + P = G_\alpha^{(T^\alpha)^{-1}g}$. Since $G_\alpha^{(T^\alpha)^{-1}g} \in L^p$, we obtain the required conclusion.

REMARK 3.10. By Corollary 3.9 $P^{\alpha,p}(g)$ is the unique polynomial $P \in \mathcal{P}_k$ such that $U_\alpha^g + P \in L^p$.

REMARK 3.11(cf. [No: Theorem 2]). We assume that $2[(\ell+1)/2] > \alpha, \ell > \max(\alpha - (n/p), \alpha - (n/2))$ if α is not odd, and $\ell = \alpha$ if α is odd. By Lemma 3.5, Proposition 3.6 and Remark 2.3, the operator $D^{\alpha,\ell}/d_{\alpha,\ell}$ is a one-to-one mapping from B_α^p to $T^\alpha(L^p)$, and

$$(T^\alpha)^{-1}\frac{D^{\alpha,\ell}}{d_{\alpha,\ell}}G_\alpha^f = f$$

for $f \in L^p$.

REMARK 3.12. The Lizorkin space Φ is defined as follows [SKM: §25]:

$$\Phi = \{\phi \in \mathcal{S} : \int \phi(x)x^\gamma dx = 0 \quad \text{for any } \gamma\}$$

where $\mathcal{S}$ is the Schwartz space. Then $\Phi \subset T^\alpha(L^p)$. Indeed, if $f \in \Phi$, then $g = \bar{\mathcal{F}}(\frac{(1+|x|^2)^{\alpha/2}}{|x|^\alpha}\mathcal{F}f) \in \Phi$ and $f = g + g * h_\alpha$ where $\bar{\mathcal{F}}$ stands for the inverse Fourier transform. Since Φ is dense in L^p(see [Li]), $T^\alpha(L^p)$ is also dense in L^p.

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