A sufficient condition for a map to be cobordant to an embedding

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Abstract

In this paper we give a proof of the theorem in [6] which asserts a sufficient condition for a map $f : M^n \to N^{2n-k}$ between compact manifolds without boundary to be cobordant to an embedding, since we did not give the details for the general case in [6].

1 Introduction

Throughout this paper, n-manifolds mean compact differentiable manifolds of dimension n. The (co-)homology is understood to have Z_2 for coefficients.

For a map $f : M^n \to N^{2n-k}$ between compact manifolds without boundary, let $w_i(f)$ be the i-th Stiefel-Whitney class of f and $f_! : H^i(M) \to H^{i+n-k}(N)$ the transfer homomorphism (or Umkehr homomorphism) of f. Further let

$$\theta(f) = f^* f_!(1) - w_{n-k}(f).$$

Then by [5, Lemma 2], $M \times \theta(f)$ is the $H^n(M) \times H^{n-k}(M)$-component of $U_M(1 \times w_{n-k}(f)) + (f \times f)^* U_N$, where $U_V \in H^{\dim V}(V \times V)$, denotes the Z_2-Thom class (or the Z_2-diagonal class) of a manifold V. Therefore, A. Haefliger [Theorem 5.2] implies that

Theorem (Haefliger) *If f is homotopic to an embedding, then*

$$\theta(f) = 0 \quad \textit{and} \quad w_{n-i}(f) = 0 \ \textit{for} \ i < k. \tag{1.1}$$

The inverse of this theorem may be hard to study. So we will study whether f is cobordant to an embedding in the sense of Stong [9] if the condition (1.1) in the above Theorem is satis-

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fied. Here a map $f_1 : M_1^n \to N_1^{n+k}$ is said to be cobordant to $f_2 : M_2^n \to N_2^{n+k}$ if there exist two cobordisms (W, M_1^n, M_2^n), $(V, N_1^{n+k}, N_2^{n+k})$ and a map $F : W \to V$ such that $F|M_i = f_i (i = 1,2)$. M. A. Aguilar and G. Pastor [1] determined the necessary and sufficient condition that a map $f : M^n \to N^{2n-k}, (k = 1,2)$ is cobordant to an embedding. In [6] we have considered cases when $k \geq 3$ and obtained following results:

Corollary 1.3 in [6] *Let $f : M^n \to N^{2n-k}$, $(k = 3,4)$ be a map. If $w_{n-i}(f) = 0$ for $0 < i < k$ and $\theta(f) = 0$, then f is cobordant to an embedding.*

Moreover we have stated the following theorem:

Theorem(Theorem 5.1' in [6]) *Let $n > 2k > 0$. Then a map $f : M^n \to N^{2n-k}$ is cobordant to an embedding if*

(1) $w_{n-i}(f) = 0$ *for* $1 \leq i < k$,

(2) $\theta(f) = 0$ *and*

(3) $w_i(M) \in f^* H^i(N)$ *for* $4i < k$.

From this theorem we obtained the following corollaries:

Corollary 1 *If $1 \leq k \leq 4, n \geq 2k+1$, a map $f : M^n \to N^{2n-k}$ is cobordant to an embedding if $w_{n-i}(f) = 0$ for $1 \leq i < k$ and $\theta(f) = 0$.*

Corollary 2 *If $5 \leq k \leq 8$, $n \geq 2k+1$, a map $f : M^n \to N^{2n-k}$ is cobordant to an embedding if*

(1) $w_{n-i}(f) = 0$ *for* $1 \leq i < k$,

(2) $\theta(f) = 0$ *and*

(3) $w_1(M) = 0$ *or* $w_1(f) = 0$.

Since in [6] we have omitted details of the proof of the above theorem for the general case, we will give the proof in this paper.

This paper is organized as follows: In §2, we recall the Stiefel-Whitney class $w(f)$ and the transfer homomorphism $f_!$ of a map $f : M^n \to N^{2n-k}$ and prepare some lemmas concerning $f_!$, $w_i(f)$'s, and the Steenrod squaring operations Sq^i's. In §3, we give the proof of the

Theorem.

2 Preliminaries

We adopt same notations and symbols as in [6]. For a manifold V, we denote by $w(V)$ and $\overline{w}(V)=w(V)^{-1}$ the total Stiefel-Whitney class and the total normal Stiefel-Whitney class of V, respectively. For a map $f: M^n \to N^{2n-k}$, the total Stiefel-Whitney class of f, $w(f)=\sum_{i\geq 0} w_i(f)$, is defined by the equation

$$w(f)=\overline{w}(M)f^*(w(N)), \tag{2.1}$$

and the transfer homomorphism $f_!: H^i(M) \to H^{i+n-k}(N)$ is defined by

$$f_!(x)=D_N f_*(x \cap [M]),$$

where D_N is the Poincaré duality and $[M] \in H_n(M)$ denotes the fundamental class of M. For $\mu=(i_1,i_2,\dots,i_p)$, let $w_\mu(V)=w_{i_1}(V)w_{i_2}(V)\dots w_{i_p}(V)$ and $|\mu|=\sum_{1\leq j\leq p} i_j$. Then R. L. W. Brown's theorem [2, p. 247] implies that

Theorem(Brown) *Let $n>2k>0$. Then a map $f: M^n \to N^{2n-k}$ is cobordant to an embedding if and only if the following conditions* (1) *and* (2) *are satisfied:*

(1) $\langle w_\mu(M)w_\lambda(f),[M]\rangle = 0$ *if* $|\mu|+|\lambda|=n$ *and* λ *has a component with* $>n-k$, *and*

(2) $\langle f^*(w_\lambda(N))w_\mu(M)f^*f_!(w_\nu(M)) - f^*(w_\lambda(N))w_\mu(M)w_\nu(M)w_{n-k}(f),[M]\rangle = 0$
for all λ,μ *and* ν *with* $|\lambda|+|\mu|+|\nu|=k$.

We denote by $v(M)=\sum_{i\geq 0} v_i(M)$ the total Wu class of M. The following relations are well-known:

$$Sq(v(M))=w(M), \tag{2.2}$$

$$Sq^i x_{n-i} = v_i x_{n-i} \quad \text{for all } x_{n-i} \in H^{n-i}(M), \tag{2.3}$$

$$Sq^i w_j(\xi)=\sum_{0\leq t\leq i}\binom{j-i+t-1}{t} w_{i-t}(\xi)w_{j+t}(\xi). \tag{2.4}$$

In the following lemmas, we list some relations among $f_!$, the Steenrod operations Sq^i and the Stiefel-Whitney classes, the first of which is seen in, eg, [3] (cf. [1]), while the last follows from the definition of $f_!$, (cf. [2]):

Lemma 1 *For a map* $f : M^n \to N^{2n-k}$ *there are relations*

(1) $f_!\left(f^*(x)y\right) = xf_!(y)$ *for* $x \in H^*(N), y \in H^*(M)$,

(2) $Sqf_!(x) = f_!\left(Sq(x)w(f)\right)$,

(3) $\langle xf_!(y),[N]\rangle = \langle f^*(x)y,[M]\rangle$ *if* $\dim x + \dim y = n$,

(4) $\langle f^*(x)yf^*f_!(z),[M]\rangle = \langle f^*(x)zf^*f_!(y),[M]\rangle$ *if* $\dim x + \dim y + \dim z = k$.

In particular $\langle f^*(x)f^*f_!(y),[M]\rangle = \langle f^*(x)yf^*f_!(1),[M]\rangle$.

Further, we have the following

Lemma 2 *Let* $f : M^n \to N^{2n-k}$ *be a map. Then*

(1) $f^*(x_i)w_{n-i}(f) = 0$ *for* $x_i \in H^i(N), (0 \le i < k)$.

(2) $f^*(y)xf^*f_!(x) = f^*(y)\sum_{t=0}^{i-1} Sq^t(x)w_{n-k+i-t}(f) + f^*(y)x^2w_{n-k}(f)$

for $x \in H^i(M), y \in H^{k-2i}(N), (0 \le 2i \le k)$.

(3) *In particular* $f^*(y)f^*f_!(1) - f^*(y)w_{n-k}(f) = 0$ *for* $y \in H^k(N)$.

Proof. See the proof of Lemma 2.2[6]. □

3 Proof of the Theorem

The following lemmas are consequences of the definition of the Wu class and the Wu's formula (2.4).

Lemma 3 *Let* t *be an integer such that* $1 \le t \le k/2$. *Let* $l \ge 0$ *and* $r \ge 1$ *be the integers defined by* $t = 2^r l + 2^{r-1}$. *Assume that* $l \ge 1$. *Then*

$$w_t(M) = Sq^{2^{r-1}} w_{2^r l}(M) + \sum_{s=1}^{2^{r-1}} a_s w_s(M) w_{t-s}(M), \tag{3.1}$$

where $a_s \in Z_2$.

Lemma 4 *If* $2 \le 2^{r-1} \le n/2$ *then*

$$\upsilon_{2^{r-1}}(M) = w_{2^{r-1}}(M) + w_{2^{r-2}}(M)^2 + \sum_{s=1}^{2^{r-2}-1} b_s w_s(M) w_{\lambda(s)}(M), \qquad (3.2)$$

where $b_s \in Z_2, |\lambda(s)| = 2^{r-1} - s$.

We postpone the proof of the above two lemmas and prove the Theorem using them. By virtue of Lemma 1(4), to prove the Theorem we have only to prove

$$(\mathrm{E}_{\lambda,\mu,\nu}) \qquad w_\lambda(M) f^* f_!\big(w_\mu(M) f^* w_\nu(N)\big) = w_\lambda(M) w_\mu(M) f^* w_\nu(N) w_{n-k}(f),$$

for $|\lambda| + |\mu| + |\nu| = k$ and $|\lambda| \le k/2$, under the assumptions (1)(2)(3) of the Theorem.

Proof of the Theorem :

We prove by induction on $|\lambda| = t$ that $(\mathrm{E}_{\lambda,\mu,\nu})$ holds for $|\lambda| + |\mu| + |\nu| = k$ and $|\lambda| = t \le k/2$, By the assumption $\theta(f) = 0, (\mathrm{E}_{(0),\mu,\nu})$ holds. Let $|\lambda| = t \ge 1$. Suppose that $(\mathrm{E}_{\lambda,\mu,\nu})$ holds for $|\lambda| \le t-1$

Case 1: $\lambda = (t)$

First we consider the case $t = 2^r l + 2^{r-1}$ *for* $l \ge 1$ *and* $r \ge 1$. Then since $2^{r-1} < t/2 \le k/4$ we have $w_i(M) \in f^* H^i(N)$ for $i \le 2^{r-1}$. Hence by the assumption for $1 \le s \le 2^{r-1}$ and $|\mu| + |\nu| = k - t$ we have

$$w_s(M) w_{t-s}(M)\big(f^* f_!\big(w_\mu(M) f^* w_\nu(N)\big) - w_\mu(M) f^* w_\nu(N) w_{n-k}(f)\big) = 0.$$

Thus denoting $w_\mu(M) f^* w_\nu(N) = x$ we have by Lemma 3

$$\begin{aligned}
w_t(M)\big(f^* f_! x - x w_{n-k}(f)\big) &= Sq^{2^{r-1}} w_{2^r l}(M)\big(f^* f_! x - x w_{n-k}(f)\big) \\
&= Sq^{2^{r-1}}\big(w_{2^r l}(M)\big(f^* f_! x - x w_{n-k}(f)\big)\big) \\
&\quad + \sum_{s=0}^{2^{r-1}-1} Sq^s w_{2^r l}(M) Sq^{2^{r-1}-s}\big(f^* f_! x - x w_{n-k}(f)\big) \\
&= \upsilon_{2^{r-1}}(M) w_{2^r l}(M)\big(f^* f_! x - x w_{n-k}(f)\big) + \sum_{|\lambda'|<t} w_{\lambda'}(M)\big(f^* f_! x_{\lambda'} - x_{\lambda'} w_{n-k}(f)\big) \\
&\quad + \sum_{i=1}^{k} y_i w_{n-k+i}(f),
\end{aligned}$$

where $x_{\lambda'} \in H^{t-|\lambda'|}(M), y_i \in H^{k-i}(M)$ and they are expressed as $\sum_{\rho,\tau} w_\sigma(M) f^* w_\tau(N)$.

Since $w_{\lambda'}(M)\left(f^* f_! x_{\lambda'} - x_{\lambda'} w_{n-k}(f)\right) = 0$ for $|\lambda'| < t$ by the induction hypothesis and $y_i w_{n-k+i}(f) = 0$ for $1 \le i \le k$ by the assumption, we have

$$w_t(M)\left(f^* f_! x - x w_{n-k}(f)\right) = v_{2^{r-1}}(M) w_{2^r l}(M)\left(f^* f_! x - x w_{n-k}(f)\right)$$
$$= w_{2^r l}(M)\left(f^* f_!\left(v_{2^{r-1}}(M)x\right) - v_{2^{r-1}}(M) x w_{n-k}(f)\right),$$

since $v_{2^{r-1}}(M) \in f^* H^*(N)$. Hence by the induction hypothesis we have

$$w_t(M)\left(f^* f_! x - x w_{n-k}(f)\right) = 0.$$

Next we consider the case $t = 2^{r-1}$. If $2t < k$ then $2^{r-2} < k/4$ and $w_i(M) \in f^* H^i(N)$ for $i \le 2^{r-2}$ hence by Lemma 4

$$
\begin{aligned}
& w_{2^{r-1}}(M)\left(f^* f_! x - x w_{n-k}(f)\right) \\
&= v_{2^{r-1}}(M)\left(f^* f_! x - x w_{n-k}(f)\right) + \sum_{s=1}^{2^{r-2}} b_s w_s(M) w_{\lambda(s)}(M)\left(f^* f_! x - x w_{n-k}(f)\right) \\
&= Sq^{2^{r-1}}\left(f^* f_! x - x w_{n-k}(f)\right) + \sum_{s=1}^{2^{r-2}} b_s w_{\lambda(s)}(M)\left(f^* f_!\left(w_s(M)x\right) - w_s(M) x w_{n-k}(f)\right) \\
&= Sq^{2^{r-1}}\left(f^* f_! x - x w_{n-k}(f)\right) \\
&= \theta(f) \sum_{s=0}^{2^{r-1}} Sq^s x w_{2^{r-1}-s}(f) + \sum_{i=1}^{k} y_i w_{n-k+i}(f) \quad \left(\text{where } y_i \in H^{k-i}(M)\right) \\
&= 0.
\end{aligned}
$$

Now we consider the remaining case $t = 2s = 2^{r-1} = k/2$. In this case $\nu = (0)$ and $|\lambda| = |\mu|$. Hence if $\mu \ne (t), (s,s)$ then $w_\mu(M) = w_p(M) w_{\mu'}(M)$ for some p, μ' such that $1 \le p < s = k/4$ and $|\mu'| < t$. Then since $w_p(M) \in f^* H^p(N)$ we have by the induction hypothesis

$$w_t(M) f^* f_!\left(w_p(M) w_{\mu'}(M)\right) = w_t(M) w_p(M) f^* f_!\left(w_{\mu'}(M)\right)$$
$$= w_{\mu'}(M) f^* f_!\left(w_t(M) w_p(M)\right) = w_{\mu'}(M) w_t(M) w_p(M) w_{n-k}(f)$$
$$= w_t(M) w_\mu(M) w_{n-k}(f).$$

If $\mu = (t) = \lambda$, then $\left(\mathrm{E}_{\lambda,\mu,(0)}\right)$ holds by Lemma 2 (2).

If $\mu = (s,s)$, then we have by Lemma 4, Lemma 2 (2), the assumption and the induction hypothesis

$$w_t(M)f^*f_!\left(w_s(M)^2\right)-w_t(M)w_s(M)^2w_{n-k}(f)$$
$$=Sq^t\left(f^*f_!\left(w_s(M)^2\right)-w_s(M)^2w_{n-k}(f)\right)+w_s(M)^2\left(f^*f_!\left(w_s(M)^2\right)-w_s(M)^2w_{n-k}(f)\right)$$
$$+\sum_{s=1}^{2^{r-2}-1}b_sw_s(M)w_{\lambda(s)}(M)\left(f^*f_!\left(w_s(M)^2\right)-w_s(M)^2w_{n-k}(f)\right)$$
$$=0.$$

□

Case2: $\lambda\neq(t)$. In this case we have

$w_\lambda(M)=w_s(M)w_{\lambda'}(M),\quad 1\le s\le t/2$ and $|\lambda'|=t-s<t$.

First we consider the case $\lambda\neq(t/2,t/2)$. Then we may assume that $s<t/2$ and by the assumption (3) we have $w_s(M)\in f^*H^s(N)$. Therefore we have by the induction hypothesis

$$w_\lambda(M)f^*f_!\left(w_\mu(M)f^*w_\nu(N)\right)=w_s(M)w_{\lambda'}(M)f^*f_!\left(w_\mu(M)f^*w_\nu(N)\right)$$
$$=w_{\lambda'}(M)f^*f_!\left(w_s(M)w_\mu(M)f^*w_\nu(N)\right)$$
$$=w_{\lambda'}(M)w_s(M)w_\mu(M)f^*w_\nu(N)w_{n-k}(f)$$
$$=w_\lambda(M)w_\mu(M)f^*w_\nu(N)w_{n-k}(f).$$

Now we consider the case $w_\lambda(M)=w_s(M)^2, s=t/2$. If t<*k/2* then $w_s(M)\in f^*H^s(N)$ by the assumption (3), therefore $\left(E_{\lambda,\mu,\nu}\right)$ holds. Hence we may assume $4s=2t=k$. Then $\nu=(0)$ since we assume $|\lambda|\le|\mu|$. Hence if $\mu\neq(t),(s,s)$ then $w_\mu(M)=w_p(M)w_{\mu'}(M)$ for some p, μ' such that $1\le p<s$ and $|\mu'|<t$. Then since $w_p(M)\in f^*H^p(N)$ we have by the induction hypothesis

$$w_s(M)^2f^*f_!\left(w_p(M)w_{\mu'}(M)\right)=w_s(M)^2w_p(M)f^*f_!\left(w_{\mu'}(M)\right)$$
$$=w_{\mu'}(M)f^*f_!\left(w_s(M)^2w_p(M)\right)=w_{\mu'}(M)w_s(M)^2w_p(M)w_{n-k}(f)$$
$$=w_s(M)^2w_\mu(M)w_{n-k}(f).$$

If $\mu=(t)$ then $\left(E_{(s,s),(t),(0)}\right)$ holds since $\left(E_{(t),(s,s),(0)}\right)$ holds by Case 1. If $\mu=(s,s)=\lambda$ then $\left(E_{(s,s),(s,s),(0)}\right)$ holds by Lemma 2 (2). Thus we complete the proof. □

Now we prove Lemma 3 and 4.

proof of Lemma 3: By Wu's formula (2.4) we have

$$Sq^{2^{r-1}} w_{2^r l}(M) = \sum_{s=0}^{2^{r-1}} \binom{2^r l - 2^{r-1} + s - 1}{s} w_{2^{r-1}-s}(M) w_{2^r l+s}(M).$$

Since $\binom{2^r l - 1}{2^{r-1}} \equiv 1 \mod 2$, we have $Sq^{2^{r-1}} w_{2^r l}(M) = w_t(M) + \sum_{s=1}^{2^{r-1}} a_s w_s(M) w_{t-s}(M)$, where $a_s \in Z_2$. □

proof of Lemma 4: For an integer $k \geq 1$ we define Z_2-submodules A_k, B_k of $H^*(M)$ as follows:

$A_k := \sum_{p_j \geq 1} Z_2 w_{p_1}(M) w_{p_2}(M) \cdots w_{p_k}(M),\ B_k := \sum_{l \geq k} A_l$.

Moreover we denote $A_2' := \sum_{p \neq q} Z_2 w_p(M) w_q(M)$.

Note that $Sq^t B_k \subset B_k$. To prove Lemma 4 it suffices to prove the following

Lemma 5 *For an integer $t \geq 2$, let r, l be the integers defined by* $t = 2^r + l,\ 1 \leq r,\ 0 \leq l < 2^r$. *Then*

$$(*)_t \quad \begin{cases} \text{if } l > 0, \text{ then } v_t(M) \in B_2, \text{ and} \\ \text{if } l = 0, \text{ then } v_t(M) \in w_{2^r}(M) + w_{2^{r-1}}(M)^2 + A_2' + B_3. \end{cases}$$

Proof. We prove the lemma by induction on *t*.

For *t* = 2, we have $v_2(M) = w_2(M) + w_1(M)^2$ and $(*)_2$ holds.

Suppose that $(*)_s$ holds for $s < t$.

If $t = 2^r + l,\ 0 < l < 2^r$, then

$$\begin{aligned} v_t(M) &= w_t(M) + \sum_{1 \leq s \leq t/2} Sq^s v_{t-s}(M) \\ &= w_t(M) + Sq^l w_{2^r}(M) + \sum_{s \neq l} Sq^s v_{t-s}(M). \end{aligned}$$

Since $\binom{2^r - 1}{l} \equiv 1 \mod 2$ we have from (2.4) $w_t(M) + Sq^l w_{2^r}(M) \in B_2$. On the other hand $v_{t-s}(M) \in B_2$ for $s \neq l$ by the induction hypothesis. Hence we have $v_t(M) \in B_2$.

If $t = 2^r$ then

$$\begin{aligned} v_t(M) &= w_t(M) + \sum_{1 \leq s \leq 2^{r-1}} Sq^s v_{t-s}(M) \\ &= w_{2^r}(M) + w_{2^{r-1}}(M)^2 + \sum_{1 \leq s \leq 2^{r-1}-1} Sq^s v_{2^r-s}(M). \end{aligned}$$

We have $v_{2^r-s}(M) \in B_2$ for $1 \leq s \leq 2^{r-1} - 1$ by the induction hypothesis. Let $s_i (i = 1, 2)$ be integers such that $0 \leq s_i < 2^{r-2}$. Since $\binom{2^{r-1} - s_i - 1}{s_i} \equiv 0 \mod 2$ for $1 \leq s_i < 2^{r-2}$ we have from (2.4) $Sq^{s_i} w_{2^{r-1}-s_i}(M) \in B_2$ for $1 \le s_i \le 2^{r-2}$. Therefore if $1 \le s_1 + s_2$ then $Sq^{s_1} w_{2^{r-1}-s_1}(M) Sq^{s_2} w_{2^{r-1}-s_2}(M)$ does not contain $w_{2^{r-1}}(M)^2$. Hence $Sq^s v_{2^r-s}(M) \in A_2' + B_3$ for $1 \le s \le 2^{r-1} - 1$. Thus we have

$v_t(M) \in w_{2^r}(M) + w_{2^{r-1}}(M)^2 + A_2' + B_3$. □

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