RECURSIVE ESTIMATION OF IMPULSE RESPONSE FUNCTION USING COVARIANCE INFORMATION IN LINEAR CONTINUOUS STOCHASTIC SYSTEMS

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Abstract This paper proposes a new recursive least-squares (RLS) estimation algorithm for an impulse response function in linear continuous-time wide-sense stationary stochastic systems. It is assumed that the input signal to the unknown impulse response function is contaminated by additive white Gaussian observation noise. The output signal from the system related with the impulse response function is observed with additive white Gaussian noise. The impulse response function is estimated recursively in terms of the variance of the white Gaussian observation noise included in the input signal, the autocovariance function of the process before the observation noise is added to the input signal, and the crosscovariance function between the output observed value and the input observed value, concerning the system based on the unknown impulse response function.

1. Introduction

The estimation problem of the impulse response function, which is classified as the nonparametric model, is one of the important quantities in the identification problem of an unknown system [1]. In the contexts of signal processing and automatic control, the Laplace transform of the impulse response function is defined by the transfer function in continuous-time systems [2]. The impulse response function is a solution of the Wiener-Hopf integral equation [3],[4]. In frequency domain [5], the spectral density function of a signal is calculated by Fourier transform of its autocorrelation function. In the relation with the Wiener-Hopf integral equation, the spectral density function for the impulse response function is calculated in terms of the crossspectral density function of the input with output of the un-

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known system and the spectral density function of the input. In time domain, on the estimation of the impulse response function with scalar input and output, the impulse response function in the Wiener-Hopf integral equation is obtained by applying white noise to the input of the unknown system [5]. However, for the system in the state of working, it might be desired to utilize a method which takes out the input and output data of the system and uses some useful information based on these data. This treatment using the sampled data is classified into the method in discrete-time systems. Also, as a different approach from above, the model-adjusting method assumes the a priori reference model of the impulse response function [6]. On the estimation of the impulse response function in linear discrete-time systems, the linear least-squares method [5], the method of steepest descent [7], the correlation method [5], and etc. are known. In the correlation method, the input and output data of the system are applied respectively to the whitening filter [5],[8] designed for the input values to the unknown system. Then, in terms of the variance of the whitened data in the input and the crosscovariance of the whitened data in the input with the processed data in the output, the impulse response function is calculated. As a consequence, in linear continuous-time systems, instead of use of white noise in the input, development of a new method, which uses some stochastic quantities related with the input and output information, might be desired.

Along above discussion, this paper designs a new RLS estimation algorithm for an unknown impulse response function by using the covariance information in linear continuous-time wide-sense stationary stochastic systems. The input signal to the impulse response function is contaminated by additive white Gaussian observation noise. The output signal from the system related with the impulse response function is observed with additive white Gaussian noise. The impulse response function is estimated recursively by the proposed algorithm in terms of the following quantities. (1) The variance of the observation noise in the input of the system related with the unknown impulse response function. (2) The autocovariance function of the process before the observation noise is added in the input of the unknown system. (3) The crosscovariance function between the output observed value degraded by the additive observation noise and the input signal process to the system. It is assumed that the autocovariance and crosscovariance functions are expressed in the semi-degenerate kernel form. The semi-degenerate kernel [9] is suitable for expressing these covariance functions by a finite sum of products of nonrandom functions.

2. Linear least-squares estimation of impulse response function

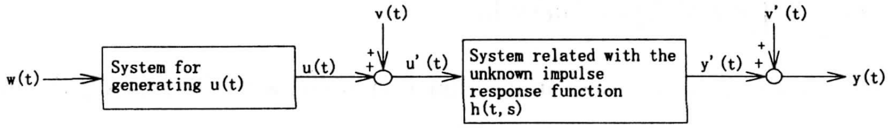

Fig.1 Block diagram concerned with the estimation problem of the impulse respones function.

Let us consider the block diagram of Fig.1 concerned with the estimation problem of the impulse response function. Let h(t,s) represent a scalar impulse response function to be estimated for an unknown system. Let w(t) represent zero-mean white Gaussian noise input to a system which generates the stochastic process u(t). Let u(t) be observed with additive zero-mean white Gaussian noise v(t) with the variance R.

$$E[v(t)v(s)] = R\delta(t-s),\ 0 \le s,t < \infty \tag{1}$$

It is assumed that u(t) is uncorrelated with v(s), i.e. E[u(t)v(s)]=0, $0\le s,t<\infty$. Let u'(t) represent a stochastic input process to the unknown system. u'(t) is given by

$$u'(t) = u(t) + v(t). \tag{2}$$

Let y'(t) represent a stochastic process in the output of the unknown system related with the impulse response function h(t,s). It is assumed that y'(t) is observed with additive zero-mean white Gaussian noise v'(t) and y'(t) is uncorrelated with v'(s), i.e. E[y'(t)v'(s)]=0, $0\le s,t<\infty$. Let y(t) represent the observed value of y'(t).

$$y(t) = y'(t) + v'(t) \tag{3}$$

The problem is to estimate the impulse response function h(t,s) of the unknown system in Fig.1. It is assumed that the objective system is asymptotically stable in linear wide-sense stationary stochastic systems. Hence, h(t,s)=h(t-s)(=h(τ), τ=t-s) and

$$\int_0^\infty |h(\tau)| d\tau < \infty. \tag{4}$$

Let y' (t) be expressed by

$$y'(t) = \int_0^t h(t,s')u'(s')ds'. \tag{5}$$

Multiplying (5) by u' (s) and taking expectation on both sides of (5), we have the Wiener-

Hopf integral equation [3]-[5].

$$E[y'(t)u'(s)] = \int_0^t h(t,s')E[u'(s')u'(s)]ds' \tag{6}$$

Let $K_{yu'}(t,s)$ represent the crosscovariance function of y(t) with u'(s), let $K_{y'u'}(t,s)$ represent the crosscovariance function of y'(t) with u'(s) and let $K_u(t,s)$ represent the autocovariance function of u(t). From the uncorrelation property of v'(t) with y'(s), the relationship $K_{yu'}(t,s)=K_{y'u'}(t,s)$ is valid. Substituting (2) into (6) and using the stochastic property of (1), we obtain

$$h(t,s)R = K_{yu'}(t,s) - \int_0^t h(t,s')K_u(s',s)ds', \ 0 \le s \le t. \tag{7}$$

It is assumed that $K_{yu'}(t,s)$, $K_u(t,s)$ and the variance R of the observation noise v(t) are given in estimating h(t,s).Here, let $K_{yu'}(t,s)$ and $K_u(t,s)$ be expressed in the semi-degenerate kernel form as follows.

$$K_{yu'}(t,s) = \begin{cases} \alpha(t)\beta^T(s), \ 0 \le s \le t \\ \gamma(t)\lambda^T(s), \ 0 \le t \le s \end{cases} \tag{8}$$

$$K_u(t,s) = \begin{cases} A(t)B^T(s), \ 0 \le s \le t \\ B(t)A^T(s), \ 0 \le t \le s \end{cases} \tag{9}$$

Here, $\alpha(t)$, $\beta(s)$, $\gamma(t)$, $\lambda(s)$, $A(t)$ and $B(s)$ are, 1 ×m, 1×m, 1×n, 1×n, 1×k and 1×k vectors respectively. On the crosscovariance function $K_{yu'}(t,s)$, it might be seen that the covariance information only for $0 \le s \le t$ suffices to be used in the RLS estimation algorithm of **Theorem 1** for h(t,s).

3. Derivation of RLS estimation algorithm for impulse response function

In this section, new RLS estimation algorithm for the impulse response function h(t,s) is proposed in **Theorem 1** for linear continuous-time wide-sense stationary stochastic systems. The algorithm is derived starting with (7) based on the invariant imbedding method [10].

Theorem 1. Let the crosscovariance function $K_{yu'}(t,s)$ between y(t) and u'(s), the autocovariance function $K_u(t,s)$ of u(t) and the variance R of the observation noise v(t) be given. Let $K_{yu'}(t,s)$ and $K_u(t,s)$ be expressed in the semi-degenerate kernel from as shown in (8) and (9). Then the RLS estimation algorithm for the impulse response function h(t,s) consists of (10)~(16). On the crosscovariance function $K_{yu'}(t,s)$, the information for $0 \le s \le t$ is used in estimating h(t,s).

$$h(t,s) = \alpha(t)J(t,s) \tag{10}$$

$$\frac{\partial J(t,s)}{\partial t} = -J(t,t)A(t)C(t,s) \tag{11}$$

$$\frac{\partial C(t,s)}{\partial t} = -C(t,t)A(t)C(t,s) \tag{12}$$

$$J(t,t) = (\beta^T(t) - r(t)A^T(t))/R \tag{13}$$

$$C(t,t) = (B^T(t) - q(t)A^T(t))/R \tag{14}$$

$$\frac{dr(t)}{dt} = J(t,t)(B(t) - A(t)q(t)), \quad r(0) = 0 \tag{15}$$

$$\frac{dq(t)}{dt} = C(t,t)(B(t) - A(t)q(t)), \quad q(0) = 0 \tag{16}$$

Proof. Substituting the expression (8) of the crosscovariance function $K_{yu'}(t,s)$ in the semi-degenerate kernel form into (7), we have

$$h(t,s)R = \alpha(t)\beta^T(s) - \int_0^t h(t,s')K_u(s',s)ds'. \tag{17}$$

Introducing an auxiliary function J(t,s), which satisfies

$$J(t,s)R = \beta^T(s) - \int_0^t J(t,s')K_u(s',s)ds', \tag{18}$$

we have (10) for h(t,s) from (17) and (18).

Differentiating (18) with respect to t, we have

$$\frac{\partial J(t,s)}{\partial t}R = -J(t,t)K_u(t,s) - \int_0^t \frac{\partial J(t,s')}{\partial t}K_u(s',s)ds'. \tag{19}$$

Substituting the semi-degenerate expression $K_u(t,s)=A(t)B^T(s)$ for $0 \le s \le t$ in (9) into (19), we have

$$\frac{\partial J(t,s)}{\partial t}R = -J(t,t)A(t)B^T(s) - \int_0^t \frac{\partial J(t,s')}{\partial t}K_u(s',s)ds'. \tag{20}$$

Introducing an auxiliary function, which satisfies

$$C(t,s)R = B^T(s) - \int_0^t C(t,s')K_u(s',s)ds', \tag{21}$$

we obtain (11) for J(t,s) from (20) and (21).

Differentiating (21) with respect to t, we have

$$\frac{\partial C(t,s)}{\partial t}R = -C(t,t)K_u(t,s) - \int_0^t \frac{\partial C(t,s')}{\partial t}K_u(s',s)ds'. \tag{22}$$

Similarly with the derivation of (11), from (9) and (22), we obtain the partial-differential equation (12) for C(t,s).

The function J(t,t) in (11) is formulated as follows. Putting s=t in (18), we have

$$J(t,t)R = \beta^T(t) - \int_0^t J(t,s')K_u(s',t)ds'. \tag{23}$$

Substituting $K_u(s', t)=B(s')A^T(t)$, $0 \le s' \le t$, from (9) into (23), we have

$$J(t,t)R = \beta^T(t) - \int_0^t J(t,s')B(s')A^T(t)ds'. \tag{24}$$

Introducing a new function r(t) defined by

$$r(t)=\int_0^t J(t,s')B(s')ds', \tag{25}$$

we obtain (13) for J(t,t).

Differentiating (25) with respect to t, we have

$$\frac{dr(t)}{dt}=J(t,t)B(t)+\int_0^t \frac{\partial J(t,s')}{\partial t}B(s')ds'. \tag{26}$$

Substituting (11) into (26), we have

$$\frac{dr(t)}{dt}=J(t,t)B(t)-J(t,t)A(t)\int_0^t C(t,s')B(s')ds'. \tag{27}$$

In (27), introducing a function q(t) defined by

$$q(t)=\int_0^t C(t,s')B(s')ds', \tag{28}$$

we obtain (15) for r(t).

Differentiating (28) with respect to t, we have

$$\frac{dq(t)}{dt}=C(t,t)B(t)+\int_0^t \frac{\partial C(t,s')}{\partial t}B(s')ds'. \tag{29}$$

Substituting (12) into (29) and using (28), we obtain (16) for q(t).

The function C(t,t) in (12) is formulated as follows. Putting s=t in (21), we have

$$C(t,t)R=B^T(t)-\int_0^t C(t,s')K_u(s',t)ds'. \tag{30}$$

Substituting $K_u(s',t)=B(s')A^T(t)$, $0\le s'\le t$, from (9) into (30), we have

$$C(t,t)R=B^T(t)-\int_0^t C(t,s')B(s')A^T(t)ds'. \tag{31}$$

Using (28) in (31), we obtain (14) for C(t,t). □

It is expected that, as the value of s becomes large the estimation accuracy for the stationary impulse response function $h(t,s)=h(t-s)=h(\tau)$, $0\le s\le t$, might be improved. This point is clarified by a succeeding numerical simulation example in section 4.

4. A Numerical Simulation Example

In this section, a numerical simulation example is demonstrated in order to show the validity of the proposed estimation algorithm of **Theorem 1**.

Let the autocovariance function $K_u(t,s)$ of (9) be given by

$$K_u(t,s) = \frac{a_2^2}{2a_1}\Gamma_w e^{-a_1|t-s|},\ a_1 = 0.85,\ a_2 = 0.9,\ \Gamma_w = 0.5^2. \tag{32}$$

Let the crosscovariance function $K_{yu'}(t,s)$, $0 \le s \le t$, of (8) be given by

$$K_{y'u}(t,s) = \frac{b_2 a_2^2}{a_1^2 - b_1^2}\Gamma_w e^{-b_1(t-s)} + \frac{b_2 a_2^2}{2a_1(b_1 - a_1)}\Gamma_w e^{-a_1(t-s)},\ 0 \le s \le t, \tag{33}$$

[11]. Let the impulse response function to be estimated be given by

$$h(t,s) = b_2 e^{-b_1 t},\ b_1 = 2,\ b_2 = 0.95. \tag{34}$$

From (9) and (32), we see that

$$A(t) = \frac{a_2^2}{2a_1}\Gamma_w e^{-a_1 t},\ B(s) = e^{a_1 s}. \tag{35}$$

From (8) and (33), we obtain expressions for $\alpha(t)$ and $\beta(s)$ as

$$\alpha(t) = \left[(b_2 R + \frac{b_2 a_2^2}{a_1^2 - b_1^2}\Gamma_w)e^{-b_1 t} \quad \frac{b_2 a_2^2}{2a_1(b_1 - a_1)}\Gamma_w e^{-a_1 t} \right],$$

$$\beta(s) = \left[e^{b_1 s} \quad e^{a_1 s} \right]. \tag{36}$$

Substituting $A(t)$, $B(t)$, $\alpha(t)$ and $\beta(t)$ into the estimation algorithm for the impulse response function h(t,s) of **Theorem 1**, we can calculate h(t,s) sequentially.

Fig.2 illustrates the true value of $h(t,s)$, s=0.5, $0.5 \le t \le 1.5$, and its estimated value (specified by the notation "+ +") for the white Gaussian observation noise sequence $N(0,0.1^2)$. Here, the values 0 and 0.1^2 in $N(0,0.1^2)$ represent the mean and the variance of the observation noise respectively. Fig.2 shows that the estimated impulse response function coincide with its true value almost precisely. Table 1 shows the mean-square values of estimation error of h(t,s) for the observation noise sequences of v(t), $N(0,0.1^2)$, $N(0,0.3^2)$, $N(0,0.5^2)$, $N(0,0.7^2)$ and $N(0,1)$, when s=0.5, s=1.0 and s=1.5. The M.S.V. is calculated on the estimation error of $h((k+j)\Delta, k\Delta)$, $0 \le j \le 1000$, $\Delta=0.001$, for each value of k, k=500, 1000, 1500. In Table 1, as the value of s in h(t,s) becomes large, the M.S.V. of the estimation error becomes small and the estimation accuracy for h(t,s) is improved. Also, the M.S.V. is not influenced almost by the noise variance R of v(t) for each corresponding value of s, s=0.5, 1.0, 1.5.

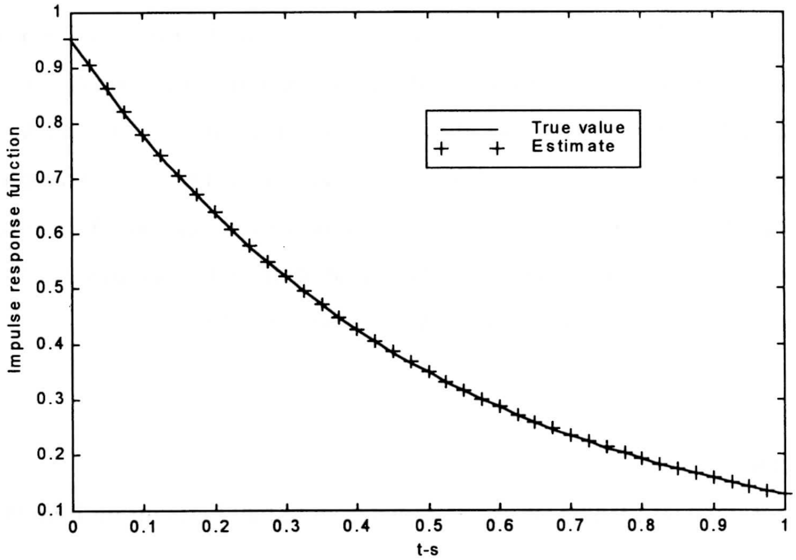

Fig.2 True value of *h(t,s)*, s=0.5, $0.5 \leq t \leq 1.5$, and its estimated value (specified by the notation "+ +") for the white Gaussian observation noise sequence $N(0,0.1^2)$.

Table 1 Mean-square values of estimation error of h(t,s) for the observation noise sequences of v(t), $N(0,0.1^2)$, $N(0,0.3^2)$, $N(0,0.5^2)$, $N(0,0.7^2)$ and $N(0.1)$, when s=0.5, s=1.0 and s=1.5.

White Gaussian observation noise	M.S.V. of estimation error for s=0.5	M.S.V. of estimation error for s=1.0	M.S.V. of estimation error for s=1.5
$N(0,0.1^2)$	8.1059539×10^{-4}	1.1346378×10^{-6}	3.0692884×10^{-9}
$N(0,0.3^2)$	7.3484486×10^{-4}	1.7405290×10^{-5}	4.2654862×10^{-7}
$N(0,0.5^2)$	2.0517957×10^{-4}	7.9874479×10^{-6}	3.1611227×10^{-7}
$N(0,0.7^2)$	6.9188386×10^{-5}	3.2167387×10^{-6}	1.5117485×10^{-7}
$N(0,1)$	1.9296446×10^{-5}	9.9808150×10^{-7}	5.1995271×10^{-8}

For references, the state-space models for generating u(t) and y'(t) are given by

$$\frac{du(t)}{dt} = -a_1 u(t) + a_2 w(t), \; E\left[w^2(t)\right] = \Gamma_w,$$

$$\frac{dy'(t)}{dt} = -b_1 y'(t) + b_2 u'(t), \; u'(t) = u(t) + v(t). \tag{37}$$

5. Conclusions

This paper has proposed the RLS estimation algorithm for the impulse response function in terms of the covariance information in linear continuous-time wide-sense stationary stochastic systems. The algorithm uses the variance of the observation noise v(t), the autocovariance function $K_u(t,s)$ of u(t) and the crosscovariance function $K_{yu'}(t,s)$ between the output observed value y(t) and u'(s). It is a characteristic that $K_u(t,s)$ and $K_{yu'}(t,s)$ are expressed in the semi-degenerate kernel form. The numerical simulation example in section 4 has shown that the proposed estimation algorithm for h(t,s) is feasible. As a result, its estimation accuracy is not influenced almost by the value of the noise variance R of v(t). Also, the estimation accuracy is improved as the value of s becomes large.

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