On Interchanging Sums and Integrals of Series of Functions

By TAKUMA KINOSHITA
Kagoshima University

§1. **Introduction.** One knows Lebesque's bounded convergence theorem ([1] Theorem 26D. 2 etc.). John W. Prott has shown that a convergent sequence of integrable functions permits exchange of lim and $\int$ if it is bracketed by two sequences which permit this exchange [2]. In this paper we shall treat series. It is assumed througout this note that all integrals are taken with respect to the same measure μ on a measure space (X, S, μ), all sets mentioned are measurable, all functions mentioned are measurable, inequalities hold almost everywhere (μ) and convergences of functions and series are either almost everywhere (μ) or in measure (μ). Proofs will be given for the case of convergence almost everywhere. (It follows that if a sequence of finife valued measurable functions converges *a. e.* to a finite limit on a set E of finite measure, then it converges in measure on E. See §22 of [1]).

§ 2. **Statement of theorems and proofs.** We shall first state and prove the lemma.

LEMMA. *If $\{f_n\}$ is a sequence of integrable functions such that $\sum_{n=1}^{\infty}\int|f_n|\,d\mu<\infty$, then the series $\sum_{n=1}^{\infty} f_n(x)$ converges a. e. to an integrable function $f(x)$ and*

$$\int fd\mu=\sum_{n=1}^{\infty}\int f_n\,d\mu \qquad \text{(See § 27.2 of [1]).}$$

PROOF. If we write $S_n=\sum_{K=1}^{n}|f_K(x)|$, then $S_1\leqq S_2\leqq\cdots\cdots\leqq S_n\leqq\cdots\cdots$.

From Fatou's lemma $\int\lim S_n d\mu=\lim\int S_n d\mu$, that is $\int\sum_{n=1}^{\infty}|f_n|\,d\mu=\sum_{n=1}^{\infty}\int|f_n|\,d\mu<\infty$.

If we write $v_n(x)=|f_n(x)|-f_n(x)$, then it follows that $\int\sum_{n=1}^{\infty}v_n d\mu=\sum_{n=1}^{\infty}\int v_n\,d\mu$,

therefere $\int\sum_{n=1}^{\infty}(|f_n|-f_n)d\mu=\sum_{n=1}^{\infty}\int(|f_n|-f_n)d\mu, \sum_{n=1}^{\infty}\int f_n d\mu=\int\sum_{n=1}^{\infty}f_n d\mu$.

THEOREM 1. *If*

(i) $\sum_{n=1}^{\infty} f_n(x)=S_1(x)$, $\sum_{n=1}^{\infty} g_n(x)=S_2(x)$, $\sum_{n=1}^{\infty} G_n(x)=S_3(x)$,

(ii) $g_n(x)\leqq f_n(x)\leqq G_n(x)$ *for* all n,

(iii) $\sum_{n=1}^{\infty}\int g_n d\mu=\int S_2 d\mu$ *and* $\sum_{n=1}^{\infty}\int G_n d\mu=\int S_3 d\mu$

with $\int S_2 d\mu$ *and* $\int S_3 d\mu$ *finite,*

then $\sum_{n=1}^{\infty}\int f_n d\mu=\int S_1 d\mu$ *and* $\int S_1 d\mu$ *is finite.*

PROOF. This follows by applying Theorem 1 of [2] to sequences $\{S_{1n}\}$, $\{S_{2n}\}$, and $\{S_{3n}\}$ defined by $S_{1n}=\sum_{K=1}^{n} f_K$, $S_{2n}=\sum_{K=1}^{n} g_K$ and $S_{3n}=\sum_{K=1}^{n} G_K$.

Received October 25, 1961.

COROLLARY 1. *If*

(i) $\sum_{n=1}^{\infty} f_n\ (x)=S_1(x)$,

(ii) $g_n \leqq f_n \leqq G_n$ *for all* n,

(iii) $\sum_{n=1}^{\infty} \int |g_n|\ d\mu<\infty,\ \ \sum_{n=1}^{\infty} \int |G_n|\ d\mu<\infty$,

then $\sum_{n=1}^{\infty} \int f_n d\mu= \int S_1 d\mu$ *(finite)*

PROOF. By Lemma, we have that the series $\sum_{n=1}^{\infty} \int g_n d\mu$ and $\sum_{n=1}^{\infty} \int G_n d\mu$ converge *a. e.* to integrable functions $S_2(x)$ and $S_3(x)$ respectively. The desired result follows from Thserem 1.

THEOREM 2. *If*

(i) $\sum_{n=1}^{\infty} f_n(x)=S_1(x),\ \sum_{n=1}^{\infty} g_n(x)=S_2(x),\ \sum_{n=1}^{\infty} G_n(x)=S_3(x)$,

(ii) $g_n \leqq f_n \leqq G_n$ *for all* n,

(iii) $\sum_{n=1}^{\infty} \int g_n d\mu= \int S_2 d\mu$ *and* $\sum_{n=1}^{\infty} \int G_n d\mu= \int S_3 d\mu$ *with* $\int S_2 d\mu$ *and* $\int S_3 d\mu$ *finite.*

(iv) $g_n \leqq 0 \leqq G_n$ *for all* n,

then

(a) $\int |\sum_{K=1}^{n} f_k - S_1|\ d\mu \to 0$;

(b) $\int_F (\sum_{K=1}^{n} f_K) d\mu \to \int_F S_1 d\mu$ *uniformly in* F*(*F *measurable)*;

(c) $\int h(\sum_{K=1}^{n} f_K) d\mu \to \int h S_1 d\mu$ *for all bounded functions* h, *uniformly in* h *for each bound.*

PROOF. (a)–(c) are equivalent, since (a) implies (c)

$$|(h\sum_{K=1}^{n} f_K - \int h S_1) d\mu| \leqq \int |h|\ |\sum_{K=1}^{n} f_K - S_1|\ d\mu \leqq M\cdot \int |\sum_{K=1}^{n} f_K - S_1|\ d\mu \to 0,$$

that is $\int h\sum_{K=1}^{n} f_K d\mu \to \int h S_1 d\mu$,

(c) implies (b)

$$\left| \int_F \sum_{K=1}^{n} f_K d\mu - \int_F S_1 d\mu \right| = \left| \int_F (\sum_{K=1}^{n} f_K - S_1) d\mu \right|,$$

in (c), if we put $h=1$, then $\int_F \sum_{K=1}^{n} f_K d\mu \to \int_F S_1 d\mu$ uniformly in F, and (b) implies (a)

let F_γ' $(\gamma\in\Gamma)$ $(F_\gamma'=x-F_\gamma)$ be a measurable set such that $\mu(F_\gamma')<\varepsilon_\gamma$ and such that the sequence $\left\{\int_{F\gamma} \sum_{K=1}^{n} f_K d\mu\right\}$ converges to $\int_{F\gamma} S_1 d\mu$ uniformly on $F_\gamma(\gamma\in\Gamma)$. If $F'=\bigcap_{\gamma\in\Gamma} F_\gamma'$, then$\mu(F')\leqq\mu(F_\gamma)<\varepsilon_\gamma$, so that $\mu(F')=0$, and it is clear that, for $x\in F$, $\left\{\int_F (\sum_{K=1}^{n} f_K) d\mu\right\}$ converges to $\int_F S_1 d\mu$. We have $\int |\sum_{K=1}^{n} f_K(x) - S_1(x)|\ d\mu \to 0$.

To prove (i)–(iv) imply (a), note that, by (ii) and (iv)

$$0 \leqq |\sum_{K=1}^{n} f_K - S_1(x)| \leqq |\sum_{K=1}^{n} f_K| + |S_1| \leqq \sum_{K=1}^{n} G_K - \sum_{K=1}^{n} g_K + S_3 - S_2 \to 2(S_3-S_2),$$

while, by (iii), ($\int \sum_{K=1}^{n} G_K - \sum_{K=1}^{n} g_K + S_3 - S_2) \to \int 2(S_3 - S_2)$ which is finite. Thus Theorem 1 applies with $|\sum_{K=1}^{n} f_K - S_1|$ for $\sum_{K=1}^{n} f_K$, 0 for $\sum_{K=1}^{n} g_K$, and $\sum_{K=1}^{n} G_K - \sum_{K=1}^{n} g_K + S_3 - S_2$ for $\sum_{K=1}^{n} G_K$. That is,

(i) $|\sum_{K=1}^{n} f_K - S_1| \to |S_1 - S_1| = 0,\ 0 \to 0$

$\sum_{K=1}^{n} G_K - \sum_{K=1}^{n} g_K + S_3 - S_2 \to 2(S_3 - S_2)$,

(ii) $0 \leq |\sum_{K=1}^{n} f_K - S_1| \leq \sum_{K=1}^{n} G_K - \sum_{K=1}^{n} g_K + S_3 - S_2$ for all n,

(iii) $\int 0 d\mu \to \int 0 d\mu$, and $\int (\sum_{K=1}^{n} G_K - \sum_{K=1}^{n} g_K + S_3 - S_2) d\mu \to 2 \int (S_3 - S_2) d\mu$

$\int 0 d\mu$ and $2 \int (S_3 - S_2) d\mu$ are finite.

Hence $\int |\sum_{K=1}^{n} f_K - S_1| d\mu \to 0$.

COROLLARY 2. *If*

(i) $\sum_{n=1}^{\infty} f_n(x) = S_1(x)$,

(ii) $g_n(x) \leq f_n(x) \leq G_n(x)$ *for all* n,

(iii) $\sum_{n=1}^{\infty} \int |g_n| d\mu < \infty$, $\sum_{n=1}^{\infty} \int |G_n| du < \infty$,

(iv) $g_n \leq 0 \leq G_n$ *for all* n,

then

(a) $\int |\sum_{K=1}^{n} f_K - S_1| du \to 0$;

(b) $\int_F (\sum_{K=1}^{n} f_K) d\mu \to \int_F S_1 du$ *uniformly in* F (F *measurable*);

(c) $\int h(\sum_{K=1}^{n} f_K) du \to \int h S_1 du$ *for all bounded functions* h, *uniformly in* h *for each bound.*

PROOF. The desired result follows from Lemma and Theorem 2.

REFERENCES

[1]. Paul R. Halmos, Measure Theory, D. Van Nostrand, New York, 1950.

[2]. John W. Pratt, "On interchanging limits and integrals," Ann. Math. Stat., Vol. 31 (1960), pp. 74~77.