A Note on an Analogue of the Wallace-Simson Theorem for Spherical Triangles

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Abstract

The famous Wallace-Simson Theorem for planar triangles is generalized to spherical triangles.

1 Introduction

For planar triangles there is a well-known old theorem due to W.Wallace (1799):

> Let P be any point on the circumcicle of a triangle ABC. Then the feet of perpendiculars from P onto the three sides of the triangle lie on a straight line.

The straight line is called a Wallace-Simson line (for example, see Coxeter and Gleitzer (1967)).

The theorem could be generalizaed into several directions. One famous example of generalization was made by J.V.Poncelet by drawing three straight lines of constant angle with three sides instead of perpendiculars (for example, see . Recently another generalization was studied by P.Pech (2005) where an analogue of the Wallace-Simson theorem in three-dimensional space.

In this paper we study an analogue of the Wallace-Simson theorem for spherical triangles. In the below we suppose that all figures lie on a unit sphere $\mathcal{S}$. Consider a spherical triangle ABC. Then the vertices $\mathrm{A}, \mathrm{B}, \mathrm{C}$ can be represented by vectors $\overrightarrow{\mathrm{A}}, \overrightarrow{\mathrm{B}}, \overrightarrow{\mathrm{C}}$ emanating from the center of $\mathcal{S}$. Since these vectors are of the unit length, we have

$$\overrightarrow{\mathrm{A}} \cdot \overrightarrow{\mathrm{A}} = \overrightarrow{\mathrm{B}} \cdot \overrightarrow{\mathrm{B}} = \overrightarrow{\mathrm{C}} \cdot \overrightarrow{\mathrm{C}} = 1,$$

where the symbol "$\cdot$" denotes the innerprouct of vectors.

As usual we write

$$a = \mathrm{BC}, b = \mathrm{CA}, c = \mathrm{AB}; \quad \alpha = \angle\mathrm{CAB}, \beta = \angle\mathrm{ABC}, \gamma = \angle\mathrm{BCA}.$$

Then

$$\cos a = \overrightarrow{\mathrm{B}} \cdot \overrightarrow{\mathrm{C}}, \ \cos b = \overrightarrow{\mathrm{C}} \cdot \overrightarrow{\mathrm{A}}, \ \cos c = \overrightarrow{\mathrm{A}} \cdot \overrightarrow{\mathrm{B}}. \tag{1}$$

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2 Method of "six edges"

In the traditional Japanese mathematics of "Edo" era called "Wasan", the method of "six edges" played an important role for study of planar triangles. In this section we search an analogous method for spherical triangles. That is, putting $x = \mathrm{PA}, y = \mathrm{PB}, z = \mathrm{PC}$ for any point P on $\mathcal{S}$, we examine a relation between six edges x, y, z, a, b, c.

If we put $\varphi_1 = \angle\mathrm{BPC}, \varphi_2 = \angle\mathrm{CPA}, \varphi_3 = \angle\mathrm{APB}$, the COS formula gives

$$\begin{cases} \cos a = \cos y \cos z + \sin y \sin z \cos \varphi_1, \\ \cos b = \cos z \cos x + \sin z \sin x \cos \varphi_2, \\ \cos c = \cos x \cos y + \sin x \sin y \cos \varphi_3 \end{cases} .$$

To simplyfy the notation we write

$$A = \cos a,\ B = \cos b,\ C = \cos c,\ X = \cos x,\ Y = \cos y,\ Z = \cos z.$$

Then we have

$$\cos\varphi_1 = \frac{A - YZ}{\sqrt{1-Y^2}\sqrt{1-Z^2}}, \cos\varphi_2 = \frac{B - ZX}{\sqrt{1-Z^2}\sqrt{1-X^2}}, \cos\varphi_3 = \frac{C - XY}{\sqrt{1-X^2}\sqrt{1-Y^2}}. \quad (2)$$

Since $\varphi_1 + \varphi_2 + \varphi_3 = 2\pi$, we see

$$\cos\varphi_3 = \cos(2\pi - (\varphi_1 + \varphi_2)) = \cos(\varphi_1 + \varphi_2) = \cos\varphi_1 \cos\varphi_2 - \sin\varphi_1 \sin\varphi_2,$$

that is,

$$\sin\varphi_1 \sin\varphi_2 = \cos\varphi_1 \cos\varphi_2 - \cos\varphi_3.$$

Hence follows

$$(1 - \cos^2\varphi_1)(1 - \cos^2\varphi_2) - (\cos\varphi_1 \cos\varphi_2 - \cos\varphi_3)^2 = 0.$$

Then substitution of (2) gives

$$\begin{aligned} &[(1-Y^2)(1-Z^2) - (A - YZ)^2][(1-Z^2)(1-X^2) - (B - ZX)^2] \\ &= [(A - YZ)(B - ZX) - (1 - Z^2)(C - XY)]^2. \end{aligned}$$

Then, dividing by a factor $1 - Z^2$, we get

$$(1-A^2)X^2 + (1-B^2)Y^2 + (1-C^2)Z^2 - 2(A-BC)YZ - 2(B-CA)ZX - 2(C-AB)XY = K, \quad (3)$$

where we put

$$K = 1 + 2ABC - (A^2 + B^2 + C^2).$$

Now the COS formula gives

$$A - BC = \sin b \sin c \cos\alpha,\ B - CA = \sin c \sin a \cos\beta,\ C - AB = \sin a \sin b \cos\gamma.$$

Furthermore, since

$$\begin{aligned} &(A - BC)^2 + (B - CA)^2 + (C - AB)^2 \\ &= (A^2 + B^2 + C^2) + (B^2C^2 + C^2A^2 + A^2B^2) - 6ABC \\ &= \{(1-B^2)(1-C^2) + (1-C^2)(1-A^2) + (1-A^2)(1-B^2)\} - 3K, \end{aligned}$$

we have

$$\begin{aligned} 3K &= \left\{(\sin b \sin c)^2 + (\sin c \sin a)^2 + (\sin a \sin b)^2\right\} \\ &\quad - \left\{(\sin b \sin c \cos\alpha)^2 + (\sin c \sin a \cos\beta)^2 + (\sin a \sin b \cos\gamma)^2\right\} \\ &= (\sin b \sin c \sin\alpha)^2 + (\sin c \sin a \sin\beta)^2 + (\sin a \sin b \sin\gamma)^2. \end{aligned} \tag{4}$$

Therefore we obtain the following method of "six edges" for spherical triangles.

Theorem 1

$$\begin{aligned} &(\sin a \cos x)^2 + (\sin b \cos y)^2 + (\sin c \cos z)^2 \\ &\quad -2(\sin b \cos y)(\sin c \cos z)\cos\alpha - 2(\sin c \cos z)(\sin a \cos x)\cos\beta - 2(\sin a \cos x)(\sin b \cos y)\cos\gamma \\ &\quad = \frac{1}{3}\left[(\sin b \sin c \sin\alpha)^2 + (\sin c \sin a \sin\beta)^2 + (\sin a \sin b \sin\gamma)^2\right] \end{aligned}$$

3 Circumcircle

Let O and R be the center and the circumradius of circumcircle of a triangle ABC. Putting $x = y = z = R$ in (3), we have

$$[3 - 2(A+B+C) + 2(BC+CA+AB) - (A^2+B^2+C^2)]\cos^2 R = K.$$

Hence it follows

$$\begin{aligned} \frac{1}{\cos^2 R} - 1 &= \frac{3 - 2(A+B+C) + 2(BC+CA+AB) - (A^2+B^2+C^2)}{K} - 1 \\ &= \frac{2(1-A)(1-B)(1-C)}{K} \end{aligned}$$

Here we substitute (4). Then

$$\tan^2 R = \frac{6(1-\cos a)(1-\cos b)(1-\cos c)}{(\sin b \sin c \sin\alpha)^2 + (\sin c \sin a \sin\beta)^2 + (\sin a \sin b \sin\gamma)^2}.$$

However the SIN formula states that

$$\frac{\sin a}{\sin\alpha} = \frac{\sin b}{\sin\beta} = \frac{\sin c}{\sin\gamma}.$$

We denote the common value by k. Then, since

$$\sin b \sin c \sin\alpha = \sin c \sin a \sin\beta = \sin a \sin b \sin\gamma = \frac{\sin a \sin b \sin c}{k},$$

it is derived that

$$\begin{aligned} \tan^2 R &= \frac{2(1-\cos a)(1-\cos b)(1-\cos c)k^2}{\sin^2 a \sin^2 b \sin^2 c} = \frac{2k^2}{(1+\cos a)(1+\cos b)(1+\cos c)} \\ &= \left(\frac{k}{2\cos\frac{a}{2}\cos\frac{b}{2}\cos\frac{c}{2}}\right)^2. \end{aligned}$$

Accordingly

$$k = 2\cos\frac{a}{2}\cos\frac{b}{2}\cos\frac{c}{2}\tan R.$$

Therefore we obtain the following result.

Lemma 1

$$\boxed{\frac{\sin a}{\sin\alpha} = \frac{\sin b}{\sin\beta} = \frac{\sin c}{\sin\gamma} = 2\cos\frac{a}{2}\cos\frac{b}{2}\cos\frac{c}{2}\tan R}$$

4 An analogue of the Wallace-Simson theorem

From a point P we draw the perpendicular onto the edge AB and let Z be its foot. Then a vector $\overrightarrow{\mathrm{Z}}$ lies on the plane spanned by two vectors $\overrightarrow{\mathrm{A}}, \overrightarrow{\mathrm{B}}$, that is, it lies on the plane which passes through two points A, B and the center of $\mathcal{S}$. Accordingly the following representation holds:

$$\overrightarrow{\mathrm{Z}} = \lambda\overrightarrow{\mathrm{A}} + \mu\overrightarrow{\mathrm{B}}. \tag{5}$$

Then, putting $x = \mathrm{PA}, y = \mathrm{PB}, z = \mathrm{PC}$, we have

$$\cos x = \overrightarrow{\mathrm{P}}\cdot\overrightarrow{\mathrm{A}},\ \cos y = \overrightarrow{\mathrm{P}}\cdot\overrightarrow{\mathrm{B}},\ \cos z = \overrightarrow{\mathrm{P}}\cdot\overrightarrow{\mathrm{C}}. \tag{6}$$

Lemma 1

$$\lambda = \frac{\pm(\cos x - \cos c\cos y)}{\sin c\cdot f(\cos x, \cos y, \cos c)},\quad \mu = \frac{\pm(\cos y - \cos c\cos x)}{\sin c\cdot f(\cos x, \cos y, \cos c)}$$

Here $f(x, y, k) = \sqrt{x^2 - 2kxy + y^2}$ と置く.

(Proof) Since the arc of a great circle PZ is orthogonal to the arc AB, the exterior product $\overrightarrow{\mathrm{P}}\times\overrightarrow{\mathrm{Z}}$ lies on the plane spanned by two vectors $\overrightarrow{\mathrm{A}}, \overrightarrow{\mathrm{B}}$. As the exterior product $\overrightarrow{\mathrm{A}}\times\overrightarrow{\mathrm{B}}$ is orthogonal to the plane, the vector $\overrightarrow{\mathrm{P}}\times\overrightarrow{\mathrm{Z}}$ is orthogonal to the vector $\overrightarrow{\mathrm{A}}\times\overrightarrow{\mathrm{B}}$. Consequently it holds that

$$(\overrightarrow{\mathrm{P}}\times\overrightarrow{\mathrm{Z}})\cdot(\overrightarrow{\mathrm{A}}\times\overrightarrow{\mathrm{B}}) = 0. \tag{7}$$

Now we use the famous formula

$$(\overrightarrow{\mathrm{P}}\times\overrightarrow{\mathrm{Z}})\cdot(\overrightarrow{\mathrm{A}}\times\overrightarrow{\mathrm{B}}) = (\overrightarrow{\mathrm{P}}\cdot\overrightarrow{\mathrm{A}})(\overrightarrow{\mathrm{Z}}\cdot\overrightarrow{\mathrm{B}}) - (\overrightarrow{\mathrm{P}}\cdot\overrightarrow{\mathrm{B}})(\overrightarrow{\mathrm{Z}}\cdot\overrightarrow{\mathrm{A}}).$$

Then (7) can be written as

$$(\overrightarrow{\mathrm{P}}\cdot\overrightarrow{\mathrm{A}})(\overrightarrow{\mathrm{Z}}\cdot\overrightarrow{\mathrm{B}}) = (\overrightarrow{\mathrm{P}}\cdot\overrightarrow{\mathrm{B}})(\overrightarrow{\mathrm{Z}}\cdot\overrightarrow{\mathrm{A}}). \tag{8}$$

Substitute (5) into (8). Then, with aid of (1) and (6), we have

$$\cos x\,(\lambda\cos c + \mu) = \cos y\,(\lambda + \mu\cos c).$$

Hence follows

$$\mu = \frac{\cos y - \cos c\cos x}{\cos x - \cos c\cos y}\,\lambda. \tag{9}$$

Now, since the point Z lies on $\mathcal{S}$, we have $\overrightarrow{\mathrm{Z}} \cdot \overrightarrow{\mathrm{Z}} = 1$. Substitution of (9) gives

$$\begin{aligned} &\frac{(\cos x - \cos c \cos y)^2}{\lambda^2} \\ = \ & (\cos x - \cos c \cos y)^2 + 2\cos c\,(\cos x - \cos c \cos y)(\cos y - \cos c \cos x) + (\cos y - \cos c \cos x)^2 \\ = \ & (1 - \cos^2 c)(\cos^2 xy, \cos c). \end{aligned}$$

Therefore we have the required result.

(Q.E.D.)

(Remark) The occurrence of the symbol "$\pm$" in λ, μ is not false, because the foot Z lies on the great circle which passes through two points A, B, and if a point Z$'$ is the opposite point of Z with respect to the center O of $\mathcal{S}$, the point Z$'$ lies on the same great circle, and the arc PZ$'$ is orthogonal to the great circle. Accordingly the point Z$'$ becomes another foot.

From the point P we draw perpendiculars onto edges BC, CA, AB, and let X, Y, Z be their feet. Moreover we consider the following function

$$\begin{aligned} F(x,y,z) = \ & 2(1 - \cos a \cos b \cos c) \cos x \cos y \cos z \\ & -(\cos a - \cos b \cos c) \cos x(\cos^2 y + \cos^2 z) \\ & -(\cos b - \cos c \cos a) \cos y(\cos^2 z + \cos^2 x) \\ & -(\cos c - \cos a \cos b) \cos z(\cos^2 x + \cos^2 y). \end{aligned} \tag{10}$$

Then we have the following theorem.

Theorem 2

The orbit of P such that three feet X, Y, Z are colinear is given by $F(x,y,z) = 0$.

References

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