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AN ELEMENTARY REMARK ON THE INTEGRAL WITH RESPECT TO EULER CHARACTERISTICS OF PROJECTIVE HYPERPLANE SECTIONS

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Dedicated to Professor Shoji Tsuboi on the occasion of his 60th birthday

ABSTRACT. In this note we remark that the degree of any term of *Chern-Schwartz-MacPherson classes* of a possibly singular projective variety X can be expressed by linear combinations of *integrals with respect to Euler characteristics* (in the sense of O.Y.Viro) of generic linear sections of X.

1. INTEGRALS

In [7] O. Y. Viro gave a very simple and useful notion of *the integral with respect to Euler characteristics*. Consider the Boolean algebra $\mathcal{B}$ generated by all the finite CW complices (or finite simplical complices), namely, the one consisting of any sets constructed by Boolean operations, that is, taking compliments, union and intersections of finitely many CW complices. Then the topological Euler characteristics χ behaves as an integer-valued *measure* with $\mathcal{B}$ being the *finitely additive* collection of *measurable sets*:

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B), \quad \chi(A \times B) = \chi(A)\chi(B), \quad A, B \in \mathcal{B}.$$

An integer valued function $\alpha : X \to \mathbb{Z}$ ($X \in \mathcal{B}$) is called *measurable* if the image of f is a finite subset of $\mathbb{Z}$ and each level set is measurable, i.e., $\alpha^{-1}(s) \in \mathcal{B}$. The *integral* of a measurable function α over X with respect to χ is defined by

$$\int_X \alpha\, d\chi := \sum_{s \in \mathbb{Z}} s \cdot \chi(\alpha^{-1}(s)).$$

For a morphism $f : X \to Y$ (in $\mathcal{B}$) the *pushforward* of α is defined to be $f_*(\alpha)(y) := \int_{f^{-1}(y)} \alpha\, d\chi$ ($y \in Y$). Then by the definition it commutes with the integral: $\int_X \alpha\, d\chi = \int_Y f_*\alpha\, d\chi$.

On the other hand, in the category of compact complex algebraic varieties, constructible sets/functions become measurable sets/functions in the above sense. Let $\mathcal{F}(X)$ denote the abelian group of all constructible functions over a variety X, that is freely generated by characteristic functions 1_W of irreducible reduced subvarieties W: $1_W(y) = 1$ if $y \in W$ otherwise 0. In [3] R. MacPherson showed that there is a unique natural transformation

$$C_* : \mathcal{F}(X) \to H_*(X; \mathbb{Z})$$

(that is, a homomorphism of abelian groups for each object X with the property $C_*(f_*\alpha) = f_*C_*(\alpha)$ for morphisms $f : X \to Y$) so that it satisfies the normalization condition: $C_*(1_X) = c(TX) \cap [X]$ for nonsinglar X. $C_*(1_X)$ is called *Chern-Schwartz-MacPherson class* of X, cf. [3], [5]. In particular, by the definition, the 0-th degree of C_* is nothing but the integral with respect to Euler characteristics: for irreducible X, taking the map $f : X \to \{pt\}$, we get

$$\text{the 0-th degree of } C_*(\alpha) = f_*C_*(\alpha) = C_*(f_*\alpha) = \int_X \alpha\, d\chi.$$

Further, as proved in [2] it holds that $C_*(\alpha \times \beta) = C_*(\alpha) \times C_*(\beta)$ (the homology cross product). So the MacPherson transformation is reasonably regarded as a "homology-valued integral".

The aim of this note is to show that in the case of complex projective varieties, not only the 0-th degree but all degrees of C_* are expressed by integrals with respect to Euler characteristics in a certain sense.

Let $X \subset \mathbb{P}^N$ be an irreducible projective variety of dimension n. For a constructible function $\alpha = \sum_W n_W 1_W \in \mathcal{F}(X)$, a linear subspace L of codimension i is called *generic with respect to* α if L is transverse to any Whitney stratification of each subvariety W supporting α. Then, we define

$$\int_{X\cap L} \alpha\, d\chi := \sum_W n_W \chi(W \cap L).$$

This number is independent of the choice of generic L. Indeed, all generic linear subspaces with respect to α form a Zariski open dense subset in the Grassmanian space, so any two generic L_0 and L_1 can be joined by a family $\{L_t\}_{0\le t\le 1}$ of generic linear subspaces. Hence the Thom's first isotopy lemma (cf.[1] for instance) shows that $W \cap L_0$ and $W \cap L_1$ are homeomorphic for each W, and thus their Euler characteristics coincide. In particular, it yields a well-defined homomorphism

$$\int_i : \mathcal{F}(X) \to \mathbb{Z}, \qquad \int_i \alpha := \int_{X\cap L} \alpha\, d\chi.$$

On one hand, for $i = 0, 1, 2\cdots, n$, the MacPherson transformation defines a homomorphism

$$\check{C}_i : \mathcal{F}(X) \to \mathbb{Z}, \quad \check{C}_i(\alpha) := \gamma^i \cap C_i(\alpha)$$

where γ is the restriction to X of the first Chern class of the tautological line bundle of the ambient projective space.

Theorem 1.1. *Let X be any irreducible projective variety of dimension n. Then it holds that for any $\alpha \in \mathcal{F}(X)$,*

$$\check{C}_i(\alpha) = \int_i \alpha + \sum_{j=i+1}^{n} a_{ij} \cdot \int_j \alpha, \tag{i}$$

where the coefficients a_{ij} *are determined by the generating function* $\left(\frac{x}{1-x}\right)^i = \sum_{j\geq i} a_{ij}x^j$. *Conversely, it also holds that*

$$\int_i \alpha = \check{C}_i(\alpha) + \sum_{j=i+1}^{n} b_{ij} \cdot \check{C}_j(\alpha), \qquad (ii)$$

where the coefficients b_{ij} *are determined by the generating function* $\left(\frac{y}{1+y}\right)^i = \sum_{j\geq i} b_{ij}y^j$.

Corollary 1.2. *The subgroup of the dual space* $\mathcal{F}(X)^*$ $(= Hom(\mathcal{F}(X), \mathbb{Z}))$ *generated by* $\int_i : \mathcal{F}(X) \to \mathbb{Z}$ $(i = 0, 1, 2\cdots, n)$ *coincides with the one generated by* $\check{C}_i : \mathcal{F}(X) \to \mathbb{Z}$ $(i = 0, 1, 2\cdots, n)$.

Remark 1.3. $\int_i : \mathcal{F}(X) \to \mathbb{Z}$ $(i = 0, 1, 2\cdots, n)$ *are linearly independent over* $\mathbb{Z}$, *and hence* $\check{C}_i : \mathcal{F}(X) \to \mathbb{Z}$ *are so.*

Example 1.4. *The lowest and highest terms of* $\check{C}_*(1_X)$ *are*

$$\begin{aligned} \check{C}_0(1_X) &= \int_0 1_X \ (= \int_X 1_X = \chi(X)), \\ \check{C}_n(1_X) &= \int_n 1_X = \int_{X\cap\mathbb{P}^{N-n}} 1_X = \text{ the degree of the variety } X \end{aligned}$$

For a possibly singular surface X, $\check{C}_1 = \int_1 + \int_2$. *For a possibly singular 3-fold,* $\check{C}_1 = \int_1 + \int_2 + \int_3$ *and* $\check{C}_2 = \int_2 + 2\int_3$. *For a possibly singular 4-fold,* $\check{C}_1 = \int_1 + \int_2 + \int_3 + \int_4$, $\check{C}_2 = \int_2 + 2\int_3 + 3\int_4$, $\check{C}_3 = \int_3 + 3\int_4$, *and so on.*

2. Proof

Let X be an irreducible projective variety of dimension n in $\mathbb{P}^N$ and set $X^{(i)} = X \cap L$, L being a generic linear subspace of codimension i. Let $\iota : X^{(i)} \to X$ be the inclusion map. Our key lemma is the following one:

Lemma 2.1. *It holds that in* $H_*(X)$

$$\gamma^i \cap C_*(1_X) = (1+\gamma)^i \cap \iota_* C_*(1_{X^{(i)}}).$$

This "adjunction formula" is a special case of Proposition 1.3 in Parusiński-Pragacz [4] and also of Proposition 3.1 in [6]. The formula can be read as a variant of "Vierdier-type Riemann-Roch" formula.

Proof of Theorem : Looking at the 0-th part of the equality in Lemma 2.1, we have

$$\begin{aligned} \check{C}_i(1_X) &= \gamma^i \cap C_i(1_X) \\ &= \text{0-th part of } (1+\gamma)^i \cap \iota_* C_*(1_{X^{(i)}}) \\ &= \sum_{k_1=0}^{i} \binom{i}{k_1} \gamma^{k_1} \cap \iota_* C_{k_1}(1_{X^{(i)}}). \end{aligned}$$

Repeat the same procedure for each term, and then

$$\begin{aligned}\gamma^{k_1} \cap \iota_* C_{k_1}(1_{X^{(i)}}) &= \iota_*(\iota^*\gamma^{k_1} \cap C_{k_1}(1_{X^{(i)}})) \\ &= \text{0-th part of } \ (1+\gamma)^{k_1} \cap \iota'_* C_*(1_{X^{(i+k_1)}}) \\ &= \sum_{k_2=0}^{k_1} \binom{k_1}{k_2} \gamma^{k_2} \cap \iota'_* C_{k_2}(1_{X^{(i+k_1)}}),\end{aligned}$$

where $\iota' : X^{(i+k_1)} \to X$ is the inclusion. Thus we have (notations of induced homomorphisms by inclusions are omitted)

$$\begin{aligned}\check{C}_i(1_X) = \gamma^i \cap C_i(1_X) &= \sum_{k_1=0}^{i} \binom{i}{k_1} \gamma^{k_1} \cap C_{k_1}(1_{X^{(i)}}) \\ = \check{C}_0(1_{X^{(i)}}) &+ \sum_{k_1=1}^{i} \binom{i}{k_1} \sum_{k_2=0}^{k_1} \binom{k_1}{k_2} \gamma^{k_2} \cap C_{k_2}(1_{X^{(i+k_1)}}) \\ &= \cdots\cdots \\ = \check{C}_0(1_{X^{(i)}}) &+ \sum_I \binom{i}{k_1}\binom{k_1}{k_2} \cdots \binom{k_{s-1}}{k_s} \check{C}_0(1_{X^{(i+k_1+\cdots+k_s)}}),\end{aligned}$$

where the sum takes over all $I = (k_1, \cdots, k_s)$ so that $i \geq k_1 \geq k_2 \geq \cdots \geq k_s > 0$ and $i + k_1 + \cdots + k_s \leq n$.

This computation is the same as follows. Set $f_0(x,t) = x$, $\phi(x,t) = t(1+x)$, and $f_s(x,t) = f_{s-1}(\phi(x,t),t)$ inductively: $f_1(x,t) = t(1+x)$, $f_2(x,t) = t(1+t(1+x))$, $\cdots$, and so on. Putting $t = x$ and taking s large enough, it holds module $\langle x^{n+1} \rangle$ that

$$\begin{aligned}f_s(x,x)^i &= x^i(1 + x(1 + \cdots + x(1 + x(1+x))\cdots))^i = x^i \sum_{k_1=0}^{i} \binom{i}{k_1} f_{s-1}(x,x)^{k_1} \\ &= \cdots \equiv x^i + \sum_I \binom{i}{k_1}\binom{k_1}{k_2} \cdots \binom{k_{s-1}}{k_s} x^{i+k_1+\cdots+k_s}.\end{aligned}$$

This is also congruent to

$$x^i(1 + x + x^2 + \cdots)^i = \left(\frac{x}{1-x}\right)^i = \sum_{j \geq i} a_{ij} x^j.$$

Therefore we have

$$\check{C}_i(1_X) = \int_i 1_X + \sum_{j=i+1}^{n} a_{ij} \int_j 1_X.$$

Hence the formula (i) in Theorem follows from the linearity of $\check{C}_i$ and $\int_i$ for any $\alpha = \sum_W n_W 1_W \in \mathcal{F}(X)$.

Also the formula (ii) follows from Lemma 2.1:

$$\begin{aligned}\int_i \alpha &= \int_{X(i)} \alpha\, d\chi = \text{0-th part of } \iota_* C_*(\alpha|_{X(i)}) \\ &= \text{0-th part of } \left(\frac{\gamma}{1+\gamma}\right)^i \cap C_*(\alpha) \\ &= \sum_{j=i}^{n} b_{ij}\gamma^j \cap C_j(\alpha) = \sum_{j=i}^{n} b_{ij}\check{C}_j(\alpha).\end{aligned}$$

This completes the proof. □

Remark 2.2. *Obviously,* $b_{ij} = (-1)^{i+j}a_{ij}$ *and the matrix* $[b_{ij}]$ *is the inverse of* $[a_{ij}]$. *In fact, substituting* $\frac{y}{1+y}$ *for* x *in the equality* $\left(\frac{x}{1-x}\right)^i = \sum_{j\geq i} a_{ij}x^j$, *we have that* $y^i = \left(\sum_{k=i}^{j} a_{ik}b_{kj}\right) y^j$ *for each pair* $i \leq j$.

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