Design of Recursive Least-Squares Fixed-Lag Smoother using Covariance Information in Linear Continuous Stochastic Systems

Department of Technology, Faculty of Education

Seiichi Nakamori

(Received 4 October, 2005)

Abstract. This paper newly designs the recursive least-squares (RLS) fixed-lag smoother and filter using the covariance information in linear continuous-time stochastic systems. It is assumed that the signal is observed with additive white observation noise and is uncorrelated with the signal. The estimators require the covariance information of the signal and the variance of the observation noise. The auto-covariance function of the signal is expressed in semi-degenerate kernel form.

Keywords. Continuous stochastic systems, Recursive least-squares fixed-lag smoother, Covariance information, Wiener-Hopf integral equation

1. Introduction

The estimation problem given in covariance information has been seen as an important research avenue in the area of detection and estimation problems [1] for communication systems. In [2-4], it is assumed that the auto-covariance function of the signal is expressed in the semi-degenerate kernel form. A semi-degenerate kernel is the function suitable for expressing a general kind of auto-covariance function by a finite sum of nonrandom functions. In a fixed-lag smoother [5], the auto-covariance function of the signal is expressed in a degenerate kernel form, not in the semi-degenerate kernel form. The degenerate kernel function cannot express the auto-covariance functions of general kinds of stochastic processes. Hence, the fixed-lag smoother using an auto-covariance function in the form of a degenerate kernel is not appropriate for estimating general kinds of stationary or non-stationary signal processes. The expression in the degenerate kernel form of the auto-covariance function is obtained through approximating the auto-covariance function by a Fourier series transformation. Hence, its approximation error causes degradation in the estimation accuracy of the fixed-lag smoother. The recursive least-squares (RLS) Wiener fixed-point smoother [6] and filter

[7] using the covariance information are designed in linear discrete-time stochastic systems. The estimators require the information of the observation matrix, the system matrix for the state variable, related with the signal, and the cross-variance function of the state variable with the observed value. The information can be obtained from the covariance function of the signal [6]. Also, it is assumed that the variance of the white observation noise is known. From this respect, in [7], [8], the fixed-lag smoothing and filtering algorithms are designed using the covariance information in linear discrete-time stochastic systems. The estimators require the information of the factorized auto-covariance function of the signal. Also, it is assumed that the variance of the white observation noise is known.

The fixed-lag smoothing algorithms in [7], [8] are sub-optimal since the impulse response function for the fixed-lag smoothing estimate is approximately obtained. The advantageous property of the semi-degenerate kernel expression of the auto-covariance function of the signal lies in the following point. It can express the auto-covariance function of general stationary or non-stationary stochastic signal processes by a finite sum of nonrandom functions without formulating the state-space model generating the signal from available data. In continuous-time stochastic systems, the RLS prediction, filtering, fixed-point smoothing and fixed-interval smoothing algorithms are shown [9] by expressing the auto-covariance function of the signal in the semi-degenerate kernel form. However, the RLS algorithm for the fixed-lag smoothing estimate remains not derived. From this respect, this paper newly designs the RLS fixed-lag smoother using the covariance information of the signal in the semi-degenerate kernel form. It is assumed that the signal is observed with additive white observation noise and the signal is uncorrelated with the signal. The algorithms are derived based on the invariant imbedding method [6].

2. Fixed-lag smoothing problem

Let an observation equation be given by

$$y(t) = z(t) + v(t), \tag{1}$$

in linear continuous-time stochastic systems, where $z(t)$ is an $m \times 1$ signal vector and $v(t)$ is a white observation noise. It is assumed that the signal and the observation noise are mutually independent stochastic processes with zero means. Let the auto-covariance function of $v(t)$ be given by

$$E[v(t)v^T(s)] = R(t)\delta(t-s), \ R(t) > 0. \tag{2}$$

Here, $\delta(\cdot)$ denotes the Dirac δ function.

Let $K_z(t,s)$ represent the auto-covariance function of the signal and let $K_z(t,s)$ be expressed in the semi-degenerate kernel form of

$$K_z(t,s)=\begin{cases} A(t)B^T(s), & 0\le s\le t, \\ B(t)A^T(s), & 0\le t\le s. \end{cases} \tag{3}$$

Here, $A(t)$ and $B(s)$ are bounded $n\times m$ matrices.

Let a fixed-lag smoothing estimate $\hat{z}(t-L,t)$ of $z(t-L)$ be given by

$$\hat{z}(t-L,t)=\int_0^t h(t,s)y(s)ds \tag{4}$$

as a linear transformation of the innovations $\{y(s),\quad 0\le s\le t\}$, where $h(t,s)$ and L are referred to be an impulse response function and the fixed Lag.

The impulse response function which minimizes the mean-square value of the fixed-lag smoothing error $z(t-L)-\hat{z}(t-L,t)$,

$$J=E[\| z(t-L)-\hat{z}(t-L,t)\|^2], \tag{5}$$

satisfies

$$K_z(t-L,s)=\int_0^t h(t,\tau)K_y(\tau,s)d\tau \tag{6}$$

by an orthogonal projection lemma [10], [11]:

$$z(t)-\hat{z}(t-L,t)\perp y(s),\ 0\le s\le t. \tag{7}$$

Here, '$\perp$' denotes the notation of the orthogonality. From (4) and (6), the impulse response function satisfies

$$h(t,s)R(s)=K_z(t-L,s)-\int_0^t h(t,\tau)K_z(\tau,s)d\tau . \tag{8}$$

3. Fixed-lag smoothing and filtering algorithms

The fixed-lag smoothing problem starting with (8) has been considered to be difficult in deriving the least-squares fixed-lag estimation equations. In this paper, for the values of t and s, we separate the impulse response function $h(t,s)$ into $h_1(t,s)$ and $h_2(t,s)$ as

$$h(t,s)=\begin{cases} h_1(t-L,s), & 0\le s\le t-L, \\ h_2(t-L,s), & t-L\le s. \end{cases} \tag{9}$$

From (4) and (9), the fixed-lag smoothing estimate is written as

$$\hat{z}(t-L,t)=\int_0^{t-L} h_1(t-L,s)y(s)ds+\int_{t-L}^{t} h_2(t-L,s)y(s)ds. \qquad (10)$$

The first term on the right hand side of (10) is the filtering estimate $\hat{z}(t-L,t-L)$ of the signal $z(t-L)$. The second term is the correction term in calculating the fixed-lag smoothing estimate $\hat{z}(t-L,t)$ of $z(t-L)$.

For $0\le s\le t-L$, it is seen that the impulse response function $h_1(t,s)$ for the filtering problem satisfies

$$h_1(t,s)R(s)=K_z(t,s)-\int_0^t h_1(t,\tau)K_z(\tau,s)d\tau. \qquad (11)$$

From (3), by $K_z(t,s)=A(t)B^T(s),\quad 0\le s\le t,$ (11) is written as

$$h_1(t,s)R(s)=A(t)B^T(s)-\int_0^t h_1(t,\tau)K_z(\tau,s)d\tau. \qquad (12)$$

Introducing an auxiliary function $J_1(t,s)$ satisfying

$$J_1(t,s)R(s)=B^T(s)-\int_0^t J_1(t,\tau)K_z(\tau,s)d\tau, \qquad (13)$$

we obtain, from (12) and (13), the impulse response function

$$h_1(t,s)=A(t)J_1(t,s). \qquad (14)$$

Differentiating (13) with respect to t, we have

$$\frac{\partial J_1(t,s)}{\partial t}R(s)=-J_1(t,t)K_z(t,s)-\int_0^t \frac{\partial J_1(t,\tau)}{\partial t}K_z(\tau,s)d\tau. \qquad (15)$$

From (3), (13) and (15), we obtain

$$\frac{\partial J_1(t,s)}{\partial t}=-J_1(t,t)A(t)J_1(t,s). \qquad (16)$$

From (13) with (3), the function $J_1(t,t)$ satisfies

$$J_1(t,t)R(t)=B^T(t)-\int_0^t J_1(t,\tau)B(\tau)d\tau A^T(t). \qquad (17)$$

Introducing a function

$$r_1(t)=\int_0^t J_1(t,\tau)B(\tau)d\tau, \qquad (18)$$

we obtain the equation for $J_1(t,t)$ as

$$J_1(t,t) = (B^T(t) - r_1(t)A^T(t))R^{-1}(t). \tag{19}$$

Differentiating (18) with respect to t, we have

$$\frac{dr_1(t)}{dt} = J_1(t,t)B(t) + \int_0^t \frac{\partial J_1(t,\tau)}{\partial t} B(\tau)d\tau. \tag{20}$$

Substituting (16) and (18) into (20), we obtain

$$\frac{dr_1(t)}{dt} = J_1(t,t)(B(t) - A(t)r_1(t)). \tag{21}$$

The initial condition on the differential equation (21) at $t = 0$ is $r_1(0) = 0$ from (18).

From (10), the filtering estimate $\hat{z}(t-L,t-L)$ of $z(t-L)$ is expressed as $\hat{z}(t-L,t-L) = \int_0^{t-L} h_1(t-L,s)y(s)ds$. The filtering estimate $\hat{z}(t,t)$ is formulated as

$$\hat{z}(t,t) = \int_0^t h_1(t,s)y(s)ds. \tag{22}$$

Substituting (14) into (22), we have

$$\hat{z}(t,t) = A(t)\int_0^t J_1(t,s)y(s)ds. \tag{23}$$

Introducing a function

$$e_1(t) = \int_0^t J_1(t,s)y(s)ds, \tag{24}$$

we obtain

$$\hat{z}(t,t) = A(t)e_1(t). \tag{25}$$

Differentiating (24) with respect to t and using (16) and (24), we obtain

$$\frac{de_1(t)}{dt} = J_1(t,t)(y(t) - A(t)e_1(t)). \tag{26}$$

The initial condition on $e_1(t)$ at $t = 0$ is $e_1(0) = 0$ from (24).

Now, let us consider the impulse response function $h_2(t-L,s)$, $t-L \le s$. From (11), $h_2(t-L,s)$ satisfies

$$h_2(t-L,s)R(s) = K_z(t-L,s) - \int_0^{t-L} h_1(t-L,\tau)K_z(\tau,s)d\tau - \int_{t-L}^t h_2(t-L,\tau)K_z(\tau,s)d\tau. \tag{27}$$

From (3) and (14) and (18), (27) is written as

$$h_2(t-L,s)R(s) = B(t-L)A^T(s) - A(t-L)\int_0^{t-L} J_1(t-L,\tau)B(\tau)d\tau A^T(s) - \int_{t-L}^{t} h_2(t-L,\tau)K_z(\tau,s)d\tau$$

$$= (B(t-L) - A(t-L)r_1(t-L))A^T(s) - \int_{t-L}^{t} h_2(t-L,\tau)K_z(\tau,s)d\tau . \quad (28)$$

Introducing an auxiliary function satisfying

$$J_2(t-L,s)R(s) = A^T(s) - \int_{t-L}^{t} J_2(t-L,\tau)K_z(\tau,s)d\tau , \quad (29)$$

we obtain

$$h_2(t-L,s) = (B(t-L) - A(t-L)r_1(t-L))J_2(t-L,s) . \quad (30)$$

Differentiating (29) with respect to t and using (3), we have

$$\frac{\partial J_2(t-L,s)}{\partial t}R(s) = -J_2(t-L,t)K_z(t,s) + J_2(t-L,t-L)K_z(t-L,s) - \int_{t-L}^{t} \frac{\partial J_2(t-L,\tau)}{\partial t}K_z(\tau,s)d\tau$$

$$= -J_2(t-L,t)A(t)B^T(s) + J_2(t-L,t-L)B(t-L)A^T(s) - \int_{t-L}^{t} \frac{\partial J_2(t-L,\tau)}{\partial t}K_z(\tau,s)d\tau . \quad (31)$$

Introducing an auxiliary function satisfying

$$J_3(t-L,s)R(s) = B^T(s) - \int_{t-L}^{t} J_3(t-L,\tau)K_z(\tau,s)d\tau , \quad (32)$$

and using (29), we obtain a partial-differential equation

$$\frac{\partial J_2(t-L,s)}{\partial t} = -J_2(t-L,t)A(t)J_3(t-L,s) + J_2(t-L,t-L)B(t-L)J_2(t-L,s) \quad (33)$$

for $J_2(t-L,s)$.

The function $J_2(t-L,t)$ in (33) is obtained as follows. Putting $s=t$ in (29) and using (3), we have

$$J_2(t-L,t)R(t)=A^T(t)-\int_{t-L}^{t} J_2(t-L,\tau)B(\tau)d\tau A^T(t). \tag{34}$$

Introducing a function

$$r_2(t)=\int_{t-L}^{t} J_2(t-L,\tau)B(\tau)d\tau, \tag{35}$$

we obtain

$$J_2(t-L,t)=(A^T(t)-r_2(t)A^T(t))R^{-1}(t). \tag{36}$$

The function $J_2(t-L,t-L)$ in (33) is obtained as follows. Putting $s=t$ in (29) and using (3), we have

$$J_2(t-L,t)R(t-L)=A^T(t-L)-\int_{t-L}^{t} J_2(t-L,\tau)A(\tau)d\tau B^T(t-L). \tag{37}$$

Introducing a function

$$r_3(t)=\int_{t-L}^{t} J_2(t-L,\tau)A(\tau)d\tau, \tag{38}$$

we obtain

$$J_2(t-L,t-L)=(A^T(t-L)-r_3(t)B^T(t-L))R^{-1}(t-L). \tag{39}$$

Next, differentiating (35) with respect to t and using (33), we have

$$\begin{aligned}\frac{dr_2(t)}{dt}&=J_2(t-L,t)B(t)-J_2(t-L,t-L)B(t-L)-\\ &J_2(t-L,t)A(t)\int_{t-L}^{t} J_3(t-L,\tau)B(\tau)d\tau+J_2(t-L,t-L)B(t-L)\int_{t-L}^{t} J_2(t-L,\tau)B(\tau)d\tau.\end{aligned} \tag{40}$$

Introducing a function

$$r_4(t)=\int_{t-L}^{t} J_3(t-L,\tau)B(\tau)d\tau, \tag{41}$$

we obtain a differential equation

$$\begin{aligned}\frac{dr_2(t)}{dt}&=J_2(t-L,t)B(t)-J_2(t-L,t-L)B(t-L)-\\ &J_2(t-L,t)A(t)r_4(t)+J_2(t-L,t-L)B(t-L)r_2(t)\end{aligned} \tag{42}$$

for $r_2(t)$. The initial condition on the differential equation (42) at $t=0$ is $r_2(0)=0$ from (35).

Next, differentiating (38) with respect to t and using (33), we have

$$\frac{dr_3(t)}{dt} = J_2(t-L,t)A(t) - J_2(t-L,t-L)A(t-L) - J_2(t-L,t)A(t)\int_{t-L}^{t} J_3(t-L,\tau)A(\tau)d\tau +$$

$$J_2(t-L,t-L)B(t-L)\int_{t-L}^{t} J_2(t-L,\tau)A(\tau)d\tau . \quad (43)$$

Using (38) and introducing a function

$$r_5(t) = \int_{t-L}^{t} J_3(t-L,\tau)A(\tau)d\tau , \quad (44)$$

we obtain

$$\frac{dr_3(t)}{dt} = J_2(t-L,t)A(t) - J_2(t-L,t-L)A(t-L) -$$
$$J_2(t-L,t)A(t)r_5(t) + J_2(t-L,t-L)B(t-L)r_3(t). \quad (45)$$

The initial condition on the differential equation (21) at $t = 0$ is $r_3(0) = 0$ from (38).

Differentiating (32) with respect to t, we have

$$\frac{\partial J_3(t-L,s)}{\partial t}R(s) = -J_3(t-L,t)K_z(t,s) + J_3(t-L,t-L)K_z(t-L,s) -$$
$$\int_{t-L}^{t} \frac{\partial J_3(t-L,\tau)}{\partial t}K_z(\tau,s)d\tau$$
$$= -J_3(t-L,t)A(t)B^T(s) + J_3(t-L,t-L)B(t-L)A^T(s) -$$
$$\int_{t-L}^{t} \frac{\partial J_3(t-L,\tau)}{\partial t}K_z(\tau,s)d\tau. \quad (46)$$

From (29) and (32), we obtain a partial-differential equation

$$\frac{\partial J_3(t-L,s)}{\partial t} = -J_3(t-L,t)A(t)J_3(t-L,s) + J_3(t-L,t-L)B(t-L)J_2(t-L,s)$$

for $J_3(t-L,s)$. (47)

Differentiating (41) with respect to t and using (35), (41) and (47), we obtain

$$\frac{dr_4(t)}{dt} = J_3(t-L,t)B(t) - J_3(t-L,t-L)B(t-L) + \int_{t-L}^{t} \frac{\partial J_3(t-L,\tau)}{\partial t}B(\tau)d\tau$$
$$= J_3(t-L,t)B(t) - J_3(t-L,t-L)B(t-L) -$$
$$J_3(t-L,t)A(t)\int_{t-L}^{t} J_3(t-L,\tau)B(\tau)d\tau + J_3(t-L,t-L)B(t-L)\int_{t-L}^{t} J_2(t-L,\tau)B(\tau)d\tau$$

$$= J_3(t-L,t)B(t) - J_3(t-L,t-L)B(t-L) - J_3(t-L,t)A(t)r_4(t) + J_3(t-L,t-L)B(t-L)r_2(t). \tag{48}$$

The initial condition on the differential equation (48) at $t=0$ is $r_4(0)=0$ from (41).

The function $J_3(t-L,t)$ in (48) is obtained as follows. Putting $s=t$ in (32) and using (3) and (41), we have

$$J_3(t-L,t)R(t) = B^T(t) - \int_{t-L}^{t} J_3(t-L,\tau)K_z(\tau,t)d\tau$$

$$= B^T(t) - \int_{t-L}^{t} J_3(t-L,\tau)B(\tau)d\tau A^T(t)$$

$$= B^T(t) - r_4(t)A^T(t). \tag{49}$$

Hence,

$$J_3(t-L,t) = (B^T(t) - r_4(t)A^T(t))R^{-1}(t). \tag{50}$$

The function $J_3(t-L,t-L)$ in (48) is obtained as follows. Putting $s=t$ in (32) and using (3) and (44), we obtain

$$J_3(t-L,t-L)R(t) = B^T(t) - \int_{t-L}^{t} J_3(t-L,\tau)K_z(\tau,t-L)d\tau$$

$$= B^T(t) - \int_{t-L}^{t} J_3(t-L,\tau)A(\tau)d\tau B^T(t-L).$$

$$= B^T(t) - r_5(t)B^T(t-L). \tag{51}$$

Hence,

$$J_3(t-L,t-L) = (B^T(t) - r_5(t)B^T(t-L))R^{-1}(t). \tag{52}$$

Differentiating (44) with respect to t and using (38), (44) and (47), we obtain

$$\frac{dr_5(t)}{dt} = J_3(t-L,t)A(t) - J_3(t-L,t-L)A(t-L) + \int_{t-L}^{t} \frac{\partial J_3(t-L,\tau)}{\partial t} A(\tau)d\tau$$

$$= J_3(t-L,t)A(t) - J_3(t-L,t-L)A(t-L) -$$

$$J_3(t-L,t)A(t)\int_{t-L}^{t} J_3(t-L,\tau)A(\tau)d\tau + J_3(t-L,t-L)B(t-L)\int_{t-L}^{t} J_2(t-L,\tau)A(\tau)d\tau$$

$$= J_3(t-L,t)A(t) - J_3(t-L,t-L)A(t-L) - J_3(t-L,t)A(t)r_5(t) + J_3(t-L,t-L)B(t-L)r_3(t). \tag{53}$$

The initial condition on the differential equation (53) at $t=0$ is $r_5(0)=0$ from (44).

From (10), (25) and (30), we have

$$\hat{z}(t-L,t) = A(t-L)e_1(t-L) + (B(t-L) - A(t-L)r_1(t-L))\int_{t-L}^{t} J_2(t-L,s)y(s)ds. \tag{54}$$

Introducing a function

$$e_2(t) = \int_{t-L}^{t} J_2(t-L,s)y(s)ds, \tag{55}$$

we obtain an equation for the fixed-lag smoothing estimate

$$\hat{z}(t-L,t) = A(t-L)e_1(t-L) + (B(t-L) - A(t-L)r_1(t-L))e_2(t). \tag{56}$$

Differentiating (55) with respect to t and using (33) and (55), we have

$$\begin{aligned}\frac{de_2(t)}{dt} &= J_2(t-L,t)y(t) - J_2(t-L,t-L)y(t-L) - \\ &J_2(t-L,t)A(t)\int_{t-L}^{t} J_3(t-L,s)y(s)ds + J_2(t-L,t-L)B(t-L)\int_{t-L}^{t} J_2(t-L,s)y(s)ds \\ &= J_2(t-L,t)y(t) - J_2(t-L,t-L)y(t-L) - \\ &\quad J_2(t-L,t)A(t)\int_{t-L}^{t} J_3(t-L,s)y(s)ds + J_2(t-L,t-L)B(t-L)e_2(t).\end{aligned} \tag{57}$$

Introducing a function

$$e_3(t) = \int_{t-L}^{t} J_3(t-L,s)y(s)ds \tag{58}$$

and using (55), we obtain a differential equation

$$\begin{aligned}\frac{de_2(t)}{dt} &= J_2(t-L,t)y(t) - J_2(t-L,t-L)y(t-L) - \\ &J_2(t-L,t)A(t)e_3(t) + J_2(t-L,t-L)B(t-L)e_2(t).\end{aligned} \tag{59}$$

The initial condition on the differential equation (59) at $t=0$ is $e_2(0)=0$ from (55).

Differentiating (58) with respect to t and using (47), (55) and (58), we obtain

$$\frac{de_3(t)}{dt} = J_3(t-L,t)y(t) - J_3(t-L,t-L)y(t-L) -$$
$$J_3(t-L,t)A(t)\int_{t-L}^{t} J_3(t-L,s)y(s)ds + J_3(t-L,t-L)B(t-L)\int_{t-L}^{t} J_2(t-L,s)y(s)ds$$
$$= J_3(t-L,t)y(t) - J_3(t-L,t-L)y(t-L) - J_3(t-L,t)A(t)e_3(t) + J_3(t-L,t-L)B(t-L)e_2(t). \quad (60)$$

The initial condition on the differential equation (60) at $t=0$ is $e_3(0)=0$ from (58).

Now, let us summarize the above results for the fixed-lag smoothing estimate and the filtering estimate in [Theorem 1].

[Theorem 1] Let the auto-covariance function $K_z(t,s)$ of the signal $z(t)$ be expressed in the semi-degenerate kernel form of (3). Let the variance $R(k)$ of the white observation noise $v(k)$ for the observation equation (1) be given. Then the RLS fixed-lag smoothing and filtering equations consist of (61)-(75) in linear continuous-time stochastic systems.
Fixed-lag smoothing estimate of the signal $z(t-L)$ in terms of the observed value $\{y(s), 0 \le s \le t\}$: $\hat{z}(t-L,t)$

$$\hat{z}(t-L,t) = A(t-L)e_1(t-L) + (B(t-L) - A(t-L)r_1(t-L))e_2(t) \quad (61)$$

$$\frac{de_2(t)}{dt} = J_2(t-L,t)y(t) - J_2(t-L,t-L)y(t-L) - J_2(t-L,t)A(t)e_3(t) + J_2(t-L,t-L)B(t-L)e_2(t), \quad e_2(0)=0 \quad (62)$$

$$\frac{de_3(t)}{dt} = J_3(t-L,t)y(t) - J_3(t-L,t-L)y(t-L) - J_3(t-L,t)A(t)e_3(t) + J_3(t-L,t-L)B(t-L)e_2(t), \quad e_3(0)=0 \quad (63)$$

$$\frac{dr_2(t)}{dt} = J_2(t-L,t)B(t) - J_2(t-L,t-L)B(t-L) - J_2(t-L,t)A(t)r_4(t) + J_2(t-L,t-L)B(t-L)r_2(t), \quad r_2(0)=0 \quad (64)$$

$$\frac{dr_3(t)}{dt} = J_2(t-L,t)A(t) - J_2(t-L,t-L)A(t-L) - J_2(t-L,t)A(t)r_5(t) + J_2(t-L,t-L)B(t-L)r_3(t), \quad r_3(0)=0 \tag{65}$$

$$\frac{dr_4(t)}{dt} = J_3(t-L,t)B(t) - J_3(t-L,t-L)B(t-L) - J_3(t-L,t)A(t)r_4(t) + J_3(t-L,t-L)B(t-L)r_2(t), \quad r_4(0)=0 \tag{66}$$

$$\frac{dr_5(t)}{dt} = J_3(t-L,t)A(t) - J_3(t-L,t-L)A(t-L) - J_3(t-L,t)A(t)r_5(t) + J_3(t-L,t-L)B(t-L)r_3(t), \quad r_5(0)=0 \tag{67}$$

$$J_2(t-L,t) = (A^T(t) - r_2(t)A^T(t))R^{-1}(t) \tag{68}$$

$$J_2(t-L,t-L) = (A^T(t-L) - r_3(t)B^T(t-L))R^{-1}(t-L) \tag{69}$$

$$J_3(t-L,t) = (B^T(t) - r_4(t)A^T(t))R^{-1}(t) \tag{70}$$

$$J_3(t-L,t-L) = (B^T(t) - r_5(t)B^T(t-L))R^{-1}(t) \tag{71}$$

Filtering estimate of the signal $z(t-L)$ in terms of the observed value $\{y(s), 0 \le s \le t-L\}$: $\hat{z}(t-L,t-L)$

$$\hat{z}(t,t) = A(t)e_1(t) \tag{72}$$

$$\frac{de_1(t)}{dt} = J_1(t,t)(y(t) - A(t)e_1(t)), \quad e_1(0)=0 \tag{73}$$

$$\frac{dr_1(t)}{dt} = J_1(t,t)(B(t) - A(t)r_1(t)), \quad r_1(0)=0 \tag{74}$$

$$J_1(t,t) = (B^T(t) - r_1(t)A^T(t))R^{-1}(t) \tag{75}$$

Let $P(t-L,t-L)$ represent the filtering error variance function of the signal

$$P(t-L,t-L) = E[(z(t-L) - \hat{z}(t-L,t-L))(z(t-L) - \hat{z}(t-L,t-L))^T].$$

Let $P(t-L,t)$ represent the fixed-lag smoothing error variance function of the signal. From the un-correlation properties between $z(t-L)-\hat{z}(t-L,t-L)$ and $\hat{z}(t-L,t-L)$ and between $z(t-L)-\hat{z}(t-L,t-L)$ and $e_2(t)$ from (55), we have

$$\begin{aligned}
P(t-L,t) &= E[(z(t-L)-\hat{z}(t-L,t))(z(t-L)-\hat{z}(t-L,t))^T] \\
&= E[(z(t-L)-\hat{z}(t-L,t-L)-(B(t-L)-A(t-L)r_1(t-L))e_2(t)) \\
&\quad (z(t-L)-\hat{z}(t-L,t-L)-(B(t-L)-A(t-L)r_1(t-L))e_2(t))^T] \\
&= E[(z(t-L)-\hat{z}(t-L,t-L)(z(t-L)-\hat{z}(t-L,t-L))^T]- \\
&\quad E[(B(t-L)-A(t-L)r_1(t-L))e_2(t))(B(t-L)-A(t-L)r_1(t-L))e_2(t))^T] \\
&= K_z(t-L,t-L)-E[\hat{z}(t-L,t-L)\hat{z}(t-L,t-L)^T]- \\
&\quad (B(t-L)-A(t-L)r_1(t-L))E[e_2(t)e_2^T(t)](B(t-L)-A(t-L)r_1(t-L))^T].
\end{aligned} \tag{76}$$

From (72), by putting

$$f_1(t)=E[e_1(t)e_1^T(t)], \tag{77}$$

the variance of the filtering estimate $\hat{z}(t-L,t-L)$ is

$$P_{\hat{z}}(t-L,t-L)=A(t-L)f_1(t-L)A^T(t-L). \tag{78}$$

Differentiating (77) with respect to t and using (18) and (24), we obtain

$$\frac{df_1(t)}{dt}=2J_1(t,t)RJ_1^T(t,t)+J_1(t,t)A(t)r_1(t)+r_1(t)A^T(t)J_1^T(t,t),\ f_1(0)=0. \tag{79}$$

$f_1(t)$ is calculated recursively by (79) together with (74) and (75).

If we put

$$f_2(t)=E[e_2(t)e_2^T(t)], \tag{80}$$

$P(t-L,t)$ is represented as

$$\begin{aligned}
P(t-L,t) &= K_z(t-L,t-L)-A(t-L)f_1(t-L)A^T(t-L)- \\
&\quad (B(t-L)-A(t-L)r_1(t-L))f_2(t)(B(t-L)-A(t-L)r_1(t-L))^T.
\end{aligned} \tag{81}$$

Since the auto-variance $K_z(t-L,t-L)$ of the signal $z(t-L)$, the filtering error variance $K_z(t-L,t-L)-A(t-L)f_1(t-L)A^T(t-L)$ of $z(t-L)$, the fixed-lag smoothing error variance $P(t-L,t)$ and $f_2(t)$ of (80) are positive semi-definite matrices,

$$0 \le P(t-L,t) \le P(t-L,t-L) \tag{82}$$

is valid. This indicates that the fixed-lag smoother might be superior in estimation accuracy to the filter.

Similarly to the derivation of (79), by introducing the functions

$$r_6(t)=\int_{t-L}^{t} J_3(t-L,s)R(s)J_2^T(t-L,s)ds,$$

$$r_7(t)=\int_{t-L}^{t}\int_{t-L}^{t} J_3(t-L,s)K_z(s,s')J_2^T(t-L,s')ds'ds$$

$$r_8(t)=\int_{t-L}^{t} J_2(t-L,s)R(s)J_2^T(t-L,s)ds$$

$$r_9(t)=\int_{t-L}^{t} J_3(t-L,s)R(s)J_3^T(t-L,s)ds$$

$$r_{10}(t)=\int_{t-L}^{t} J_2(t-L,s)\phi_1^T(t,s)ds$$

$$r_{11}(t)=\int_{t-L}^{t} J_3(t-L,s)\phi_2^T(t,s)ds$$

$$r_{12}(t)=\int_{t-L}^{t} J_2(t-L,s)\phi_2^T(t,s)ds$$

$$\phi_1(t,s)=\int_{t-L}^{t} J_2(t-L,s')K_z(s',s)ds'$$

$$\phi_2(t,s)=\int_{t-L}^{t} J_3(t-L,s')K_z(s',s)ds',$$

$f_2(t)$ is calculated recursively in terms of (64)-(71) and (83)-(92) recursively.

$$\frac{df_2(t)}{dt}=2J_2(t-L,t)R(t)J_2^T(t-L,t)+J_2(t-L,t)A(t)r_2^T(t)-$$
$$2J_2(t-L,t-L)R(t-L)J_2^T(t-L,t-L)-J_2(t-L,t-L)B(t-L)r_3^T(t)-$$

$$J_2(t-L,t)A(t)r_6(t)-J_2(t-L,t)r_7(t)+J_2(t-L,t-L)B(t-L)f_2(t)+$$
$$r_2(t)A^T(t)J_2^T(t-L,t)-r_3(t)B^T(t-L)J_2^T(t-L,t-L)-$$
$$r_6^T(t)A^T(t)J_2^T(t-L,t)-r_7^T(t)J_2^T(t-L,t)+f_2^T(t)B^T(t-L)J_2^T(t-L,t-L), \quad f_2(0)=0 \quad (83)$$

$$\frac{dr_6(t)}{dt}=J_3(t-L,t)R(t)J_2^T(t-L,t)-J_3(t-L,t-L)R(t-L)J_2^T(t-L,t-L)-$$
$$J_3(t-L,t)A(t)r_6(t)+J_3(t-L,t-L)B(t-L)r_8(t)- \quad (84)$$
$$r_9(t)A^T(t)J_2^T(t-L,t)+r_6(t)B^T(t-L)J_2^T(t-L,t-L), \quad r_6(0)=0$$

$$\frac{dr_7(t)}{dt}=J_3(t-L,t)\phi_1^T(t,t)-J_3(t-L,t-L)\phi_1^T(t,t-L)-$$
$$J_3(t-L,t)A(t)r_7(t)+J_3(t-L,t-L)B(t-L)r_{10}(t)+ \quad (85)$$
$$r_4(t)A^T(t)J_2^T(t-L,t)-r_5(t)B^T(t-L)J_2^T(t-L,t-L)-$$
$$r_{11}(t)A^T(t)J_2^T(t-L,t)+r_7(t)B^T(t-L)J_2^T(t-L,t-L), \quad r_7(0)=0$$

$$\frac{dr_8(t)}{dt}=J_2(t-L,t)R(t)J_2^T(t-L,t)-J_2(t-L,t-L)R(t-L)J_2^T(t-L,t-L)-$$
$$J_2(t-L,t)A(t)r_6(t)+J_2(t-L,t-L)B(t-L)r_8(t)- \quad (86)$$
$$r_6^T(t)A^T(t)J_2^T(t-L,t)+r_8(t)B^T(t-L)J_2^T(t-L,t-L), \quad r_8(0)=0$$

$$\frac{dr_9(t)}{dt}=J_3(t-L,t)R(t)J_3^T(t-L,t)-J_3(t-L,t-L)R(t-L)J_3^T(t-L,t-L)-$$
$$J_3(t-L,t)A(t)r_9(t)+J_3(t-L,t-L)B(t-L)r_6(t)- \quad (87)$$
$$r_9(t)A^T(t)J_3^T(t-L,t)+r_6(t)B^T(t-L)J_3^T(t-L,t-L), \quad r_9(0)=0$$

$$\frac{dr_{10}(t)}{dt}=J_2(t-L,t)\phi_1^T(t,t)-J_2(t-L,t-L)\phi_1^T(t,t-L)- \quad (88)$$
$$J_2(t-L,t)A(t)r_7(t)+J_2(t-L,t-L)B(t-L)r_{10}(t), \quad r_{10}(0)=0$$

$$\frac{dr_{11}(t)}{dt}=J_3(t-L,t)\phi_2^T(t,t)-J_3(t-L,t-L)\phi_2^T(t,t-L)- \quad (89)$$
$$J_3(t-L,t)A(t)r_{11}(t)+J_3(t-L,t-L)B(t-L)r_{12}(t), \quad r_{11}(0)=0$$

$$\frac{dr_{12}(t)}{dt} = J_2(t-L,t)\phi_2^T(t,t) - J_2(t-L,t-L)\phi_2^T(t,t-L) - J_2(t-L,t)A(t)r_{11}(t) + J_2(t-L,t-L)B(t-L)r_{12}(t), \quad r_{12}(0)=0 \tag{90}$$

$$\phi_1(t,t) = r_2(t)A^T(t), \quad \phi_1(t,t-L) = r_3(t)B^T(t-L) \tag{91}$$

$$\phi_2(t,t) = r_4(t)A^T(t), \quad \phi_2(t,t-L) = r_5(t)B^T(t-L) \tag{92}$$

In section 4, the estimation characteristic of the proposed RLS fixed-lag smoother is shown.

4. A numerical simulation example

Let a scalar observation equation be given by

$$y(t) = z(t) + v(t). \tag{93}$$

Let the observation noise $v(t)$ be a zero-mean white Gaussian process with the variance R, $N(0,R)$. Let the autocovariance function of the signal $z(t)$ be given by

$$K_z(t,s) = \Lambda e^{-a|t-s|}, \text{ a=0.4861}, \ \Lambda = 9. \tag{94}$$

From (94), the functions $A(t)$ and $B(s)$ in (3) are expressed as follows:

$$A(t) = \Lambda e^{-at}, \ B(s) = e^{as}. \tag{95}$$

If we substitute (95) into the fixed-lag smoothing algorithm of [Theorem 1], we can calculate the fixed-lag smoothing estimate and the filtering estimate recursively.

Fig.1 illustrates the signal $z(t)$ and the fixed-lag smoothing estimate $\hat{z}(t-0.005,t)$ for the white Gaussian observation noise $N(0,0.5^2)$ by the RLS fixed-lag smoother in [Theorem 1]. Fig.2 illustrates the mean-square values (MSVs) of the fixed-lag smoothing and filtering errors by the RLS estimators in [Theorem 1] for the observation noises $N(0,0.3^2)$, $N(0,0.5^2)$, $N(0,0.7^2)$ and $N(0,1)$ vs. the fixed-lag L, $0 \le L \le 10$. For $L=0$, the MSV of the filtering error is shown. The MSVs of the fixed-lag smoothing and filtering errors are evaluated by $\sum_{i=1}^{300}(z(\Delta i) - \hat{z}(\Delta i, \Delta i + L))^2/(300)$ and $\sum_{i=1}^{300}(z(\Delta i) - \hat{z}(\Delta i, \Delta i))^2/300$, $\Delta = 0.001$. Here,

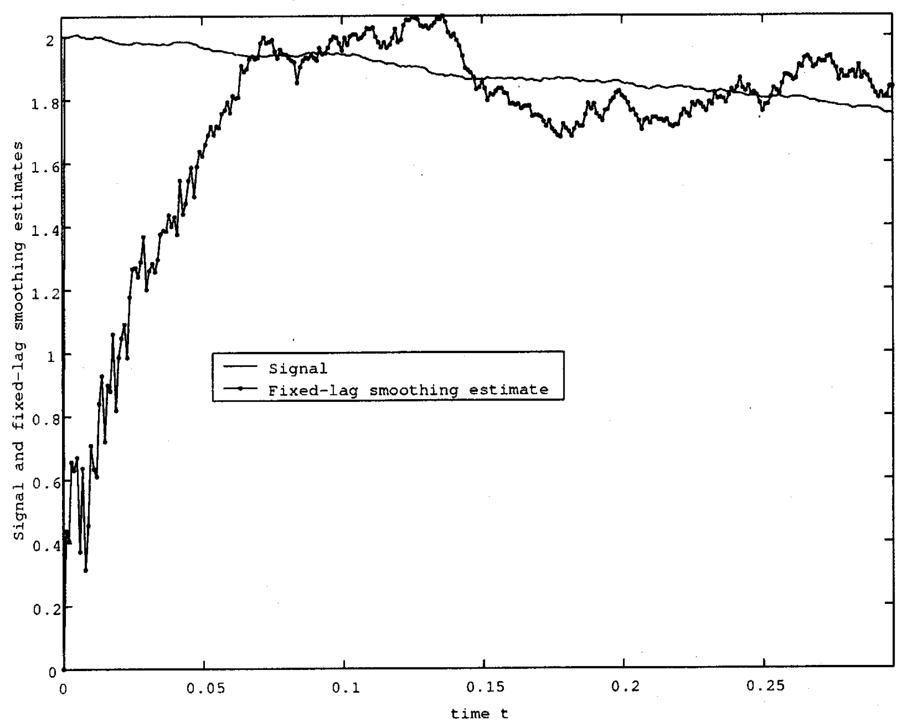

Fig.1 Signal $z(t)$ and the fixed-lag smoothing estimate $\hat{z}(t-0.005,t)$ by the RLS fixed-lag smoother in [Theorem 1] vs. t for the white Gaussian observation noise $N(0,0.5^2)$.

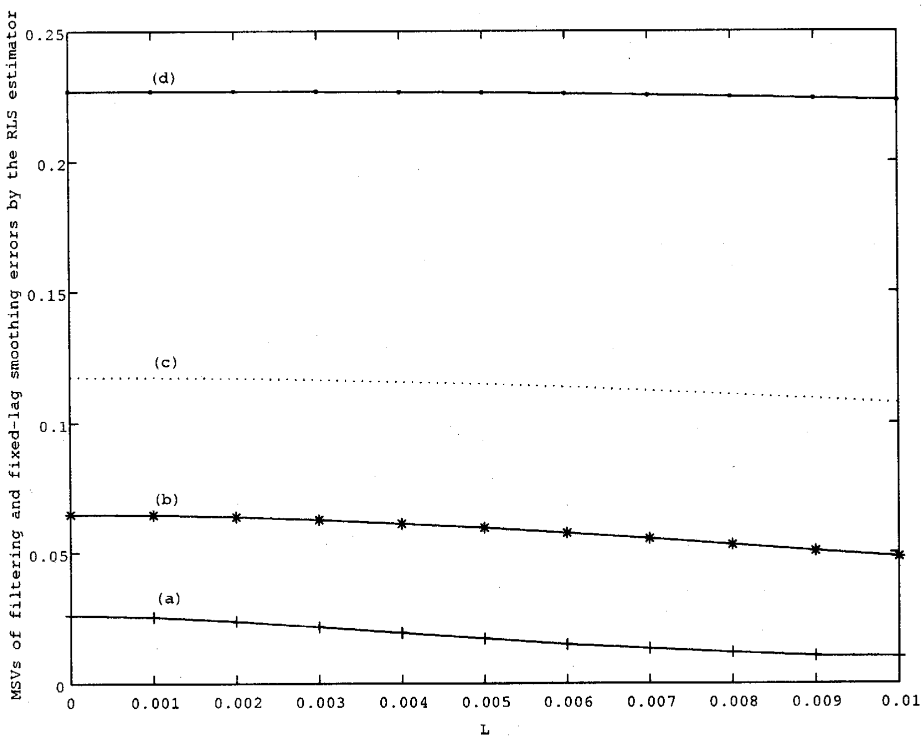

Fig.2 Mean-square values of the filtering and fixed-lag smoothing errors by the RLS fixed-lag smoother in [Theorem 1] for the observation noises $N(0,0.3^2)$, $N(0,0.5^2)$, $N(0,0.7^2)$ and $N(0,1)$ vs. L, $0 \le L \le 10$.

for the numerical integration of the differential equations, the fourth-order Runge-Kutta method is used.

For references, the state-space model, which generate the signal, is given by

$$\frac{dz(t)}{dt} = -az(t) + w(t), \quad E[w(t)w(s)] = 2a\Lambda\delta(t-s). \tag{96}$$

5. Conclusions

In this paper, the RLS fixed-lag smoother using the covariance information of the signal in the form of the semi-degenerate kernel has been devised in linear continuous-time system. The simulation results have shown that the fixed-lag smoothing algorithm is feasible. As the value of the observation noise variance becomes small, the estimation accuracy of the fixed-lag smoother is improved.

References

[1] H. L. Van Trees, Detection, Estimation, and Modulation Theory, Part 1, Wiley New York, 1968.

[2] S. Nakamori, Design of a fixed-point smoother based on an innovations theory for white Gaussian plus coloured observation noise, International Journal of Systems Science 22 (12) (1991) 2573-2584.

[3] S. Nakamori, Design of predictor using covariance information in continuous-time stochastic systems with nonlinear observation mechanism, Signal Processing 68 (1998) 183-193.

[4] S. Nakamori, Design of estimators using covariance information in discrete-time stochastic systems with nonlinear observation mechanism, IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences E82-A (7) (1999) 1292-1304.

[5] S. Nakamori, M. Sugisaka, Design of linear fixed-lag smoother by covariance information, Transactions of SICE 14 (4) (1978) 405-412.

[6] S. Nakamori, Recursive estimation technique of signal from output measurement data in linear discrete-time systems,IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences E782-A (5) (1995) 600-607.

[7] S. Nakamori, New design of recursive Wiener fixed-lag smoother in linear discrete-time stochastic systems, Applied Mathematics and Computation 159 (2004) 303-315.

[8] S. Nakamori, New design of fixed-lag smoother using covariance information in linear discrete-time systems, Applied Mathematics and Computation 156 (2004) 415-426.

[9] M. Sugisaka, The design of on-line least-squares estimators given covariance specifications via an imbedding method, Applied Mathematics and Computation 13 (1983) 55-85.
[10] T. Kailath, Lectures on Linear Least-Squares Estimation, Springer-Verlag, New York, 1976.
[11] A.P. Sage, J.L. Melsa, Estimation Theory with Applications to Communications and Control, McGraw-Hill, New York, 1971.