COMPACTIFICATION OF TO-SPACES

COMPACTIFICATION OF T_0-SPACES

By

Mitsunobu SHIRAKI

(Received August 31, 1968)

R. S. PIERCE has given a beautiful characterization of compact T_1-spaces by introducing a concept which is called a covering ideal (see [1]).

In this paper, we shall consider the compactification of T_0-spaces generalizing his method.

DEFINITION 1. Let $\boldsymbol{C}$ be the collection of all open covers of a topological space X, Then *an open filter Y of X is said to be under* $\boldsymbol{C}$ if $Y \cap Z \neq \phi$ for each $Z \in \boldsymbol{C}$.

DEFINITION 2. An order relation in the collection of open filters in X is defined as follows: $Y_1 < Y_2$ if and only if for each $U \in Y_1$ there is at least a $V \in Y_2$ such that $U \subset V$.

LEMMA (PIERCE). *Suppose X is a T_0-space which has more than two elements, and $\boldsymbol{C}$ is the collection of all finite open covers of X. Suppose $\mathfrak{D}$ is a collection of finite systems of open sets satisfying the following conditions.*

(*a*) *If P, $Q \in \mathfrak{D}$, then $P \cup Q \in \mathfrak{D}$.*

(*b*) $\boldsymbol{C} \cap \mathfrak{D} = \phi$.

Then there is an open filter Y which has the following properties.

(*i*) *Y is minimal in the set of open filters under $\boldsymbol{C}$.*

(*ii*) *$P \cap Y = \phi$ for each $P \in \mathfrak{D}$.*

THEOREM. *Suppose X is a T_0-space which has more than two elements. Then there are a space S and an imbedding $f : X \to S$ satisfying the following conditions*:

(1) *S is a compact T_0-space.*

(2) *$f(X)$ is every where dense in S.*

(3) *If X is a compact T_0-space, then $f(X) = S$.*

(4) *Let $\varphi : X \to T$ be a continuous mapping of X to a compact Hausdorff space T. Then the mapping $\varphi \circ f^{-1} : f(X) \to T$ is extended to a continuous mapping of S to T.*

PROOF. Let $\boldsymbol{C}$ be the collection of all finite open covers of X, and X_x be the open neighborhood filter of an element x of X.

Consider a set

$$S=\{X_x \mid x \in X\} \cup \{F : \textit{open filter} \mid F \textit{ is minimal under } \boldsymbol{C}\}.$$

The topology in S is defined by taking the family of the sets $S(U)$ such that

$$S(U)=\{Y \in S \mid U \in Y\},$$

where U is open in X, as an open basis.

1° The space S is T_0. To see this we suppose that $Y, Z \in S$ and $X \neq Z$. Then there is an open set U such that $U \in Y$ and $U \notin Z$. So $S(U) \ni Y$ and $S(U) \not\ni Z$. Hence S is a T_0-space.

2° We define a mapping $f : X \to S$ by $f(x)=X_x$ for $x \in X$. Since X is a T_0-space, f is a bijection such that

$$f(U) \ni X_x \Leftrightarrow U \ni x \Leftrightarrow U \in X_x \Leftrightarrow X_x \in S(U),$$

for any open set U. Since

$$f(U)=S(U) \cap f(X),$$

f is an imbedding.

3° For each $Y \in S$ and $U \in Y$, we have $S(U) \ni Y$. While $x \in U$ implies $X_x \ni U$ and $X_x \in S(U)$, hence

$$S(U) \cap f(X) \neq \phi.$$

Therefore $f(X)$ is dense in S.

4° S is a compact space. To prove this, let $\{S(U_\lambda) \mid U_\lambda \in Q\}$ be any open cover of S. And let $\mathfrak{D}$ be the collection of all nonempty finite subsets of Q. S is compact if $\boldsymbol{C} \cap \mathfrak{D} \neq \phi$ is proved. Suppose

$$\boldsymbol{C} \cap \mathfrak{D}=\phi.$$

Then clearly $\mathfrak{D}$ satisfies the conditions (a) and (b) of the previous lemma. Hence there is an open filter Y satisfying (i) and (ii) of the lemma. From (i) and the definition of S, $Y \in S$ follows. And from (ii) we have

$$Q \ni U \Rightarrow Y \cap \{U\}=\phi \Rightarrow Y \not\ni U \Rightarrow S(U) \not\ni Y.$$

This contradicts to the assumption that $\{S(U_\lambda) \mid U_\lambda \in Q\}$ is a cover of S. Therefore S is a compact space.

5° If X is a compact T_0-space, then $f(X)=S$. To see this we suppose $f(X) \neq S$, then there is a $Y \in S$ such that $Y \neq X_x$ for any X_x. Since Y is minimal under $\boldsymbol{C}$, we have $Y \not\supset X_x$. Hence there is a neighborhood $U(x)$ of x such that $X_x \ni U(x)$ and $Y \not\ni U(x)$.

Then

$$K=\{U(x)\mid x\in X\}$$

is an open cover of X. From compactness of X, K has a finite subcover

$$L=\{U(x_1),\ U(x_2),\ \cdots,\ U(x_n)\}.$$

Then

$$X=\bigcup_{i=1}^{n}\{U(x_i)\mid U(x_i)\in L\},$$

that is, $L\in \boldsymbol{C}$. But since $U(x_i)\notin Y$ for $U(x_i)\in L$, we have

$$L\cap Y=\phi.$$

This contradicts to the fact that Y is under $\boldsymbol{C}$.

6° Let $\varphi : X\rightarrow T$ be a continuous mapping of X to a compact Hausdorff space T. We define a mapping

$$g : S\rightarrow T$$

as follows:
If $X_x\in f(X)$, then

$$g(X_x)=\varphi(x).$$

Next, let $Y\in S-f(X)$. We consider an open symmetric base

$$\mathcal{U}=\{V_\alpha\mid \alpha\in A\}$$

for the uniform structure of T. If $\alpha\in A$, there is a finite cover

$$\{V_\alpha(x_1),\ V_\alpha(x_2),\ \cdots,\ V_\alpha(x_n)\}$$

of T. Since φ is continuous,

$$K=\{\varphi^{-1}(V_\alpha(x_1)),\ \varphi^{-1}(V_\alpha(x_2)),\ \cdots,\ \varphi^{-1}(V_\alpha(x_n))\}$$

is a finite open cover of X. Therefore $K\in\boldsymbol{C}$. Since Y is under $\boldsymbol{C}$ there is $\varphi^{-1}(V_\alpha(x_\alpha))\in K$ such that $\varphi^{-1}(V_\alpha(x_\alpha))\in Y$. Thus for each $\alpha\in A$ we have

$$Y\supset\{\varphi^{-1}(V_\alpha(x_\alpha))\mid\alpha\in A\}.$$

Y is a filter and T is compact. Then we have

$$\cap\{V_\alpha(x_\alpha)^-\mid\alpha\in A\}\neq\phi.$$

We define that $g(Y)$ is an element of the above intersection.

Now the mapping g defined as above satisfies

$$\varphi\circ f^{-1}(X_x)=\varphi(x)=g(X_x)$$

for $X_x \in f(X)$. Hence we have

$$g \circ f^{-1} = g | f(X).$$

Now we need only to prove the continuity of the mapping g. Let V_β be any element of $\mathcal{U}$. And take a $V_\alpha \in \mathcal{U}$ such that

$$V_\beta \supset V_\alpha^4.$$

If Y belongs to $S - f(X)$, then since Y is under $\boldsymbol{C}$, there is a $W = \varphi^{-1}(V_\alpha(x_\alpha))$ such that

$$Y \ni \varphi^{-1}(V_\alpha(x_\alpha)).$$

We take $S(W)$ as an open neighborhood of Y. If $Z \in S(W)$, and if $Z \in S - f(X)$, then

$$Z \ni W = \varphi^{-1}(V_\alpha(x_\alpha)).$$

From the definition of $g(Z)$

$$g(Z) \in V_\alpha(x_\alpha)^- \subset V_\alpha^2(x_\alpha),$$

while

$$g(Y) \in V_\alpha(x_\alpha)^- \subset V_\alpha^2(x_\alpha).$$

Since V_α^2 is symmetric

$$x_\alpha \in V_\alpha^2(g(Y)),$$

hence

$$g(Z) \in V_\alpha^4(g(Y)) \subset V_\beta(g(Y)).$$

If $Z = X_x \in f(X)$, then

$$X_x \ni W = \varphi^{-1}(V_\alpha(x_\alpha)),$$

hence

$$g(X_x) = \varphi(x) \in V_\alpha(x_\alpha),$$

and

$$g(X_x) \in V_\alpha(x_\alpha) \subset V_\alpha^3(g(Y)) \subset V_\beta(g(Y)).$$

Thus g is continuous at Y.

Reference

[1] R. S. PIERCE: Coverings of a topological space. Trans. Amer. Math. Soc. Vol. 77, (1954) 281-298.