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On Graphs Having Exact Two Vertices with the Same Degree

Koukichi SAKAI *

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Abstract

As well known, every graph has at least two vertices with the same degree. The purpose of this note is to determine the graphs having exact two vertices with the same degree and to state some properties of these graphs. Especially we show that for every integer $n \geq 2$ there exists exactly one connected graph of order n having exact two vertices with the same degree.

Key words: Graph, Graph having exact two vertices with the same degree, Degree sequence, Graphical sequence.

1 Degree sequences

In this note we use freely the terminology and notation concerning graphs in G.Chartrand and L.Lesniak [1]. For any positive integer n and non-negative integer m with $m < n$, we use the following notation:

$[n] := \{1, 2, 3, ..., n\}$ $\quad [m, n] := \{m, m + 1, ..., n\}$.

A sequence $s : s_1, s_2, ..., s_n$ of nonnegative integers is said to be *graphical* if there exists a graph $G, V(G) = \{v_1, v_2, ..., v_n\}$, of order n whose degree sequence is s, that is, deg $v_j = s_j$ for all $j \in [n]$.

In this section, for any integer $n \geq 2$ we shall determine the graphical sequences $s : s_1, s_2, ..., s_n$ with the following property:

(1.1) $\quad n - 1 \geq s_1 > s_2 > ... > s_{k-1} > s_k = s_{k+1} > s_{k+2} > ... > s_n \geq 0$

for some $k \in [n - 1]$.

For the sake of brevity any sequence $s : s_1, s_2, ..., s_n$ of non-negative integers with the property (1.1) is said to be $(n,2)$-*admissible*. Let $s : s_1, s_2, ..., s_n$ be $(n,2)$-admissible. If $s_1 = n-2$ then $s_n = 0$. Moreover if s is graphical and $s_1 = n - 1$ then s_n must be equal to 1. So to our aim it suffices to consider the following two types of $(n,2)$-admissible sequences:

(1.2) $\quad n - 1, n - 2, ..., k + 1, k, k, k - 1, ..., 2, 1$ for some $k \in [n - 1]$,

(1.3) $\quad n - 2, n - 3, ..., k + 1, k, k, k - 1, ..., 1, 0$ for some $k \in [0, n - 2]$.

We denote by $s_n(n - 1; k)$ and $s_n(n - 2; k)$ the $(n,2)$-admissible sequence given in the form (1.2) and (1.3) respectively. The next lemma, noted in [1] and [2], plays the essential role in our

* Department of Mathematics and Computer Science, Faculty of Science, Kagoshima University, Kagoshima 890-0065, Japan.

discussion.

Lemma 1.1 *A sequence $s : s_1, s_2, ..., s_n$ of non-negative integer with $s_1 \geq s_2 \geq ... \geq s_n, n \geq 2, s_1 \geq 1$, is graphical if and only if the following sequence $h(s)$ with $n-1$ terms is graphical:*

$h(s) : s_2 - 1, s_3 - 1, ..., s_{t+1} - 1, s_{t+2}, s_{t+3}, ..., s_n$

where $t = s_1$.

In general for any sequence $s : s_1, s_2, ..., s_n$ of integers with n terms, $n \geq 2$, we define the following four kinds of sequences $c(s)$ with n terms, $h(s)$ with $n-1$ terms, $p(s)$ and $z(s)$ with $n+1$ terms respectively:

$c(s) : n - 1 - s_n, n - 1 - s_{n-1}, ..., n - 1 - s_2, n - 1 - s_1$

$h(s) : s_2 - 1, s_3 - 1, ..., s_n - 1$

$p(s) : n, s_1 + 1, s_2 + 1, ..., s_n + 1$

$z(s) : s_1, s_2, ..., s_n, 0.$

Lemma 1.2

(1) *Any $(n, 2)$-admissible sequence s is graphical if and only if so is $c(s)$.*

(2) *$s_n(n-1; k)$ is graphical if and only if so is $s_n(n-2; n-1-k)$.*

(3) *A sequence $s : s_1, s_2, ..., s_n$ of positive integers is graphical if and only if so is $z(s)$.*

(4) *$s_n(n-1; k+1) = p(s_{n-1}(n-3; k))$ for any $k \in [0, n-3]$.*

(5) *$s_n(n-2; k) = z(s_{n-1}(n-2; k))$ for any $k \in [n-2]$.*

Proof (1) follows from the fact that if s is a degree sequence of a graph G of order n then $c(s)$ is the degree sequence of the complement graph of G. Since $c(s_n(n-1; k)) = s_n(n-2; n-1-k)$, (2) is obvious from (1). (3) is a consequence of the fact that any vertex with degree 0 of a graph is isolated. (4) and (5) are obvious. □

Lemma 1.3 *For any integer $n \geq 3$, we have*

(1) *$s_n(n-1; n-1)$ is not graphical.*

(2) *$s_n(n-2; 0)$ is not graphical.*

(3) *$s = s_n(n-1; k), k \in [n-2]$, is graphical if and only if $h(s) = s_{n-1}(n-3; k-1)$ is graphical.*

(4) *$s = s_{n-1}(n-2; k), k \in [n-3]$ is graphical if and only if so is $z(s) = s_n(n-2; k)$.*

Proof (1) $h(s_n(n-1; n-1)) = (n-2, n-3, ..., 2, 1, 0)$ is not graphical, because any finite sequence consisting of mutually distinct non-negative integers is not graphical. So (1) follows from Lemma 1.1. Since $s_n(n-2; 0) = c(s_n(n-1; n-1))$, (2) is obvious from (1) and Lemma 1.2 (2). (3) and (4) follows immediately from Lemma 1.1 and Lemma 1.2(3) respectively. □

Now we denote by $GS(n, 2)$ the set of all graphical $(n,2)$-admissible sequences, and $GS_n(n-1)$ and $GS_n(n-2)$ be the set of all graphical sequences in $\{s_n(n-1; k); k \in [n-1]\}$ and $\{s_n(n-2; k); k \in [0, n-2]\}$ respectively. Then we have

$GS(n, 2) = GS_n(n-1) \cup GS_n(n-2)$

$GS_n(n-2) = \{c(s); s \in GS_n(n-1)\}.$

Combining Lemmas 1.2 and 1.3, we have

Lemma 1.4 *For any integer* $n \geq 3, GS(n,2)$ *is completely determined from* $GS(n-1,2)$ *in the following way :*

(1) $GS_n(n-1) = \{p(s); s \in GS_{n-1}(n-3)\}$.

(2) $GS_n(n-2) = \{z(s); s \in GS_{n-1}(n-2)\}$.

The next is seen easily.

Lemma 1.5 $GS(2,2)$ *is given as follows:*

(1) $GS_2(1)$ *and* $GS_2(0)$ *consists of only one sequence* $s(2) = (1,1)$ *and* $c(s(2)) = (0,0)$ *respectively.*

(2) *The complete graph* K_2 *of order* 2 *is the only one graph with degree sequence* $s(2)$.

From Lemmas 1.4 and 1.5, we can obtain explicitly sequences in $GS(n,2)$.

Theorem 1.6 *For any integer* $n \geq 2$, $GS(n,2)$ *consists of two sequences* $s(n) = s_n(n-1; m)$ *and* $c(s(n)) = s_n(n-2; n-m-1)$, *where* $m =$ *the floor of* $\frac{n}{2}$,*i.e.,*

$$s(n) = \begin{cases} (2m, 2m-1, 2m-2, ..., m+1, m, m, m-1, ..., 1) & \textit{for } n = 2m+1 \\ (2m-1, 2m-2, ..., m+1, m, m, m-1, ..., 1) & \textit{for } n = 2m. \end{cases}$$

Remark 1.7 $GS(3,2)$ consists of the following two sequences:

$s(3) = p(c(s(2)) = (2,1,1) \quad c(s(3)) = z(s(2)) = (1,1,0)$.

We note that the path P_3 of order 3 is the only one graph whose degree sequence is $s(3)$.

Remark 1.8 Let $n \geq 4$. From Lemma 1.4 and Theorem 1.6 we have

(1.4) $\quad s(n) = p(z(s(n-2)))$.

2 Graphs corresponding to $s(n)$

In this section let n be any integer with $n \geq 2$ and we construct the graphs whose degree sequence is $s(n)$ given in Theorem 1.6. At first we note that this graph is uniquely determined by $s(n)$. Let $s(n) : s_1, s_2, ..., s_n$ and let G and H be any graphs with degree sequence $s(n)$. The vertex sets $V(G) = \{v_k; k \in [n]\}$ and $V(H) = \{u_k; k \in [n]\}$ are labeled as $s_k =$ deg v_k = deg u_k for all $k \in [n]$. We define the degree preserving map ϕ_n from $V(G)$ onto $V(H)$ by $\phi_n(v_k) = u_k, k \in [n]$. Then we have

Lemma 2.1 ϕ_n *is an isomorphism from* G *to* H.

Proof We show by the induction on n. As noted in Lemma 1.5 and Remark 1.7,ϕ_2 and ϕ_3 are isomorphic. Let $n \geq 4$. From (1.4) we see that the subgraphs $G' = (G - v_1) - v_n$ and $H' = (H - u_1) - u_n$ admit the degree sequence $s(n-2)$. Moreover the restriction of ϕ_n to $V(G')$ is identical with ϕ_{n-2}. By the inductive hypothesis ϕ_{n-2} is an isomorphism from G' to H'. Since deg v_1= deg $u_1 = n-1$, and v_1 and u_1 are adjacent to all the other vertices in G and H respectively, we conclude that ϕ_n is isomorphic. □

Theorem 2.2 *For every integer $n \geq 2$ there exists exactly one graph G whose degree sequence is $s(n)$. Namely the graphs of order n admitting exact two vertices with the same degree are G and its complement.*

We denote by G_n the graph with degree sequence $s(n)$. For any graphical sequence $s : s_1, s_2, ..., s_n$ let G be a graph of order n whose degree sequence is s. Then the union $G \cup N_1$ of G and the empty graph N_1 of order 1 has the degree sequence $z(s)$, and the join $G + N_1$ of G and N_1 has the degree sequence $p(s)$. Hence by virtue of (1.4) and Theorem 2.2, G_n is constructed inductively in the following way.

Theorem 2.3
(1) $G_2 = K_2$ *and* $G_3 = P_3$.
(2) $G_n = (G_{n-2} \cup N_1) + N_1$ *for every integer* $n \geq 4$.
(3) *The complement graph of* G_n *is* $G_{n-1} + N_1$ *for every integer* $n \geq 3$.

Theorem 2.4 *G_n is a connected graph and its size $q(G_n)$ is*
$q(G_n) = \frac{1}{2}(\frac{n(n-1)}{2} + m)$, *i.e.,*
$q(G_{2m}) = m^2, \quad q(G_{2m+1}) = m^2 + m.$

In what follows let the vertex set $V(G_n) = \{v_1, v_2, ..., v_m, , v_{m+1}, ..., v_n\}$ be labeled as:

$$\deg\ v_k = \begin{cases} k & \text{for } k \in [m] \\ k-1 & \text{for } k \in [m+1, n] \end{cases}$$

where m is the floor of $\frac{n}{2}$. Then from Theorem 2.3 we have

Theorem 2.5 *The adjacency matrix $A(G_n) = (a_{ij})$ of G_n is given as follows:*

$$a_{ij} = \begin{cases} 0 & \text{for } i = j \in [n] \\ 0 & \text{for } 1 \leq j \leq n - i, i \in [n] \\ 1 & \text{for } n - i < j \leq n, j \neq i, i \in [n]. \end{cases}$$

Remark 2.6 Two vertices v_m, v_{m+1} of degree m are adjacent if and only if n is even.

Example $A(G_n)$, for $n = 6$ and 7, are given in the following form:

$$A(G_6) = \begin{pmatrix} 0&0&0&0&0&1 \\ 0&0&0&0&1&1 \\ 0&0&0&1&1&1 \\ 0&0&1&0&1&1 \\ 0&1&1&1&0&1 \\ 1&1&1&1&1&0 \end{pmatrix}, \quad A(G_7) = \begin{pmatrix} 0&0&0&0&0&0&1 \\ 0&0&0&0&0&1&1 \\ 0&0&0&0&1&1&1 \\ 0&0&0&0&1&1&1 \\ 0&0&1&1&0&1&1 \\ 0&1&1&1&1&0&1 \\ 1&1&1&1&1&1&0 \end{pmatrix}$$

3 Properties of G_n

In this section we state some properties of G_n without proof, which are obtained from Theorem 2.3. Throughout this section let n be any integer with $n \geq 2$, and m be the floor of

$\frac{n}{2}$. We denote by $K(v; u_1, u_2, ..., u_n)$ the star graph $K_{1,n}$ with the center vertex v and the end vertices $u_1, u_2, ..., u_n$, and by $N(V)$ the empty graph with the vertex set V.

3.1 Subgraphs

The maximal complete subgraph of G_n is K_{m+1}. So G_n is planar if and only if $n \leq 7$. G_n contains the star $K(v_n; v_1, v_2, ..., v_{n-1})$ and the path P_n as its spanning trees. Moreover G_n is decomposed into mutually edge-disjoint spanning subgraphs $F_1, F_2, ..., F_m$ defined by

$F_k = K(v_{n+1-k}; v_k, v_{k+1}, ..., v_{n-k}) \cup N(v_1, v_n, v_2, v_{n-1}, ..., v_{k-1}, v_{n+2-k})$

for $k \in [m]$.

The center of $G_n = N(v_n)$, and the periphery of G_n is the union of G_{n-2} and $N(v_1)$, which is the complement of G_{n-1}. We see that the radius of G_n is equal to 1 and the diameter of G_n is equal to 2.

3.2 Colorings

The chromatic number χ and the edge chromatic number χ_1 of G_n are given by

$\chi(G_n) = m + 1$ and $\chi_1(G_n) = n - 1$.

The chromatic polynomial $P(G_n, k)$ is given in the following form:

$P(G_{2m}, k) = k((k-1)(k-2) \times ... \times (k-m+1))^2(k-m)$,

$P(G_{2m+1}, k) = k((k-1)(k-2) \times ... \times (k-m+1)(k-m))^2$.

3.3 Coverings

The vertex covering number α and the independence number β of G_n are given by

$\alpha(G_n) = m$ and $\beta(G_{2m}) = m$, $\beta(G_{2m+1}) = m + 1$.

The edge covering number α_1 and the edge independence number β_1 of G_n are given by

$\alpha_1(G_{2m}) = m$, $\alpha_1(G_{2m+1}) = m + 1$ and $\beta_1(G_n) = m$.

Especially G_{2m} contains a 1-factor.

3.4 Characteristic polynomial of $A(G_n)$

We put $P_n(\lambda) = \det\,(\lambda E - A(G_n))$. Then we have

$P_2(\lambda) = \lambda^2 - 1$ and $P_3(\lambda) = \lambda^3 - 2\lambda$,

$P_4(\lambda) = \lambda^4 - 4\lambda^2 - 2\lambda + 1$,

$P_5(\lambda) = \lambda^5 - 6\lambda^3 - 4\lambda^2 + 2\lambda$.

In general we have the following recurrence formula:

$P_n(\lambda) = (2\lambda^2 + 2\lambda - 1)P_{n-2}(\lambda) - \lambda^2(\lambda+1)^2 P_{n-4}(\lambda)$ for $n \geq 6$.

We see that $\lambda = -1$ [resp. $\lambda = 0$] is a proper value of $A(G_n)$ for even [resp. odd] n.

References

[1] C.Chartrand and L. Lesniak: Graphs & Digraphs, Chapman & Hall, London, 1996.

[2] N.Hartsfield and G. Ringel: Pearls in graph theory, Academic Press, 1990.