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Abstract

Let d and n be any positive integers. A positive integer a is said to be (n, d)-generative if a has at least one n-partition $\{p_1, p_2, \cdots, p_n\}$ with the following property:

(*) *every number $x, d \leq x \leq a - d$, is expressed by sum of some p_j's.*

By $G(n, d)$ denote the set of all (n, d)-generative numbers. The purpose of this paper is to determine any members in $G(n, d)$ and in a special subset $SG(n, d)$(see Definition 1) of $G(n, d)$.

Key words: generative number, strongly generative number, partition of number.

1 Introduction

The author has met the following interesting exercise in A.Tucker's book [1, p.422]:

(A) *Show that any set of 16 positive integers (not all distinct) summing to 30 has a subset summing to k, for* $k = 1, 2, 3, ..., 29$.

This exercise suggests a problem:

(B) *For any given positive integer n, find all numbers a, $a \geq n$, such that any set of n positive numbers (not all distinct) summing to a has a subset summing to k, for* $k = 1, 2, 3, ..., a - 1$.

Furthermore we propose a problem in connection with (A) and (B):

(C) *For any given positive integer n, find all numbers a, $a \geq n$, with the following property: there exists at least one set of n positive numbers (not all distinct) summing to a which has a subset summing to k, for* $k = 1, 2, 3, ..., a - 1$.

In this paper we shall consider some problems including (B) and (C) as special cases. To state the problems we prepare some notations used in the paper. Since we deal with only positive integers, "number" means always "positive integer" and any variables $n, d, j, \ldots$ named by small letters express positive integers unless otherwise noted. For any m, n with $m \leq n$, we use the following notation:

$[m, n] = \{m, m + 1, \cdots, n - 1, n\}$, and $[m] = [1, m]$.

The cardinal number of any finite set A is denoted by $|A|$. By *n-set* we mean a collection of n numbers which are not all distinct. So for any n-set P and m-set Q, the union $P \cup Q$ is understood as the $(n+m)$-set of all numbers in P or Q, *e.g.*, $\{1, 1, 2, 3\} \cup \{2, 3\} = \{1, 1, 2, 2, 3, 3\}$.

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Let $P = \{p_1, p_2, \cdots, p_n\}$ be any n-set. Unless otherwise noted, we assume always $p_1 \leq p_2 \leq p_3 \leq \cdots \leq p_n$ and if a number p appears exactly s times in P, this is shortly denoted by $p^{(s)}$, *e.g.*, $\{1, 2, 2, 2, 4, 4\} = \{1, 2^{(3)}, 4^{(2)}\}$. For any n-set P we use the following notation:

$sum(P)$ = the sum of all numbers in P,

$ps(P)$ = the set of $sum(Q)$ for all proper subsets Q of P.

For any fixed number d, an n-set P is called a *(n,d)-set* if $p \geq d$ for all $p \in P$. For any given number a, a (n,d)-partition of a is a (n,d)-set P with $a = sum(P)$. A (n,d)-set P is said to be *d-generative* if $ps(P) = [d, a-d]$, where $a = sum(P)$. For example, it is seen easily that $\{1, 2, 2^2, 2^3, ..., 2^{n-1}\}$ is a 1-generative $(n,1)$-partition of $2^n - 1$, and $\{2, 3, 2^2, 2^3, 2^4, ..., 2^{n-1}\}$ is a 2-generative $(n,2)$-partition of $2^n + 1$.

In what follows let n and d be any fixed numbers with $n > d$. Then any d-generative (n,d)-set P contains the following $(d+1,d)$-sets $A_1(d)$ or $A_2(d)$:

$A_1(d) := [d, 2d]$, $A_2(d) := \{d^{(2)}\} \cup [d+1, 2d-1]$.

Note that $A_1(d)$ and $A_2(d)$ are d-generative. For the sake of brevity, any (n,d)-set is said to be *d-admissible* if it contains $A_1(d)$ or $A_2(d)$. When $n > d+1$, any d-admissible (n,d)-set P is written in the form: $P = A(d) \cup Q_k$, where $k = n-d-1$, Q_k is a (k,d)-set and $A(d)$ is $A_1(d)$ or $A_2(d)$.

Definition 1. (1) A number a is said to be *(n,d)-generative* if a has at least one d-generative (n,d)-partition. It is denoted by $G(n,d)$ the set of all (n,d)-generative numbers.

(2) A number a is said to be *strongly (n,d)-generative* if every d-admissible (n,d)-partition of a is d-generative. It is denoted by $SG(n,d)$ the set of all strongly (n,d)-generative numbers.

Under the above definition, the exercise (A) asserts that $30 \in SG(16,1)$, and the problems (B) and (C) are one to find any numbers in $SG(n,1)$ and $G(n,1)$ respectively.

The aim of this paper is to determine explicitly any numbers in $G(n,d)$ and $SG(n,d)$ for any n, d with $n > d \geq 1$. In section 2 we prepare some lemmas used in our aim. $G(n,d)$ and $SG(n,d)$ are given in sections 3-4 and 5 respectively.

Remark 1. By the definition we have

$$G(d+1,d) = SG(d+1,d) = \{s_2, s_1\},$$

where $s_1 = sum(A_1(d)) = \frac{3}{2}d(d+1)$ and $s_2 = sum(A_2(d)) = \frac{1}{2}d(3d+1)$.

Remark 2. $G(n,d)$ and $SG(n,d)$ are defined for $n > d$. But even if $n \leq d$, the following cases have meaningful:

$G(1,1) = SG(1,1) = \{1\}$, $G(2,d) = SG(2,d) = \{2d, 2d+1\}$ for any d.

2 Lemmas

In this section we prepare some lemmas concerning with d-generativity. Throughout the section let $n > d$.

Lemma 2.1. *Let $P = \{p_1, p_2, \cdots, p_n\}$ be any d-generative (n,d)-set, and assume that $p_j = p_{j+1}$ for some $j \in [n-1]$. Then $Q = \{p_1, p_2, \cdots, p_j, 2p_j, p_{j+2}, \cdots, p_n\}$, obtained from P replacing p_{j+1} by $2p_j$, is d-generative.*

Proof. Put $p = p_j$, $a = sum(P)$, and $P' = P \setminus \{p_j, p_{j+1}\}$. Since P is d-generative, Q is d-admissible and $[d, d+p] \subset ps(Q)$. Let $x \in [d+p+1, a+p-d]$. Then $x-p \in [d+1, a-d] \subset ps(P)$ and $x-p$ is expressed in the form: $x-p = \alpha p + \beta p + q$, where α and β are in $\{0,1\}$, and $q \in ps(P')$. If $(\alpha, \beta) = (1,1)$ then $x = p + 2p + q$. If $(\alpha, \beta) = (1,0)$ or $(0,1)$ then $x = 2p + q$. Finally If $(\alpha, \beta) = (0,0)$ then $x = p + q$. Therefore in any cases $x \in ps(Q)$. This completes the proof. □

By the same method in the proof of the above, we see

Lemma 2.2. *Let $P = A_1(d) \cup P_k$ be any (n,d)-set with $a = sum(P)$, where $P_k = \{p_1, p_2, \cdots, p_k\}$. Moreover put $Q_k = \{p_1, p_2, \cdots, p_{k-1}, p_k + d\}$ and consider the (n,d)-partition $Q = A_2(d) \cup Q_k$ of a. If Q is d-generative then so is P.* □

The next lemma plays the most essential roles in our discussion.

Lemma 2.3. *Let P be a d-generative (n,d)-set P with $a = sum(P)$, c be any number with $c \geq d$, and consider a $(n+1,d)$-set $Q = P \cup \{c\}$. Then we have:*

(1) *when $d = 1$, Q is 1-generative if and only if $c \leq a+1$.*

(2) *when $d = 2$, Q is 2-generative if and only if $c \leq a-3$ or $c = a-1$.*

(3) *when $d \geq 3$, Q is d-generative if and only if $c \leq a - 2d + 1$.*

Proof. Let $c \in [d, a-d]$. Then $ps(Q) = [d, a-d] \cup [d+c, a-d+c]$. Hence $ps(Q) = [d, a+c-1]$ if and only if $d + c \leq a - d + 1$, *i.e.*, $c \leq a - 2d + 1$. Next let $a - d < c$. Then $ps(Q) = [d, a-d] \cup \{a\} \cup \{c\} \cup [d+c, a+c-d]$. If $d \geq 3$, then $ps(Q)$ does not contain either $c+1$ or $c-1$ according to $c = a-d+1$ or $a-d+1 < c$. So we have (3). If $d = 2$, then $ps(Q) = [2, a+c-2]$ if and only if $c = a-1$. If $d = 1$, we see that $ps(Q) = [a+c-1]$ if and only if $c = a$ or $c = a+1$. Hence we have (1) and (2). □

Under the notations in Lemma 2.3, if $c > a - 2d + 1$ and $d \geq 3$, Q is not d-generative. But if $a - 2d + 1 < c \leq a - d + 1$ then we can make a d-generative $(n+m, d)$-set $P \cup \{c\} \cup R$ for some $(m-1, c)$-set R, where m is at most d. For example let $d = 4$, P be a 4-generative $(n,4)$-set with $a = sum(P)$, and let $c = a - 5$. Consider the following sets:
$P_1 = P \cup \{c\}$, $P_2 = P_1 \cup \{a-3\}$ and $P_3 = P_2 \cup \{a-2\}$.
Then P_1 and P_2 are not 4-generative and P_3 is 4-generative.

Lemma 2.4. *Let $d \geq 3$, and P be a d-generative (n,d)-set P with $a = sum(P)$. For any m we put $c(m) = a - 2d + 1 + m$, $P_m = P \cup \{c(m)\}$ and $Q_m = P_m \cup R(m)$, where $R(m)$ is any set of distinct numbers greater than $c(m)$. Then we have:*

(1) *if $m \leq d-1$ and Q_m is d-generative, then $I_m = [a-d+1, a-d+m] \subset R(m)$ and the $(n+m+1, d)$-set $P_m \cup I_m$ is d-generative.*

(2) *if $m = d$ and Q_d is d-generative, then $I_d = [a-d+2, a-1] \subset R(d)$ and the $(n+d-1, d)$-set $P_m \cup I_d$ is d-generative.*

(3) *if $m > d$ then Q_m is not d-generative for any $R(m)$.*

Proof. Let $m > d$. Then $a - d + 1 < c(m)$, and $a - d + 1$ does not belong to both $ps(P_m)$ and $ps(Q_m)$. So Q_m is not d-generative for any $R(m)$. This proves (3). Next assume that Q_m is d-generative and $m \leq d$. Then $ps(P_m) = [d, a-d] \cup [d + c(m), a - d + c(m)] =$

$[d, a-d] \cup [a-d+m+1, 2a-3d+m+1]$ or $ps(P_m) = [d, a-d+1] \cup [a, 2a-2d+1]$ according to $m \leq d-1$ or $m = d$. If $m \leq d-1$ then $I_m \cap ps(P_m) = empty, I_m \subset ps(Q_m)$. So $I_m \subset ps(R(m))$. Moreover every number in I_m is not expressed as a sum of distinct numbers in $R(m)$. Hence $I_m \subset R(m)$. Similarly $R(d)$ contains I_d if $m = d$. Thus we get the first assertions in (1) and (2). The others are seen easily. □

Lemma 2.5. *Let $m(n,d)$ and $sm(n,d)$ be the smallest number in $G(n,d)$ and in $SG(n,d)$ respectively. Then we have*

$$m(n,d) = sm(n,d) = \frac{1}{2}d(3d+1) + (n-d-1)d = \frac{1}{2}d(d+2n-1).$$

Proof. For $n = d+1$ it is obvious from Remark 1. Let $n > d+1, P = A_2(d) \cup \{d^{(n-d-1)}\}$ and $s = sum(P)$. Evidently s is the smallest number with d-admissible (n,d)-partition, and P is only one d-admissible (n,d)-partition of s. Moreover by Lemma 2.3, P is d-generative. Hence $m(n,d) = sm(n,d) = s$. □

3 $G(n,1)$ and $G(n,2)$

Let us denote the maximum number in $G(n,d)$ by $M(n,d)$. The determination of $M(n,d)$ is the most important works for our aim. From the fact that $|P|$ is at most $2^n - 2$ for any (n,d)-set P, we can get easily $M(n,d)$ for the cases $d \leq 2$. So in this section we shall characterize any numbers in $G(n,1)$ and $G(n,2)$.

Lemma 3.1. $M(n,1) = 2^n - 1$ *and* $M(n,2) = 2^n + 1$.

Proof. It is noted in introduction that $2^n - 1 \in G(n,1)$ and $2^n + 1 \in G(n,2)$. For any d-generative (n,d)-set P with $a = sum(P)$, $|ps(P)| = a - 2d + 1 \leq 2^n - 2$. Hence $a \leq 2^n + 2d - 3$. Especially $M(n,1) \leq 2^n - 1$ and $M(n,2) \leq 2^n + 1$. This completes the proof. □

Notice that $M(n,1)$ admits a unique 1-generative $(n,1)$-partition $\{1, 2, 2^2, 2^3, \cdots, 2^n - 1\}$ and $M(n,2)$ admits a unique 2-generative $(n,2)$-partition $\{2, 3, 2^2, 2^3, \cdots, 2^n - 1\}$. From this fact we have

Lemma 3.2. $2^n \notin G(n,2)$ *for any* $n > 2$.

Proof. For $n = 3$, the assertion is true(see Remark 1). So we suppose that 2^n has a 2-generative $(n,2)$-partition $P = \{p_1, p_2, \cdots, p_n\}$ for some $n \geq 4$. If a number p appears at least twice in P then $2^n + p \in G(n,2)$ by Lemma 2.1. This is a contradiction: $M(n,2) = 2^n + 1 < 2^n + p$. Hence $p_1 = 2, p_2 = 3, p_3 = 4 < p_4 < \cdots < p_{n-1} < p_n$, and $p_{n-1} \leq 2^{n-1} - 1$. Since $|ps(P)| = 2^n - 3$, for every $x \in [2, 2^n - 2]$ except $x = 2^{n-1}$ there exists a unique subset $P(x)$ of P with $x = sum(P(x))$. Especially $[2, 2^{n-1} - 1] \subseteq ps(P_{n-1})$, where $P_{n-1} = P \backslash \{p_n\}$. Hence $P_{n-1} = \{2, 3, 2^2, 2^3, \cdots, 2^{n-2}\}, sum(P_{n-1}) = 2^{n-1} + 1 \in G(n-1, 2)$, and $p_n = 2^n - sum(P_{n-1}) = 2^{n-1} - 1$. As $p_n = sum(P_{n-1}) - 2$, P is not 2-generative by Lemma 2.3(2), which is a contradiction. This completes the proof. □

Lemma 3.3.

(1) $[a+1, 2a+1] \subseteq G(n+1, 1)$ *for any* $a \in G(n,1)$.

(2) $[a+2, 2a-3] \cup \{2a-1\} \subseteq G(n+1, 2)$ *for any* $a \in G(n,2)$.

Proof. Each assertion is an immediate consequence from Lemma 2.3. □

Theorem A.

(1) $G(n,1) = [n, 2^n - 1]$ $(n > 1)$.

(2) $G(n,2) = [2n+1, 2^n - 1] \cup \{2^n + 1\}$ $(n > 2)$.

Proof. From Lemmas 3.1-3.3 and Lemma 2.5, the assertion is shown easily by the induction on n. □

4 $G(n,d)$

In this section let $d \geq 3$ and $n > d$.

Lemma 4.1. $G(d+2,d) = [\frac{3}{2}d(d+1), 3d^2 + d + 1]$.

Proof. Recall from Remark 1 that $G(d+1,d) = \{s_2, s_1\}$, where $s_1 = \frac{3}{2}d(d+1)$ and $s_2 = \frac{1}{2}d(3d+1)$. Let $a \in G(d+2,d)$. Then a has a d-generative $(d+2,d)$-partition $P = A(d) \cup \{c\}$ for some $c \geq d$, where $A(d)$ is $A_1(d)$ or $A_2(d)$. By Lemma 2.3(3), $d \leq c \leq sum(A(d)) - 2d + 1$, and hence $a \in [s+d, 2s-2d+1]$ where $s = s_1$ or $s = s_2$. As $(2s_2 - 2d + 1) - (s_1 + d) = s_2 + 1 > 0$, we have $G(d+2,d) = [s_2 + d, 2s_1 - 2d + 1] = [\frac{3}{2}d(d+1), 3d^2 + d + 1]$. □

We prepare some lemmas in order to determine the maximum number $M(n,d)$ in $G(n,d)$.

Lemma 4.2. *Let* $\{M_n(d)\}(n \geq d+1)$ *be the sequence defined by* $M_{d+1}(d) = \frac{3}{2}d(d+1)$ *and the recurrence relations:* $M_{n+1}(d) = 2M_n(d) - 2d + 1$ *for any* $n > d$. *Then we have:*

(1) $M(n,d) = M_n(d)$ *for* $n = d+1, d+2$.

(2) $M_n(d) \in G(n,d)$.

(3) $M_n(d) = (3d^2 - d + 2)2^{n-d-2} + 2d - 1$.

Proof. (1) is obvious from Remark 1 and Lemma 4.1. Lemma 2.3(3) teaches us that $2a - 2d + 1 \in G(n+1,d)$ for any $a \in G(n,d)$. Using repeatedly this fact, we have (2). (3) is seen by the induction on n. □

Let $n \geq d+2, k = n - d - 1 \geq 2$ and let $P = A_1(d) \cup Q_k$ be a d-generative (n,d)-partition of $M(n,d)$. By Lemma 2.1 Q_k consists of distinct k numbers greater then $2d$. Let $h, 0 \leq h \leq k-1$, be the maximal number among $|Q|$, where Q is any proper subset of Q_k such that $A_1(d) \cup Q$ is d-generative. We understand as $h = 0$ if such Q does not exist. Let Q_h with $|Q_h| = h$ be a proper subset of Q_k such that $P_h = A_1(d) \cup Q_h$ is d-generative. When $h = 0$, let $Q_0 = empty$. Here we put:

$a(h) = sum(P_h)$

$c(Q_h) = Q_k \setminus Q_h$

$c(h) =$ the smallest number in $c(Q_h)$

$c(h,m) = a(h) - 2d + m + 1$ for any m with $1 \leq m \leq d$.

Under the these notations the next is derived from Lemma 2.4 and the maximality of h.

Lemma 4.3. *Let $0 \leq h \leq k-2$. Then we have the following cases depending on m with $m \leq d$:*

(a) *when $m \leq d-1$*

$$\begin{aligned} &k = h + m + 1 \text{ and } k - d \leq h \\ &c(h) = c(h,m) \\ &c(Q_h) = \{c(h)\} \cup [a(h) - d + 1, a(h) - d + m] \\ &M(n,d) = (m+2)(a(h) - d + \tfrac{1}{2}(m+1)). \end{aligned}$$

(b) *when $m = d$*

$$\begin{aligned} &k = h + d - 1 \\ &c(h) = c(h,d) = a(h) - d + 1 \\ &c(Q_h) = [a(h) - d + 1, a(h) - 1] \\ &M(n,d) = d(a(h) - \tfrac{1}{2}(d-1)). \end{aligned}$$

□

Theorem B. *$M(n,d) = M_n(d)$ for any $d \geq 3$ and $n > d$, where $M_n(d)$ is in Lemma 4.2.*

Proof. We use freely the above notations concerning $M(n,d)$ and in Lemma 4.3. We prove the assertion by the induction on n. It is true for $n = d+1, d+2$. Let $n > d+2$, *i.e.*, $k \geq 2$. Note that $M_n(d) \leq M(n,d)$ by Lemma 4.2. Now suppose $h \leq k-2$. Then from Lemma 4.3 and the inductive hypothesis it follows: $M(n,d) \leq R(h,m,d)$, where

$$R(h,m,d) = \begin{cases} (m+2)(M_{d+1+h}(d) - d + \frac{1}{2}(m+1)) & \text{when } \; m \leq d-1 \\ d(M_{d+1+h}(d) - \frac{1}{2}(d-1)) & \text{when } \; m = d. \end{cases}$$

But it is seen that $R(h,m,d) < M_n(d)$ for each case, which is a contradiction. Hence $h = k-1$ and $M(n,d) = a(k-1) + c(k-1) \leq M_{n-1}(d) + (M_{n-1}(d) - 2d + 1 = M_n(d)$. Therefore $M(n,d) = M_n(d)$ and $c(k-1) = M(n-1,d) - 2d + 1$. □

Corollary B-1. *Let $d \geq 3$ and $n > d+1$.*

(1) *$M(n,d)$ has a unique d-generative (n,d)-partition $A_1(d) \cup \{q_1, q_2, \cdots, q_k\}$, where $k = n - d - 1$ and $q_j = M(d+j, d) - 2d + 1$ for $j \in [k]$.*

(2) $M(n+1,d) = 2M(n,d) - 2d + 1$. □

Theorem C. *$G(n,d) = [m(n,d), M(n,d)]$ for any $d \geq 3$ and $n \geq d+2$, where*

$$\begin{aligned} &m(n,d) = \tfrac{1}{2}d(d + 2n - 1), \\ &M(n,d) = (3d^2 - d + 2)2^{n-d-2} + 2d - 1. \end{aligned}$$

Proof. Lemma 2.3(3) teaches us that $[a+d, 2a-2d+1] \subset G(n+1,d)$ *for any* $a \in G(n,d)$. From this fact, Lemma 2.5, Theorem B and Colloraly B-1, the assertion is obtained by the induction on n. □

5 $SG(n,d)$

Theorem D. *$SG(n,d)$ is given as follows:*

(1) $SG(n,1) = [n, 2n-1] \quad (n > 1)$

(2) $SG(n,2) = [2n+1, 4n-5] \cup \{4n-3\} \quad (n > 2, n \neq 5)$

(3) $SG(5,2) = [11,15]$

(4) $SG(d+1,d) = \{\frac{1}{2}d(3d+1), \frac{3}{2}d(d+1)\}$

(5) $SG(n,d) = [sm(n,d), s(n,d)] \quad (n > d+1, d > 2)$, *where*

$$sm(n,d) = d(d+2n-1)/2,$$
$$s(n,d) = d^2 + (2n-5)d + 1.$$ □

The proof of the above theorem is divided into some lemmas. From Lemma 2.2 it follows that any member in $SG(n,d)$ is characterized as follows.

Lemma 5.1. *$a \in SG(n,d)$ if and only if every (n,d)-partition of a containing $A_2(d)$ is d-generative.* □

As $SG(d+1,d)$ is known in Remark 1, let $n \geq d+2$, and introduce the following numbers depending on n and d:

$$k = n - d - 1$$
$$s(d) = sum(A_2(d)) = \tfrac{1}{2}d(3d+1)$$
$$s(n,1) = 2n-1$$
$$s(n,d) = 2s(d) + 2(k-2)d + 1 = d^2 + (2n-5)d + 1 \quad (d \geq 2)$$
$$m_j(1) = s(1) + j = j + 2$$
$$m_j(d) = s(d) + (j-3)d + 1 \quad (d \geq 2).$$

Recall from Lemma 2.5 that the smallest number $sm(n,d)$ in $SG(n,d)$ is given in the form: $sm(n,d) = s(d) + kd$. Let $s \in SG(n,d)$ and put $c = s - sm(n-1,d)$. Then we get the (n,d)-partition $P_n(s) = P_{n-1} \cup \{c\}$ of s where $P_{n-1} = A_2(d) \cup \{d^{(n-d-2)}\}$. Since P_{n-1} is d-generative, $P_n(a)$ is d-generative if and only if c satisfies the condition in Lemma 2.3 for $a = sum(P_{n-1}) = sm(n-1,d)$. So we get

Lemma 5.2. *Let $s \in SG(n,d)$.*

(1) *when $d = 1$, $s \leq s(n,1)$.*

(2) *when $d = 2$, $s \leq s(n,2) = 4n-5$ or $s = 4n-3$, and $s \neq 4n-4$.*

(3) *when $d \geq 3$, $s \leq s(n,d)$.* □

Lemma 5.3. *Let $Q_k = \{q_1, q_2, \cdots, q_k\}$ be any (k,d)-set and let $P_k = A_2(d) \cup Q_k$. Here consider the following condition depending on d:*
(S_d) $\quad q_j \leq m_j(d)$ for any $j \in [k]$.
If Q_k satisfies (S_d) then P_k is d-generative.

Proof. Let $d \geq 2$. For any $j \in [k-1]$ put $Q_j = \{q_1, q_2, \cdots, q_j\}$ and $P_j = A_2(d) \cup Q_j$. We prove by the induction on $j \in [k]$. By Lemma 2.3 P_1 is d-generative if $q_1 \leq s(d) - 2d + 1 = m_1(d)$. Suppose that P_j is d-generative, and put $a_j = sum(P_j)$. Then $a_j \geq s(d) + jd$ and $m_{j+1}(d) = s(d) + (j-2)d + 1 \leq a_j - 2d + 1$. So if $q_{j+1} \leq m_{j+1}(d)$ it follows from Lemma 2.3 that P_{j+1} is d-generative. The assertion in the case of $d = 1$ is proved by the same way. □

Lemma 5.4. *Let $a \in [sm(n,d), s(n,d)]$ and $P = A_2(d) \cup \{q_1, q_2, \cdots, q_k\}$ be any (n,d)-partition of a. Then $q_j \leq m_j(d)$ for any $j \in [k]$. Moreover if $d = 2$, $a = s(n,2) + 2 = 4n - 3$ and $n \geq 6$, then $q_j \leq m_j(2)$ for any $j \in [k-1]$.*

Proof. We prove for the case $d \geq 2$. It suffices to prove for $a = s(n,d)$. Suppose $q_j > m_j(d)$ for some $j \in [k]$. Then we have the following contradiction:

$$sum(P) - a \geq s(d) + (j-1)d + (k-j+1)(m_j(d)+1) - a = (k-j)(s(d) + (j-5)d + 2) + 1 > 0.$$

□

From Lemmas 5.2-5.4, it follows that $SG(n,d) = [sm(n,d), s(n,d)]$ for any $n > d+1$ except $d = 2$, $[sm(n,2), s(n,2)] \subset SG(n,2)$ and $4n - 4 \notin SG(n,2)$. So by Lemma 5.2(2) it remains to see whether $4n - 3 \in SG(n,2)$ or not.

Lemma 5.5.

(1) $4n - 3 \in SG(n,2)$ *for any* $n \geq 6$.

(2) $13 \in SG(4,2)$ *and* $17 \notin SG(5,2)$.

Proof. Put $f(n) = 4n - 3$. Let $n \geq 6$ and $P = A_2(2) \cup \{q_1, q_2, \cdots, q_k\}$ be any $(n,2)$-partition of $f(n)$. If $q_k \leq m_k(2)$, then P is 2-generative by Lemmas 5.3-5.4(2).The other partitions P are $A_2(2) \cup \{2^{(n-4)}, m_k(2) + 2\}$ or $A_2(2) \cup \{2^{(n-5)}, 3, m_k(2) + 1\}$. These are 2-generative by Lemma 2.3. Hence we have (1). Note $f(4) = 13$ and $f(5) = 17$. $13 \in SG(4,2)$ is seen easily. A $(5,2)$-partition $P = \{2,2,3,5,5\}$ of 17 is not 2-generative, because $6 \notin ps(P)$. Hence $17 \notin SG(5,2)$. □

Theorem D follows from Lemmas 5.2-5.5.

Reference

[1] A.Tucker: *Applied Combinatorics* (3rd edition), John Wiley & Sons, New York (1995).