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A Note on Maxcliques Covered by Another Maxcliques in Regular Graphs

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Abstract

Let G be a graph with the edge set $E(G)$, let $\theta_m(G)$ be the number of maxcliques in G, and $\theta_1(G)$ be the minimum number of maxcliques of G by which $E(G)$ is covered. We say that a maxclique Q of G is *non-proper* if for every edge e of Q there is at least one another maxclique Q' such that $e \in E(Q')$. If G contains at least one non-proper maxclique, then $\theta_1(G) < \theta_m(G)$.

In this note we determine the structure of non-proper maxcliques order $r-1$ in r-regular graphs, and characterize any r-regular graphs G with $\theta_1(G) < \theta_m(G)$ for $r = 4$ and 5.

Key words: regular graph, maxclique, edge maxclique cover.

1 Introduction

In this note the terminology and notion concerning graphs follow Chartrand and Lesniak [1] unless otherwise stated, and any graphs are always finite and simple. For any graph G we denote by $V(G)$ and $E(G)$ the vertex set and the edge set respectively. For any $u \in V(G)$ we denote by $N(u)$ the neighborhood of u. A complete subgraph Q of G is called a *clique*, and especially it is called a *maxclique* if it is not properly contained in another cliques. A family $\mathbf{F}$ of distinct maxcliques of G is called an *edge maxclique cover* of a subgraph H if $E(H) \subseteq \cup\{E(Q); Q \in \mathbf{F}\}$.

An edge e of G is said to be *proper* if it is contained a unique maxclique. A maxclique Q is said to be *proper* if $E(Q)$ contains at least one proper edge, otherwise is called *non-proper*.

Here we introduce the following numbers for G:

$\theta_m(G) =$ the number of all maxcliques of G,

$\theta_1(G) =$min $\{|\mathbf{F}|; \mathbf{F}$ is any edge maxclique cover of $G\}$.

As noted in [2], for any graph G, $\theta_1(G) = \theta_m(G)$ if and only if every maxclique of G is proper. Any graphs with this property are studied in Wallis et al. [4] and called maximal clique irreducible graphs.

The aim of this note is to study on edge maxclique covers of non-proper maxcliques of order $r-1$ in any connected r-regular graphs, and is to characterize any r-regular graphs G with $\theta_1 G) < \theta_m(G)$ for $r = 4$ and 5.

For any positive integers m, n with $m \leq n$ we use the following notation:
$[m, n] = \{m, m+1, m+2, \cdots, n-1, n\}$ and $[n] = [1, n]$.

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2 Non-proper maxcliques of order $r-1$ in r-regular graphs

Let G be any connected r-regular graph with $r \geq 3$, and Q and Q_1 be any maxcliques in G with $E(Q) \cap E(Q_1) \neq \emptyset$. Then we note that

(2.1) $|V(Q)| + |N(u) \setminus V(Q)| = r+1$ for any $u \in V(Q)$,

(2.2) $|V(Q)| + |V(Q_1) \setminus V(Q)| \leq r+1$,

For the sake of brevity we use freely the following notation for Q:

$$N_Q(z) = N(z) \cap V(Q) \text{ for any } z \in V(G),$$
$$Z(Q) = \{z \in V(G) \setminus V(Q); |N_Q(z)| \geq 2\},$$
$$E(Q,u) = \{e \in E(Q); e \text{ is incident with } u\} \text{ for } u \in V(Q).$$

For any maxclique Q' we write as $Q' =< W_1, W_2, \cdots, W_k >_c$ if $V(Q')$ is partitioned into subsets $\{W_j; j \in [k]\}$. Under these notations we have

Lemma 2.1. *If Q is non-proper, then for any distinct $u, v \in V(Q)$ there is a $z \in Z(Q)$ for which $N(z)$ contains both u and v.* □

The above is an immediate consequence of non-properness of Q. Here we prove

Theorem 2.2. *Any maxclique Q of order r in any connected r-regular graph G is proper.*
Proof. Suppose Q is non-proper, and let $u \in V(Q)$. Since $N(u) \setminus V(Q)$ is a singleton $\{z\}$, it follows from Lemma 2.1 that $V(Q) = N(z)$. Hence $< z, N(z) >_c$ is a clique containing properly Q. But this contradicts to the maximality of Q. □

By the above theorem any triangles in 3-regular graphs are proper. Hence we have

Corollary 2.3. *$\theta_1(G) = \theta_m(G)$ for any connected 3-regular graphs G.* □

In what follows let Q be non-proper. Any maxclique Q_1 with $E(Q) \cap E(Q_1) \neq \emptyset$ is characterized by the next lemma, which is seen easily from the maximality of Q_1.

Lemma 2.4. *Let Q_1 be given in the form: $Q_1 =< Z_1, W_1 >_c$, where $Z_1 = V(Q_1) \setminus V(Q)$ and $W_1 = V(Q) \cap V(Q_1)$. Then we have the following two cases* (a) *and* (b):

(a) $W_1 = N_Q(z_1)$ *for some* $z_1 \in Z_1$ *and* $Z_1 = \{z \in Z(Q); z \in N(z_1)$ *and* $N_Q(z) = W_1\}$,

(b) $W_1 = \cap\{N_Q(z); z \in Z_1\}$, *and* Z_1 is the maxclique in $\{z \in Z(Q); W_1 \subset N_Q(z)\}$.

Conversely for any $Z_1 \subset Z(Q)$ and $W_1 \subset V(Q)$ with $2 \leq |W_1| < |V(Q)|$ if they satisfy the conditions (a) *or* (b), *then $< Z_1, W_1 >_c$ is a maxclique containing some edges of Q.* □

If $Q_1 =< Z_1, W_1 >_c$ satisfy the conditions (b) in the above. Then it is covered by the family of cliques $\{< z, N_Q(z) >_c; z \in Z_1\}$, and hence it covered by the maxcliques with the conditions (a). So we may assume that any non-proper maxclique Q is covered by maxcliques with (a).

In $Z(Q)$ define an equivalence relation (*) as follows: for any z and z' in $Z(Q)$, $z(*)z'$ if $N_Q(z) = N_Q(z')$ and z and z' are adjacent. For any $z \in Z(Q)$ we denote $[z]$ by the equivalence class belonging to z. Then from Lemma 2.4 any maxcliques $Q(z)$ covering some edges of Q is given in the form: $Q(z) :=< [z], N_Q(z) >_c$ for some $z \in Z(Q)$. For any $u \in V(Q)$ we put

$Z(Q,u) = N(u) \cap Z(Q)$,

$d(Q,u) =$ the number of distinct (*)-equivalence classes $[z]$ for $z \in Z(Q,u)$.

If $Z(Q,u)$ consists of a single (*)-equivalent class $[z]$ for some $u \in V(Q)$, then $V(Q) = N_Q(z)$ by Lemma 2.1 and the maxclique $< [z], N_Q(z) >_c$ contains properly Q, which contradicts to the maximality of Q. So for any $u \in V(Q)$ we have

(2.3) $d(Q,u) \geq 2$,

(2.4) $E(Q,u)$ is covered by the family $\{< [z], N_Q(z) >_c; z \in Z(Q,u)\}$ of maxcliques.

Theorem 2.5. *Suppose that $d(Q,u) = 2$ for all $u \in V(Q)$. Then*

(1) *$Z(Q)$ consists of three (*)-equivalent classes $\{[z_j]; j \in [3]\}$,*

(2) *Q is covered by the three maxcliques $< [z_j], N_Q(z_j) >_c, j \in [3]$*

(3) *$V(Q)$ is partitioned into three subsets $\{W_j; j \in [3]\}$ as follows:*

$N_Q(z_1) = W_1 \cup W_3, N_Q(z_2) = W_2 \cup W_3$ and $N_Q(z_3) = W_1 \cup W_2$.

Proof. Let $u \in V(Q)$. By the hypothesis $E(Q,u)$ is covered by $Q_j =< [z_j], N_Q(z_j) >, j = 1, 2$, where $Z(Q,u) = \{[z_1], [z_2]\}$. Here we put $W_3 = N_Q(z_1) \cap N_Q(z_2), W_1 = N_Q(z_1) \setminus V(Q_2)$ and $W_2 = N_Q(z_2) \setminus V(Q_1)$. Then $\{W_j; j \in [3]\}$ is a 3-partition of $V(Q)$. Let $v \in W_1$ and $w \in W_2$. By the hypothesis $Z(Q,v) = \{[z_1], [z_3]\}$ for some $z_3 \in Z(Q)$ and $Z(Q,v)$ is covered by Q_1 and $Q_3 =< [z_3], N_Q(z_3) >_c$. So $W_2 \subset N_Q(z_3)$ and $Z(Q,w) = \{[z_2], [z_3]\}$. Since $Z(Q,w)$ is covered by Q_2 and Q_3, $W_1 \subset N_Q(z_3)$. Moreover as $E(Q,u')$ is covered by Q_1 and Q_2 for any $u' \in W_3$, we have $N_Q(z_3) = W_1 \cup W_2$. This completes the proof. □

Theorem 2.6. *Let Q be any maxclique of order $r-1$ in any connected r-regular graph G. Then Q is non-proper if and only if $V(Q)$ has a 3-partition $\{W_j; j \in [3]\}$ and there is a 3-set $\{z_j; j \in [3]\}$ in $Z(Q)$ such that Q has an edge maxclique cover $\{Q_j; j \in [3]\}$ given in the form: $Q_i =< z_i, W_j, W_k >_c$ for any distinct $i, j, k \in [3]$.*

Proof. Let Q be non-proper. Since $d(Q,u) = |Z(Q,u)| = |N(u) \setminus V(Q)| = 2$ for all $u \in V(Q)$, it follows from (2.4) that $E(Q,u)$ is covered by exact two maxcliques. Hence under the notation in Theorem 2.5, $[z_j]$ is a singleton $\{z_j\}$ for $j \in [3]$. The maximality of Q_j's is obvious. This completes the proof. □

3 r-Regular graphs containing non-proper maxcliques

In this section for $r \in [4,5]$ we consider r-regular graphs G with $\theta_1(G) < \theta_m(G)$. By virtue of theorem 2.6, we can determine explicitly any non-proper maxcliques of order $r-1$ for $r \in [4,5]$. In order to state the following theorems, we introduce the notion of quasi-induced subgraphs following [3]. For two disjoint subsets V_1 and V_2 of $V(G)$, we define a subgraph H of G as follows: $V(H) = V_1 \cup V_2$ and $E(H)$ consists of all edges uv in G such that $u \in V_1$ and $v \in V_1 \cup V_2$. This H is called a *quasi-induced subgraph*, and V_1 and V_2 are called the base set and the neighborhood set respectively. In Figs. 1-2 any vertices in the base [resp. neighborhood] set are denoted by black circles • [resp. circles ∘], the base set induces a non-proper maxclique.

Theorem 3.1. *For any 4-regular graph G, $\theta_1(G) < \theta_m(G)$ if and only if G contains at least one quasi-induced subgraph isomorphic to the graph in* Fig.1. □

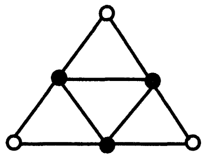
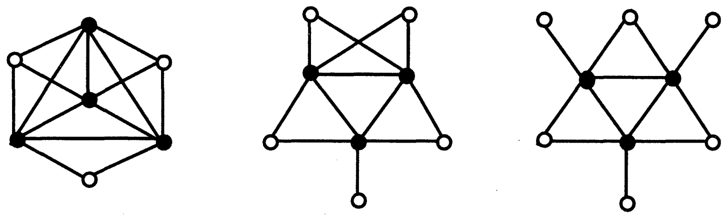


Fig.1

Any non-proper maxcliques of order 3 in 5-regular graphs are shown in Fig. 2(b)-(c).

Theorem 3.2. *For any connected 5-regular graph G, $\theta_1(G) < \theta_m(G)$ if and only if G contains at least one quasi-induced subgraphs isomorphic to the graphs in* Fig.2 (a)-(c). □



(a) (b) (c)

Fig.2

References

[1] L. Chartrand and L. Lesniak: *Graphs & Digraphs*, Chapman & Hall, 1996.

[2] K. Sakai: *On set representations and intersection numbers of some graphs,* Rep. Fac. Sci. Kagoshima Univ. **33**(2000), 39-46.

[3] M.Tsuchiya: *On antichain intersection numbers, total cliques covers and regular graphs,* Discrete Math.**127** (1994), 305-318.

[4] W.D. Wallis and Guo-Hui Zhang: *On maximal clique irreducible graphs*, J.Combin. Math. Combin. Comput., **8**(1990), 187-193.