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ON THE WOLVERTON AND WAGNER'S ASYMPTOTICALLY OPTIMAL DISCRIMINANT FUNCTION

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1. Introduction

Suppose we have an observation x, which may be a scalar or a vector, and we know apriori that it should have come from either of two populations π_1 and π_2, which have the probability density functions $f_1(x)$ and $f_2(x)$ respectively and apriori probabilities q_1 and q_2 respectively. We assume that the losses due to two kinds of misclassification are same, where one misclassification is that if the observation is actually from π_1 we classify it as coming from π_2 and the other is that if the observation is actually from π_2 we classify it as coming from π_1. Then according to the Bayes procedure, if

$$\frac{q_1 f_1(x)}{q_1 f_1(x) + q_2 f_2(x)} \geqq \frac{q_2 f_2(x)}{q_1 f_1(x) + q_2 f_2(x)}$$

then we decide that the observation has come from π_1, and otherwise we decide that the observation has come from π_2. Equivalently if

$$D(x) = q_1 f_1(x) - q_2 f_2(x) \geqq 0$$

then we decide that it has come from π_1 and otherwise we decide that it has come from π_2.

The purpose of this paper is to discuss the statistical properties of the estimate of $D(x)$ constructed by Wolverton and Wagner [6] by using the results in Yamato [7].

Let $X_1^1, X_2^1, X_3^1, \ldots.$ and $X_1^2, X_2^2, X_3^2, \ldots.$ be sequences of independent, identically distributed m-dimensional random vectors in the m-dimensional Euclidian space E_m, which have the probability density functions $f_1(x)$ and $f_2(x)$ respectively. Let $\rho_1, \rho_2, \rho_3 \ldots.$ be a sequence of independent identically distributed random variables with

$\Pr(\rho_i=1)=q_1$ and $\Pr(\rho_i=0)=q_2$ $(i=1,2,3,\ldots.)$.

We assume that X_i^1, X_j^2, ρ_k are mutually independent for all $i=1, 2, \ldots., j=1, 2, \ldots.$ and $k=1, 2, \ldots.$ In this paper we consider a sequential estimation of $D(x)$ with a scheme that we observe X_i^1 when $\rho_i=1$ and X_i^2 when $\rho_i=0$. Wolverton and Wagner [6] considered an estimate of $D(x)$ under the same sampling scheme given by

$$D_n(x) = \frac{1}{n} \sum_{j=1}^{n} \left[\rho_j \frac{1}{h_j^m} K\left(\frac{x - X_j^1}{h_j}\right) - (1 - \rho_j) \frac{1}{h_j^m} K\left(\frac{x - X_j^2}{h_j}\right)\right]$$

and showed that under a certain condition on $f_1(x)$, $f_2(x)$, h_n, $k(\cdot)$

$$\int_{Em} |D_n(x) - D(x)|^2 \, dx$$

converges to 0 in probability (with probability 1) and then $P_{D_n}(e)$ converges to $P_D(e)$ in probability (with probability 1), where $P_d(e)$ denote the probability of misclassification by using a discriminant function $d(x)$.

In the following sections we shall discuss the asymptotic unbiasedness, asymptotically uniform unbiasedness, consistency, uniform consistency and asymptotic normality of $D_n(x)$ by using Yamato[7]. Concerning its asymptotically uniform unbiasedness, Wolverton and Wagner[6] proved it in Lemma 2 under the assumption that $f_1(x)$ and $f_2(x)$ are uniformly continuous. In section 2 we shall, however, generalize it for continuous probability density functions $f_1(x)$ and $f_2(x)$ and moreover at the continuous point x of $f_1(x)$ and $f_2(x)$.

In section 3 we shall treat the limits of the variance and the mean square error of $D_n(x)$ and the limit of nh_n^m Var $[D_n(x)]$.

In section 4 we shall treat the uniform consistency of $D_n(x)$.

In section 5 we shall treat the limit distribution of $D_n(x)$.

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2. Asymtotic unbiasedness

Theorem 1. *We suppose that the probability density functions $f_1(x)$ and $f_2(x)$ are continuous and that $\{h_n\}$ is a sequence of monotone decreasing positive numbers such that*

$$\lim_{n\to\infty} h_n = 0 \tag{2.1}$$

Let $K(y)$ be a measurable function satisfying

$$\sup_{y\in Em} |K(y)| < \infty \tag{2.2}$$

$$\int_{Em} K(y)\, dy = 1 \tag{2.3}$$

$$\int_{Em} |K(y)|\, dy < \infty \tag{2.4}$$

where E_m denotes the m-dimensional Euclidian space and let $\{X_i^1\}$, $\{X_i^2\}$, $\{\rho_i\}$ be mutually independent sequences of random variables and vectors as described in section 1. *Then*

$$D_n(x) = \frac{1}{n}\sum_{j=1}^{n}\left[\rho_j \frac{1}{h_j^m} K\left(\frac{x - X_j^1}{h_j}\right) - (1-\rho_j)\frac{1}{h_j^m} K\left(\frac{x - X_j^2}{h_j}\right)\right] \tag{2.5}$$

is an asymptotically unbiased estimate of $D(x)$.

The following corollary can be found in Wolverton and Wagner [6], which we need to prove Theorem 5. In the following corollary, we assume the uniform continuity of the probability density function, which is satisfied when a population characteristic function is absolutely integrable.

Corollary 1. *If we assume the uniform continuity of the probability density functions* $f_1(x)$ *and* $f_2(x)$ *in Theorem* 1, *then we have*

$$\lim_{n\to\infty} \sup_{x\in E_m} |E D_n(x) - D(x)| = 0 \tag{2.6}$$

Theorem 2. *We suppose that* $\{h_n\}$ *is a sequence of monotone decreasing positive numbers satisfying* (2. 1) *and that the measurable function* $K(y)$ *satisfies* (2.2), (2.3), (2.4) *and*

$$\lim_{y\to\infty} |y|^m |K(y)| = 0 \tag{2.7}$$

Let $\{X_i^1\}$, $\{X_i^2\}$, $\{\rho_i\}$ *be mutually independent sequences of random variables and vectors as described in section* 1. *Then* $D_n(x)$ *is asymptotically unbiased at the point* x *such that both* $f_1(x)$ *and* $f_2(x)$ *are continuous.*

By applying Theorem 1, Corollary 1 and Theorem 2 in Yamato[7] on an inequality

$$\begin{aligned} &|E D_n(x) - D(x)| \\ &\leqq q_1|E\hat{f}_1(x) - f_1(x)| + q_2|E\hat{f}_2(x) - f_2(x)| \end{aligned} \tag{2.8}$$

we can easily obtain Theorem 1, Corollary 1 and Theorem 2, where

$$\hat{f}_1(x) = \frac{1}{n}\sum_{j=1}^{n}\frac{1}{h_j^m}K\left(\frac{x - X_j^1}{h_j}\right) \tag{2.9}$$

$$\hat{f}_2(x) = \frac{1}{n}\sum_{j=1}^{n}\frac{1}{h_j^m}K\left(\frac{x - X_j^1}{h_j}\right). \tag{2.10}$$

3. Consistency

Theorem 3. *We suppose that the probability density functions* $f_1(x)$ and $f_2(x)$ *are continuous and that* $\{h_n\}$ *is a sequence of monotone decreasing positive numbers satisfying* (2.1) *and*

$$\lim_{n\to\infty} n h_n^m = \infty . \tag{3.1}$$

Let the measurable function $K(y)$ *satisfy* (2.2) *and* (2.4) *and let* $\{X_i^1\}$, $\{X_i^2\}$, $\{\rho_i\}$ *be mutually independent sequences of random vectors and variables as described in section* 1. *Then we have*

$$\lim_{n\to\infty} \mathrm{Var}[D_n(x)] = 0 \qquad \text{at all points } x \in E_m. \tag{3.2}$$

Furthermore if $K(y)$ satisfy (2.3), *then we have*

(3.3)
$$\lim_{n\to\infty} E|\boldsymbol{D}_n(\boldsymbol{x}) - \boldsymbol{D}(\boldsymbol{x})|^2 = 0 .$$

Proof. We shall note at first that

(3.4)
$$\begin{aligned}&\mathrm{Var}[\boldsymbol{D}_n(\boldsymbol{x})]\\ &= E\,\mathrm{I}_1^2 + E\,\mathrm{I}_2^2 + E\,\mathrm{I}_3^2 + E\,\mathrm{I}_4^2 - 2\,E\,\mathrm{I}_2\,\mathrm{I}_4\end{aligned}$$

where

(3.5)
$$\begin{aligned}
\mathrm{I}_1 &= \frac{1}{n}\sum_{j=1}^{n} \rho_j \frac{1}{h_j^m}\left[K\left(\frac{\boldsymbol{x}-\boldsymbol{X}_j^1}{h_j}\right) - E\,K\left(\frac{\boldsymbol{x}-\boldsymbol{X}_j^1}{h_j}\right)\right]\\
\mathrm{I}_2 &= \frac{1}{n}\sum_{j=1}^{n} (\rho_j - q_1) \frac{1}{h_j^m} E\,K\left(\frac{\boldsymbol{x}-\boldsymbol{X}_j^1}{h_j}\right)\\
\mathrm{I}_3 &= \frac{1}{n}\sum_{j=1}^{n} (1-\rho_j) \frac{1}{h_j^m}\left[K\left(\frac{\boldsymbol{x}-\boldsymbol{X}_j^2}{h_j}\right) - E\,K\left(\frac{\boldsymbol{x}-\boldsymbol{X}_j^2}{h_j}\right)\right]\\
\mathrm{I}_4 &= \frac{1}{n}\sum_{j=1}^{n} (1-\rho_j - q_2) \frac{1}{h_j^m} E\,K\left(\frac{\boldsymbol{x}-\boldsymbol{X}_j^2}{h_j}\right).
\end{aligned}$$

We can show easily that

(3.6)
$$\begin{aligned}
E\,\mathrm{I}_1^2 &= q_1\,\mathrm{Var}[\hat{\boldsymbol{f}}_1(\boldsymbol{x})]\\
E\,\mathrm{I}_2^2 &\leqq \frac{1}{n} q_1 q_2 \,||\boldsymbol{f}_1||\left\{\int_{E_m} |K(\boldsymbol{y})|\,d\,\boldsymbol{y}\right\}^2\\
E\,\mathrm{I}_3^2 &= q_2\,\mathrm{Var}[\hat{\boldsymbol{f}}_2(\boldsymbol{x})]\\
E\,\mathrm{I}_4^2 &\leqq \frac{1}{n} q_1 q_2 \,||\boldsymbol{f}_2||\left\{\int_{E_m} |K(\boldsymbol{y})|\,d\,\boldsymbol{y}^2\right\}^2 .
\end{aligned}$$

where $||\boldsymbol{f}_1|| = \max \boldsymbol{f}_1(\boldsymbol{x})$ and $||\boldsymbol{f}_2|| = \max \boldsymbol{f}_2(\boldsymbol{x})$, whose existence is secured by the continuity of $\boldsymbol{f}_1(\boldsymbol{x})$ and $\boldsymbol{f}_2(\boldsymbol{x})$. By applying (3.6) and the Schwarz's inequality on (3.4), we have

(3.7)
$$\mathrm{Var}[\boldsymbol{D}_n(\boldsymbol{x})] = q_1\,\mathrm{Var}[\hat{\boldsymbol{f}}_1(\boldsymbol{x})] + q_2\,\mathrm{Var}[\hat{\boldsymbol{f}}_2(\boldsymbol{x})] + 0\left(\frac{1}{n}\right).$$

Theorem 3 in Yamato[7] implies that the right side of (3.7) tends to zero as n tends to ∞. Thus (3.2) was established.

Next, it is obvious that

(3.8)
$$E|\boldsymbol{D}_n(\boldsymbol{x}) - \boldsymbol{D}(\boldsymbol{x})|^2 = \mathrm{Var}[\boldsymbol{D}_n(\boldsymbol{x})] + |E\,\boldsymbol{D}_n(\boldsymbol{x}) - \boldsymbol{D}(\boldsymbol{x})|^2 .$$

The combination of (3.2), (3.8) and Theorem 1 leads us to (3.3), thus proving the theorem.

This theorem furnishes a sufficient condition for $D_n(x)$ to be consistent. In Theorem 3, if we assume furthermore that $K(y)$ satisfies (2.7), then we have that both $\mathrm{Var}[D_n(x)]$ and $E|D_n(x)-D(x)|^2$ converge to zero at all points x at which both probability density funcions $f_1(x)$ and $f_2(x)$ are continuous.

Theorem 4. *We suppose that the probability density functions $f_1(x)$ and $f_2(x)$ are continuous and that for the sequence of monotone decreasing positive numbers $\{h_n\}$* satisfying (2.1) *there exists a limit with*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \frac{h_n^m}{h_j^m} = a \qquad (0 \leqq a \leqq 1). \tag{3.9}$$

Let the measurable function $K(y)$ satisfy (2.2) *and* (2.4) *and let $\{X_i^1\}$, $\{X_i^2\}$, $\{\rho_i\}$ be mutually independent sequences of random variables and vectors as described in section* 1. *Then for $D_n(x)$ defined by* (2.5) *we have*

$$\lim_{n\to\infty} n\, h_n^m \,\mathrm{Var}\,[D_n(x)] = a\,\{q_1 f_1(x) + q_2 f_2(x)\}\int_{Em} |K(y)|^2\, dy\,. \tag{3.10}$$

Proof. It follows from (3.7) that

$$n\, h_n^m \,\mathrm{Var}\,[D_n(x)] = q_1 \cdot n\, h_n^m \,\mathrm{Var}\,[\hat{f}_1(x)] + q_2\, n\, h_n^m \,\mathrm{Var}\,[\hat{f}_2(x)] + 0\,(h_n^m)\,. \tag{3.11}$$

Hence by applying Theorem 4 in Yamato[7] on (3.11) we have (3.10). Thus the theorem is proved.

4. Uniform consistency

Theorem 5. *We suppose that the probability density functions $f_1(x)$ and $f_2(x)$ are uniformly continuous and that a sequence of monotone decreasing postive numbers $\{h_n\}$ satisfy* (2.1) *and*

$$\lim_{n\to\infty} n^{1/2} h_n^m = \infty\,. \tag{4.1}$$

Let the measurable function $K(y)$ satisfy (2.3) *and* (2.4), *its Fourier transform*

$$k(u) = \int_{Em} e^{iu'y} K(y)\, dy \tag{4.2}$$

be absolutely integrable and $k(u)$ be nondecreasing in negative part and nonincreasing in positive part for each argument.

Let $\{X_i^1\}$, $\{X_i^2\}$, $\{\rho_i\}$ be mutually independent sequences of random variables and vectors as described in section 1. *Then for $D_n(x)$ defined by* (2.5) *we have*

(4.3) $$\sup_x |D_n(x) - D(x)| \xrightarrow{P} 0$$

where (4.2) *denotes that* $\sup|D_n(x)-D(x)|$ *converges to zero in probability as n tends to* ∞.

Proof. In terms of $k(u)$, the Fourier transform of $K(y)$, we have

(4.4) $$D_n(x) - ED_n(x)$$

$$= \frac{1}{(2\pi)^m}\int_{Em}\left\{\frac{1}{n}\sum_{j=1}^{n}[\rho_j e^{iu'X_j^1} - q_1\varphi_1(u)]\,k(h_j u)\right\}e^{-iu'x}\,du$$

$$- \frac{1}{(2\pi)^m}\int_{Em}\left\{\frac{1}{n}\sum_{j=1}^{n}[(1-\rho_j)\,e^{iu'X_j^2} - q_2\varphi_2(u)]\,k(h_j u)\right\}e^{-iu'x}\,du$$

where $\varphi_1(u)$ and $\varphi_2(u)$ are the characteristic functions of $f_1(x)$ and $f_2(x)$ respectively. Therefore we have

(4.5) $$\sup_x|D_n(x) - ED_n(x)|$$

$$\leqq \frac{1}{(2\pi)^m}\int_{Em}\left|\frac{1}{n}\sum_{j=1}^{n}[\rho_j e^{iu'X_j^1} - q_1\varphi_1(u)]\,k(h_j u)\right|du$$

$$+ \frac{1}{(2\pi)^m}\int_{Em}\left|\frac{1}{n}\sum_{j=1}^{n}[(1-\rho_j)\,e^{iu'X_j^2} - q_2\varphi_2(u)]\,k(h_j u)\right|du\,.$$

By applying the Schwartz's inequality on (4.5)

(4.6) $$E\sup_x|D_n(x) - ED_n(x)|$$

$$\leqq \frac{1}{(2\pi)^m}\int_{Em}\left\{\frac{1}{n^2}\sum_{j=1}^{n}E|\rho_j e^{iu'X_j^1} - q_1\varphi_1(u)|^2\cdot|k(h_j u)|^2\right\}^{1/2}du$$

$$+ \frac{1}{(2\pi)^m}\int_{Em}\left\{\frac{1}{n^2}\sum_{j=1}^{n}E|(1-\rho_j)\,e^{iu'X_j^2} - q_2\varphi_2(u)|^2\cdot|k(h_j u)|^2\right\}^{1/2}du\,.$$

Since
$$E|\rho_j e^{iu'X_j^1} - q_1\varphi_1(u)|^2 \leqq 1\,,$$
$$E|(1-\rho_j)e^{iu'X_j^2} - q_2\varphi_2(u)|^2 \leqq 1\,,$$

$\{h_n\}$ is the sequence of monotone decreasing positive numbers and $k(u)$ is nondecreasing in negative part and nonincreasing in positive part for each argument, by (4.5) we have

(4.6) $$E\sup_x|D_n(x) - ED_n(x)|$$

$$\leqq \frac{1}{(2\pi)^m n}\int_{Em}\left\{n|k(h_n u)|^2\right\}^{1/2}du$$

$$+\frac{1}{(2\pi)^m n}\int_{Em}\left\{n|k(h_n \boldsymbol{u})|^2\right\}^{1/2} d\boldsymbol{u}$$

$$=\frac{2}{n^{1/2} h_n^m (2\pi)^m}\int_{Em}|k(\boldsymbol{u})|d\boldsymbol{u}.$$

By applying (4.1) on (4.6), we have

$$\lim_{n\to\infty} E\sup_{\boldsymbol{x}}|\boldsymbol{D}_n(\boldsymbol{x})-E\boldsymbol{D}_n(\boldsymbol{x})|=0. \tag{4.7}$$

It follows from (4.7) and Markov's inequality that

$$\sup_{\boldsymbol{x}}|\boldsymbol{D}_n(\boldsymbol{x})-E\boldsymbol{D}_n(\boldsymbol{x})|\xrightarrow{P} 0. \tag{4.8}$$

Finally we remark the inequality

$$\sup_{\boldsymbol{x}}|\boldsymbol{D}_n(\boldsymbol{x})-\boldsymbol{D}(\boldsymbol{x})| \tag{4.9}$$

$$\leqq\sup_{\boldsymbol{x}}|\boldsymbol{D}_n(\boldsymbol{x})-E\boldsymbol{D}_n(\boldsymbol{x})|+\sup_{\boldsymbol{x}}|E\boldsymbol{D}_n(\boldsymbol{x})-\boldsymbol{D}(\boldsymbol{x})|.$$

By applying Corollay 1 and (4.8) on (4.9), we have (4.3). Thus the theorem is proved.

5. Asymptotic normality

Theorem 6. *We suppose that the probability density functions* $f_1(\boldsymbol{x})$ *and* $f_2(\boldsymbol{x})$ *are continuous and that for the sequence of monotone decreasing positive numbers* $\{h_n\}$ *satisfying* (2.1) *and* (3.1) *there exists a non zero limit with* (3.9). *Let the measurable function* $K(\boldsymbol{y})$ *satisfy* (2.2) *and* (2.4) *and let* $\{\boldsymbol{X}_i^1\}$, $\{\boldsymbol{X}_i^2\}$, $\{\rho_i\}$ *be mutually independent sequences of random variables and vectors as described in section* 1. *Then for* $\boldsymbol{D}_n(\boldsymbol{x})$ *defined by* (2.5) *the distribution function of*

$$\frac{\boldsymbol{D}_n(\boldsymbol{x})-E\boldsymbol{D}_n(\boldsymbol{x})}{\sqrt{\mathrm{Var}[\boldsymbol{D}_n(\boldsymbol{x})]}} \tag{5.1}$$

converges to the standardized normal distribution function at all points $\boldsymbol{x}$.

$h_n=1/n^{r/m}$ $(0<r<1/2)$ and $h_n=1/(\log n)^{1/m}$ are examples of sequences of monotone decreasing positive numbers satisfying (2.1), (3.1), (4.1) and (3.9) with $\alpha=1/(r+1)$ and $\alpha=1$ respectively.

Proof. If we put for any fixed $\boldsymbol{x}$

$$V_j^1=\frac{1}{h_j^m}K\left(\frac{\boldsymbol{x}-\boldsymbol{X}_j^1}{h_j}\right)\quad (j=1,2,3,\cdots)$$

(5.2)

$$V_j^2=\frac{1}{h_j^m}K\left(\frac{\boldsymbol{x}-\boldsymbol{X}_j^2}{h_j}\right)\quad (j=1,2,3,\cdots)$$

then $\{\rho_j V_j^1-(1-\rho_j)V_j^2\}$ ($j=1, 2, 3, \dots$) is a sequence of independent random variables and we have

(5.3)
$$\frac{D_n(x)-ED_n(x)}{\sqrt{\mathrm{Var}[D_n(x)]}} = \frac{\sum_{j=1}^{n}\{\rho_j V_j^1-(1-\rho_j)V_j^2-q_1 EV_j^1+q_2 EV_j^2\}}{\sqrt{\mathrm{Var}\left[\sum_{j=1}^{n}\{\rho_j V_j^1-(1-\rho_j)V_j^2\}\right]}}$$

Therefore by virtue of Lyapunov's condition it is enough to show that

(5.4)
$$\lim_{n\to\infty}\frac{\sum_{j=1}^{n} E|\rho_j V_j^1-(1-\rho_j)V_j^2-q_1 EV_j^1+q_2 EV_j^2|^3}{\left(\mathrm{Var}\left[\sum_{j=1}^{n}\{\rho_j V_j^1-(1-\rho_j)V_j^2\}\right]\right)^{3/2}}=0.$$

From Theorem 4 we have

(5.5)
$$\begin{aligned}&\frac{h_n^m}{n}\mathrm{Var}\left[\sum_{j=n}^{n}\{\rho_j V_j^1-(1-\rho_j)V_j^2\}\right]\\ &= n h_n^m \mathrm{Var}[D_n(x)]\\ &\to \alpha\{q_1 f_1(x)+q_2 f_2(x)\}\int_{Em} K^2(y)dy\ (n\to\infty).\end{aligned}$$

On the other hand, by an inequality

(5.6)
$$(a+b+c+d)^3 \leqq 16(a^3+b^3+c^3+d^3) \text{ for } a, b, c, d \geqq 0$$

we have

(5.7)
$$\begin{aligned}&\sum_{j=1}^{n} E|\rho_j V_j^1-(1-\rho_j)V_j^2-q_1 EV_j^1+q_2 EV_j^2|^3\\ &\leqq \sum_{j=1}^{n} E\{|\rho_j(V_j^1-EV_j^1)|+|(\rho_j-q_1)EV_j^1|\\ &\quad+|(1-\rho_j)(V_j^2-EV_j^2)|+|(1-\rho_j-q_2)EV_j^2|\}^3\\ &\leqq 16\Big\{q_1\sum_{j=1}^{n} E|V_j^1-EV_j^1|^3+q_2\sum_{j=1}^{n} E|V_j^2-EV_j^2|^3\\ &\quad+q_1 q_2(q_1^2+q_2^2)\Big(\sum_{j=1}^{n} E|V_j^1|^3+\sum_{j=1}^{n} E|V_j^2|^3\Big)\Big\}.\end{aligned}$$

By (5.6) and (5.9) in Yamato[7], it turns out that the right hand side of (5.7) is smaller than

$$64\{q_1\|f_1\|+q_2\|f_2\|\}\frac{n}{h_n^{2m}}\int_{Em}|K(z)|^3dz$$

$$+64n\{q_1\|f_1\|^3+q_2\|f_2\|^3\}\left\{\int_{Em}|K(z)|dz\right\}^3$$

$$+16q_1q_2(q_1^2+q_2^2)\cdot n\cdot\{\|f_1\|^3+\|f_2\|^3\}\left\{\int_{Em}|K(z)|dz\right\}^3 .$$

Hence we have

$$\lim_{n\to\infty}\frac{\sum_{j=1}^{n}E|\rho_jV_j^2-(1-\rho_j)V_j^2-q_1EV_j^1+q_2EV_j^2|^3}{\left(\mathrm{Var}\left[\sum_{j=1}^{n}\left\{\rho_jV_j^1-(1-\rho_j)V_j^1\right\}\right]\right)^{3/2}} \tag{5.9}$$

$$\leqq\lim_{n\to\infty}\frac{1}{\left(\frac{h_n^m}{n}\mathrm{Var}\left[\sum_{j=1}^{n}\{\rho_jV_j^1-(1-\rho_j)V_j^2\}\right]\right)^{3/2}}$$

$$\times\Big(64\{q_1\|f_1\|+q_2\|f_2\|\}\frac{1}{(nh_n^m)^{1/2}}\int_{Em}|K(z)|^3dz$$

$$+64\frac{h_n^{3m/2}}{n^{1/2}}\{q_1\|f_1\|^3+q_2\|f_2\|^3\}\left\{\int_{Em}|K(z)|dz\right\}^3$$

$$+16q_1q_2(q_1^2+q_2^2)\frac{h_n^{3m/2}}{n^{1/2}}\{\|f_1\|^3+\|f_2\|^3\}\left\{\int_{Em}|K(z)|dz\right\}^3\Big).$$

By applying (2.1), (2.2), (2.4), (3.1) and (5.5) on (5.9) we have (5.4), which leads us to the completion of the theorem.

Thus we have obtained the asymptotic normality of $\boldsymbol{D}_n(\boldsymbol{x})$. We considered its property under the assumption that for $\{h_n\}$ there exists a non zero limit with (3.9) and the auther wishes to develop the asymptotic normality of $\boldsymbol{D}_n(\boldsymbol{x})$ without this assumption on another occasion.

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Errata

Page	Line	Wrong	Corrected
13	↑9	pifferentiable	differentiable