ASYMPTOTIC BEHAVIOR OF THE SOLUTIONS OF SOME THIRD ORDER DIFFERENTIAL EQUATIONS

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ASYMPTOTIC BEHAVIOR OF THE SOLUTIONS OF SOME THIRD ORDER DIFFERENTIAL EQUATIONS

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1. Introduction.

We consider here the differetial equations

$$\dddot{x}+a\ddot{x}+g(x)\dot{x}+h(x)=e(t,x,\dot{x},\ddot{x})\left(\dot{x}=\frac{dx}{dt}\right) \tag{1.1}$$

$$\dddot{x}+p(t)\ddot{x}+q(t)g(\dot{x})+h(x)=e(t,x,\dot{x},\ddot{x}), \tag{1.2}$$

where a is a positive constant and e,g,h′,p′, and q′ are continuous and real valued functions for all x and t.

In [3] SWIK considered the behavior as $t\to\infty$ of solutions of the differential equations

$$\dddot{x}+a\ddot{x}+g(x)\dot{x}+h(x)=e(t) \tag{1.3}$$

$$\dddot{x}+p(t)\ddot{x}+q(t)g(\dot{x})+h(x)=e(t), \tag{1.4}$$

and he has shown that every solution of (1, 3) and (1.4) satisfies

$$x(t)\to 0,\ \dot{x}(t)\to 0,\ \ddot{x}(t)\to 0 \tag{1.5}$$

as t→∞ under some conditions and here in order to obtain the ultimate bondedness for solutions of (1.3) and (1.4),
he required that the conditions

$$\int_0^t |e(s)|ds\leqq E_0 \quad \text{for all } t. \tag{1.6}$$

It will be shown here that the condition (1.6) is replaced by the condition

$$|e(t,x,y,z)|\leqq\bar{e}(t) \quad \text{for all t, and all } (x,y,z)\in R^3, \text{ and}$$

$$\int_0^t \bar{e}(s)ds\leqq\infty \qquad \textit{for all } t,$$

under which *every* solution of (1.1) and (1.2) satisfies (1.5) under the same conditions as in K,E, SWICK[3], [4]. The results of T. HARA [1], [2] and M. YAMAMOTO [5], [6] are interesting for us.

I wish to express my hearty thanks to M, TECH, T, HARA and DR, M, YAMAMOTO for their

invaluable suggestions and attentions.

2. *Theorems.*

The following result is well known, see [7].

Lemma **2.1.**

Consider the system $\dot{X}=F(t,X)+G(t,X)$, *where* $\int_0^t \|G(s,X)\|ds$ *is bounded for all* t *whenever* X *belongs to any compact subset of* R^n. *Suppose that there exists a nonnegative Lyapunov function* $W(t,X)$ *on* $I\times Q$, $Q\subset R^n$, *such that with respect to this system,* $\dot{W}(t,X)\leqq -V(X)$, *where* $V(x)$ *is positive definite with respect to a closed set* Ω *in the space* Q, *Moreover suppose* $F(t,X)$ *is bounded and*

(a) $F(t,X)\to H(X)$ *for* $X \varepsilon \Omega$ *as* $t\to\infty$,

$F(t,X)\to H(X)$ *(uniformly) for* $X \varepsilon \tilde{\Omega}$ *as* $to\to\infty$,

where $\tilde{\Omega}$ *is any compact set in* Ω,

(*b*) *for each* $\varepsilon>0$ *and* $Y\in\Omega$, *there exist* $\delta(\varepsilon, Y)$ *and* $T(\varepsilon, Y)$ *such that if* $\|X-Y\|<\delta(\varepsilon, Y)$ $\to\|F(t,X)-F(t,Y)\|<\varepsilon$ *for* $t\geqq T(\varepsilon,Y)$.

Then every bounded solution of $\dot{X}=F(t)+G(t,X)$ *approaches the largest semi-invariant set of the system* $\dot{X}=H(X)$ *contained in* Ω *as* $t\to\infty$.

Theorem 1.

Assume that there exist positive constants b, c *and* E_0, *and a positive function* $e(t)$ *which satisfy the following conditions,*

(i) $G(x)/x\geqq b$ $(x\neq 0)$

where $G(X)=\int_0^x g(u)du$,

(ii) $h'(x)\leqq c$(*for all* x) *and* $ab>c$,

(iii) $h(x)sgn\, x>0$ $(x\neq 0)$,

(iv) $|e(t,x,y,z)|\leqq\bar{e}(t)$, $(\forall(x,y,z)\varepsilon R^3)$

and $E(t)=\int_0^t \bar{e}(s)ds<+\infty$ *for all* t.

Then every solution of (1.1) *satisfies* (1.5) *as* $t\to\infty$.

THEOREM 2.

If there exist positive constants δ_0, δ_1, a, b, c, K, L *and* E_0, *and positive function* e (t) *which satisfy the conditions;*

(i) $h(x)/x\geqq\delta_0$ $(|x|\geqq K)$,

(ii) $h'(x)\leqq c$(*for all* x,),

(iii) $g(y)/y\geqq b(y\neq 0)$,

(iv) $1\leqq\delta\leqq q(t)$ *and* $q'(t)\geqq 0$, $(t\geqq 0)$,

(v) $a\leqq p(t)\leqq L$ $(t\geqq 0)$,

(vi) $|e(t,x,y,z)|\leqq\bar{e}(t)$, (*for any* $(x,y,z,)\varepsilon R^3$)

and $E(t)=\int_0^t \bar{e}(s)ds\leqq E_0$ (*for* $t\geqq 0$),

(vii) $h(x)\, sgn\, x>0$ $(x\neq 0)$,

and that there exist α *and* δ_3 *satisfying*
$b/c>\alpha>1/a$ *and* $(1/2)p'(t)\leqq\varepsilon_3\leqq(\delta_1 b\text{-}\alpha c)$ *for* $t\geqq 0$.

Then every solution of (1.2) *satisfies* (1.5).

3. Proof of Theorem 1.

The equation (1.1) is equivalent to the system

$$
(3.1)\qquad \begin{cases} \dot{x}=y \\ \dot{y}=z-ay-G(x) \\ \dot{z}=e(t,x,y,z-ay-G(x))-h(x)\,. \end{cases}
$$

Let β be a constant such that $b>\beta>c/a$, and we define the Lyapunov function $W(t, x, y, z)$ as

$$
2\,W(t,x,y,z)=e^{-2E(t)}[V(x,y,z)+k]\,,
$$

where $V(x,y,z)=2a\int_0^x h(u)du+2\beta\int_0^x G(u)\,du+\beta y^2+z^2+2h(x)y-2\beta xz$, and k is a positive constant to be determined later in the proof.

Lemma 3.1.

There exist continuous functions $\alpha(r)$, $\beta(r)$ *which satisfy the following conditions*:

(i) $\alpha(\|X\|)\leqq W(t,X)\leqq\beta(\|X\|)$ (*for all* X *and* $t\geqq 0$)

where $X=(x,y,z)$ *and* $\|X\|=\sqrt{x^2+y^2+z^2}$,

(ii) $\alpha(r)\geqq 0$ *for* $r\geqq 0$ *and* $\alpha(r)\to\infty$ *as* $r\to\infty$.

Proof of Lemma 3.1.

From the condition (i) of Theorem, 1, we have

$$
2\int_0^x G(u)du\geqq bx^2
$$

and thus

$$
\begin{aligned}
V(x,y,z)&\geqq 2a\int_0^x h(u)du+b\beta x^2+\beta y^2+z^2+2h(x)y-2\beta xz \\
&=b\beta x^2-2\beta xz+z^2+\beta\Big(y+\frac{h(x)}{\beta}\Big)^2+\frac{2}{\beta}\int_0^x[a\beta-h'(u)]h(u)du\,.
\end{aligned}
$$

The first three terms can be written as

$$
\beta\Big(bx^2-2xz+\frac{1}{\beta}z^2\Big)=\beta(x,z)A\binom{x}{z}
$$

where $\quad A=\begin{pmatrix}\sqrt{b} & -1 \\ -1 & 1/\sqrt{\beta}\end{pmatrix}$

and since $b>\beta$, the eigenvalues λ, λ_2 of A are both positive real constants.
If we define the constant d as

$$d=\min(\lambda_1,\lambda_2),$$

then

$$b\beta x^2-2\beta xz+z^2\geqq d\beta(x^2+z^2),$$

so that

$$V(x,y,z)\geqq d\beta(x^2+z^2)+\mu y^2+(\beta-\mu)\Big(y+\frac{h(x)}{\beta-\mu}\Big)^2+2a\int_0^x\Big\{a-\frac{h'(\mu)}{\beta-\mu}\Big\}h(u)du,$$

where μ is a constant satisfying

$$0<\mu\leqq\beta-\frac{c}{a}.$$

Then we have

$$V(x,y,z,\geqq d\beta(x^2+z^2)+\mu\, y^2.$$

Setting $l=\min(d\beta,\mu)$, we have

$$V(x,y,z)\geqq l(x^2+y^2+z^2)\equiv\alpha(\|X\|)\quad\text{where } \|X\|=\sqrt{x^2+y^2+z^2}.$$

Now if we define the function $h^*(X)=\max\limits_{-|X|\leqq\xi\leqq|X|}|h(\xi)|$ on $(-\infty,+\infty)$, then

$$h^*(0)=0,\ h(x)\leqq h^*(x),\ \text{and } h^*(x)\leqq h^*(y)\ \text{for } 0\leqq x\leqq y.$$

Likewise let we define $g^*(X)=\max\limits_{-|X|\leqq\xi\leqq|X|}|g(\xi)|$ on $(-\infty,+\infty)$, then

$$g^*(0)=0,\ g(x)\leqq g^*(x),\ \text{and } g^*(x)\leqq g^*(y)\ \text{for } 0\leqq x\leqq y.$$

Thus we can define the continuous non-decreasing functions

$$H^*(r)=\int_0^r h^*(s)ds,$$

$$G^*(r)=\int_0^r g^*(s)\,ds\ \text{for } r\in[0,+\infty).$$

Now setting $X_0=(x_0,\ y,\ z)$ for $(x_0,\ y,\ z)\in R^3$, we have

$$|x_0|\leqq\sqrt{x+y^2+z^2}=\|X_0\|$$

and

$$V(x,y,z)\leqq 2a\int_0^x|h(u)|du+2\beta\int_0^x|G(u)|du+\beta y^2+z^2+2|h(x)|\,|y|+2\beta|x|\,|z|$$

$$\leqq 2aH^*(|x|)+2\beta G^*(|x|)+\beta y^2+z^2+2h^*(|x|)|y|+2\beta|x|\,|z|$$

$$\leqq 2aH^*(\|X\|)+2bG^*(\|X\|)+(b+1)(\|X\|^2)+2h^*(\|X\|)\|X\|$$

$$\equiv\beta(\|X\|).$$

Thus the proof of Lemma 3.1 is completed.

LEMMA 3.2

Let $(x\ (t),\ y\ (t),\ z\ (t)\ )$ *be any solution of* (3.1), *then along this solution*

$$\dot{W}_{(3.1)}(t,x,y,z)\leqq -2e^{-2E_0}(a\beta-c)y^2 \quad for\ t\geqq 0\,.$$

PROOF OF LEMMA 3.2.

$$\dot{W}_{(3.1)}(t,x,y,z)=-2\bar{e}(t)e^{-2E(t)}[V(x,y,z)+k]+e^{-2E(t)}\{2ah(x)\dot{x}$$
$$+2\beta G(x)\dot{x}+2\beta y\dot{y}+2z\dot{z}+2h'(x)\dot{x}y+2h(x)\dot{y}-2\beta\dot{x}z$$
$$-2\beta x\dot{z}\}\,,$$

and from the assumptions of Theorem 1

$$2ah(x)\dot{x}+2G(x)\dot{x}+2\beta y\dot{y}+2z\dot{z}+2h'(x)\dot{x}y+2h(x)\dot{y}-2\beta\dot{x}z-2\beta x\dot{z}$$
$$\leqq 2\{-(a\beta-c)y^2-(G(x)-\beta x)\,h(x)+\bar{e}(t)\,|z|+\beta\bar{e}(t)\,|x|\,\}\,.$$

Thus we have

$$\dot{W}_{(3.1)}(t,x,y,z)=-2\bar{e}(t)e^{-2E(t)}\{Vx(t),y(t),z(t))+k-|z|-\beta|x|\}$$
$$-2e^{-2E(t)}\{(a\beta-c)y^2+(G(x)-\beta x)h(X)\}\,,$$

and

$$V(x(t),y(t),z(t))+k-|z|-\beta|x|\geqq d\beta(x^2+z^2)+k-|z|-\beta|x|$$
$$=d\beta\Big(|x|-\frac{1}{2d}\Big)^2+d\beta\Big(|z|-\frac{1}{2d\beta}\Big)^2+k-\Big(\frac{1}{4d\beta}+\frac{\beta}{4d}\Big)\,.$$

Setting the constant $k\geqq\Big(\frac{\beta}{4d}+\frac{1}{4d\beta}\Big)$, we have the following inequality

$$W_{(3.1)}(t,x,y,z)\leqq -2e^{-2E(t)}\{(a\beta-c)y^2+(G(x)-\beta x)h(x)\}$$
$$\leqq -2e^{-2E(t)}(a\beta-c)y^2\equiv -W_1(t,x,y,z)\,.$$

Q.E.D.

The function $W_1(t,x,y,z)$ is positive definite with respect to the closed set Ω in the space R^3, where $\Omega=\{(x,y,z)\varepsilon R^3;\ y=0\}$.
In the system (3.1), we set

$$(3.2)\qquad F(t,X)=\begin{pmatrix} y \\ z-ya-G(x) \\ -h(x)\end{pmatrix},\quad G(t,X)=\begin{pmatrix}0\\0\\e(t,x,y,z-ay-G(x))\end{pmatrix},$$

and we take the function

$$(3.3)\qquad H(t,X)=\begin{pmatrix}0\\ z-G(x)\\ -h(x)\end{pmatrix},$$

then the condition (a) and (b) of Lemma 1.1 are satisfied and since $F\ (t,\ X)$ is independent of t, and $h(x)$, $G(x)$ are continuous, it follows that $\|F(t,X)\|$ is bounded for all t on any compact subset of R^3. Moreover from the assumptions of Theorem 1,

$$\int_0^t \|G(s,X)\| \, ds \text{ is bounded for all } t.$$

It follows from Lemma 1.1 that every solution of (3.1) approaches the largest semi-invariant set of $\dot{X}=H(X)$ contained in Ω as $t\to\infty$. From (3.3), $X=H(X)$ is the system

$$\begin{cases} \dot{x}=0 \\ \dot{y}=z-G(x) \\ \dot{z}=-h(x), \end{cases}$$

and therefore

$$\begin{cases} x=c_1 \\ y=-h(c_1)\dfrac{(t-t_0)^2}{2}+c_2(t-t_0)-G(c_1)(t-t_0)+c_3 \\ z=-h(c_1)(t-t_0)+c_2. \end{cases}$$

To remain in Ω,

$$h(c_1)=0, \; G(c_1)=c_2 \text{ and } c_3=0,$$

and we have $c_1=0$, $c_2=0$ and $c_3=0$.
From the assumption (ii) of Theorem 1, it follows that

$$x(t)\to 0, \; \dot{x}(t)\to 0, \; \ddot{x}(t)\to 0, \text{ as } t\to\infty,$$

which completes the proof of Theorem 1.

4. Proof of Theorem 2.

The equation (1.2) is equivalent to the system

$$(4.1) \qquad \begin{cases} \dot{x}=y, \\ \dot{y}=z, \\ \dot{z}=e(t,x,y,z)-p(t)z-q(t)g(y)-h(x), \end{cases}$$

and we define the Lyapunov function

$$W(t,x,y,z)=e^{-E(t)}\{V(t,x,y,z)+\bar{k}\}$$

where

$$V(t,x,y,z)=\int_0^x h(u)du+aq(t)\int_0^y g(u)du+ah(x)y+\frac{1}{2}p(t)y^2+\frac{1}{2}az^2+yz$$

and $\bar{k}$ is a positive constant to be determined later in the proof.

LEMMA 4.1

There exist continuous functions $a(r)$ and $b(r)$ with the properties;

$$a(r) \geqq 0 \quad \textit{for } r \geqq 0\,,$$
$$b(r) \geqq 0 \quad \textit{for } r \geqq 0\,,$$

and

$$a(\|X\|) \leqq W(t,X) \leqq b(\|X\|) \qquad \textit{for } t \geqq 0\,,$$

where $X=(x,y,z) \in R^3$, $\|X\|=\sqrt{x^2+y^2+z^2}$

Proof of Lemma 4.1. If we set

$$H(x)=\int_0^x h(u)du,\ G(y)=\int_0^y g(u)du\,,$$

$$V(t,x,y,z)=[H(x)+q(t)G(y)+h(x)y]+\frac{1}{2}p(t)y^2+\frac{1}{2}az^2+yz\,,$$

the last three terms may be written as

$$\frac{1}{2}p(t)y^2+\frac{1}{2}az^2+yz=\frac{1}{2}(y,z)A\begin{pmatrix} y \\ z \end{pmatrix}$$

were $A=\begin{pmatrix} p(t) & 1 \\ 1 & a \end{pmatrix}$.

And from the assumptions of Theorem 2, both of the eigenvalues of this matrix are positive real and greater than the constant $\dfrac{2a-1}{2(L+a+1)}$,
therefore

$$\frac{1}{2}p(t)y^2+\frac{1}{2}az^2+yz \geqq \frac{aa-1}{2(L+a+1)}(y^2+z^2)\,.$$

For the remainder terms, there exists a positive constant M such that

$$H(x)+aq(t)G(y)+ah(x)y \geqq H(x)+\frac{1}{2}abq(t)y^2+ah(x)y$$

$$\geqq H(x)+\frac{1}{2}ab\delta_1 y^2+ah(x)y$$

$$=\frac{1}{2}\ \frac{a}{b\delta_1}(b\delta_1 y+h(x))^2+\int_0^x \left[1-\frac{ah'(x)}{b\delta_1}\right]h(u)du$$

$$\geqq \frac{1}{2}\ \delta_4 x^2-M$$

for all $(x,y,z) \in R^3$ and $t \geqq 0$,
where

$$\delta_4=1-\frac{ac}{b\delta_1}\,,$$

hence

$$V(t,x,y,z) \geqq \delta(x^2+y^2+z^2)-M$$

where $\delta=\min\left[\frac{aa\quad -1}{2(L+a+1)}, \frac{1}{2}\delta_4\right]$.

The remainder of the proof is likewise as the last half of the proof of Lemma 3.1.

LEMMA 4.2

Under the hyphothesis of Theorem 2, *there exists a positive constant* $\tilde{\delta}$ *such that along any solution* $(x(t), y(t), z(t))$ *of* (4.1),

$$\dot{V}_{(4.1)}(t,x,y,z) \leqq -\tilde{\delta}(y^2+z^2)+ae(t,x,y,z)z+e(t,x,y,z)y$$

PROOF OF LEMMA 4.2.

$$\dot{V}_{(4.1)}(t,x,y,z)$$
$$=\{1-p(t)\}z^2-\{q(t)g(y)y-a\ h'(x)y^2\}+a\ q'(t)\int_0^y g(u)du+\frac{1}{2}p'(t)y^2+aze(t,x,y,z)$$
$$+ye(t,x,y,z)\ .$$

From the assumptions of Theorem 2,

$$q(t)g(y)y-ah'(x)y^2 \geqq q(t)by^2-ah'(x)y^2$$
$$\geqq q(t)by^2-acy^2$$
$$\geqq(\delta b- ac)y^2\ ,$$

$$\frac{1}{2}p'(t)y^2-\{q(t)g(y)y-h'(x)y^2\} \leqq \frac{1}{2}p'(t)y^2-(\delta_1 b- ac)y^2$$

$$\leqq \delta_5 y^2$$

where $\delta_5=\frac{1}{2}\delta_3-(\delta_1 b-ac)>0$,

therefor

$$V_{(4.1)}(t,x,y,z) \leqq \delta_6 z^2+\delta_5 y^2+aq'(t)\int_0^y g(u)du+aze(t,x,y,z)+ye(t,x,y,z)$$
$$\leqq -\tilde{\delta}(y^2+z^2)+ye(t,x,y,z)+aze(t,x,y,z)$$

where $\delta_6=1-a\mathrm{a}+1-ap(t)$ and $\delta=-\max(\delta_5,\delta_6)$. Thus the proof is completed.

LEMMA 4.3

Under the assumptions of Theorem 2, there exists a positive constant $\hat{\delta}$ *such that along the solution of* (4.1)

$$\dot{W}_{(4.1)}(t,x,y,z) \leqq -\hat{\delta}(y^2+z^2)\ .$$

PROOF OF LEMMA 4.3.

$$\begin{aligned}\dot{W}_{(4.1)}(t,x,y,z)&=-\tilde{e}(t)e^{-E(t)}\{V(t,x,y,z)+\tilde{k}\}+e^{-E(t)}\dot{V}_{(4.1)}(t,x,y,z)\\&\leqq-\tilde{e}(t)e^{-E(t)}\{\delta(x^2+y^2+z^2)-M+\tilde{k}\}+e^{-E(t)}\{-\tilde{\delta}(y^2+z^2)\\&\quad+aze(t,x,y,z)+ye(t,x,y,z)\}\\&\leqq-\tilde{e}(t)e^{-E(t)}\{\delta(x^2+y^2+z^2)-M+\tilde{k}\}+e^{-E(t)}\{-\tilde{\delta}(y^2+z^2)+|a||z|\tilde{e}(t)\\&\quad+|y|\tilde{e}(t)\}\\&\leqq-\tilde{e}(t)e^{-E(t)}\{\delta(x^2+y^2+z^2)-|a||z|-|y|-M+\tilde{k}\}-\tilde{\delta}e^{-E(t)}(y^2+z^2)\}\,.\end{aligned}$$

And

$$\begin{aligned}&\delta(x^2+y^2+z^2)-|a||z|-|y|-M+\tilde{k}\\&=\delta x^2+\delta\Big(|y|-\frac{1}{2\delta}\Big)^2+\delta\Big(|z|-\frac{|a|}{2\delta}\Big)^2-\frac{1}{4\delta}-\frac{|a|^2}{4\delta}-M+\tilde{k}\,.\end{aligned}$$

Take the constant $\tilde{k}$ as follows:

$$\tilde{k}\geqq M+\frac{1}{4\delta}(1+|a|^2)\,,$$

we have

$$\begin{aligned}\dot{W}_{(4.1)}(t,x,y,z)&\leqq-\tilde{\delta}e^{E(t)}(y^2+z^2)\\&\leqq-\tilde{\delta}e^{-E_0}(y^2+z^2)\\&\equiv-\hat{\delta}(y^2+z^2)\qquad(\hat{\delta}=\tilde{\delta}e^{-E_0})\,.\end{aligned}$$

The remainder of the proof of Theorem 2 can be easily proved in a similar fashion as of Theorem 1.

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