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§ 1. Introduction

B. Eckmann and P.J. Hilton [1] gived the definition of homotopy groups with coefficients in a given abelian group. On the other hand Y. Katuta [2] defined a new kind of homotopy groups with an arbitrary finitely generated abelian group as the coefficient group. The latter definition was then shown to satisfy the Hurewicz isomorphism theorem, the universal coefficient theorem and the others. In this note, we shall define the relative homotopy group with coefficient group G, where G is a finitely generated abelian group. And our definition will be shown to satisfy some properties.

§ 2. Preliminaries

We shall fix some notations:

CX	the reduced cone over X
$\Sigma^i X$	the i-fold reduced suspension of X
$X \vee Y$	the one point union of X and Y
C_f	the reduced mapping cone
Cf	the reduced cone of a map f
$\Sigma^i f$	the i-fold suspension of a map f
$f_m: S^1 \to S^1$	for an integer to denote a map of degree m
$F(X;\ Y), F(X,A;\ Y,B)$	the funtion space with compact open topology
$[X;\ Y],[X,A;\ Y,B]$	the set of homotopy classes
$\Omega(X,A)$	the function space $F\ (I,\ \mathbf{0};\ X,\ A)$, where the base point of I is 1.

Define a map (by a map we mean a base point preserving continuous function)

$$\bar{\tau}: F(CZ,Z;\ X,A) \to F(Z;\ \Omega(X,A))$$

by taking $[(\bar{\tau} f)\ (z)]\ (t) = f\ (z,\ t)$ where $f \in F\ (CZ,\ Z;\ X,\ A)$ *and* $t \in I$.
Then a function

$$\tau: [CZ,Z;\ X,A] \to [Z;\ \Omega(X,A)]$$

is obtained as follows, $\tau[f] = [\bar{\tau}\ (f)]$ where $[f] \in [CZ,\ Z;\ X,\ A]$. We have the following facts. (We list up 2.1 ~ 2.4. without proof)

LEMMA 2.1. *τ is a bijection and* $\tau[0]=[0]$.

LEMMA 2.2. *There are commutative diagrams*

$$\text{(i)}\quad \begin{array}{ccc} [CZ',Z';X,A] & \xrightarrow{\tau} & [Z';\Omega(X,A)] \\ \downarrow{\scriptstyle (C\psi)^*} & & \downarrow{\scriptstyle (\psi)^*} \\ [CZ,Z;X,A] & \xrightarrow{\tau} & [Z,\Omega(X,A)] \end{array} \qquad \text{(ii)}\quad \begin{array}{ccc} [CZ,Z;X,A] & \xrightarrow{\tau} & [Z,)X,A)] \\ \downarrow{\scriptstyle (\varphi)_*} & & \downarrow{\scriptstyle (\Omega\varphi)_*} \\ [CZ,Z;X',A'] & \xrightarrow{\tau} & [Z,\Omega(X',A')] \end{array},$$

where $\psi:Z\to Z'$ *and* φ: $(X, A)\to(X',A')$.

LEMMA 2.3. *Let Z be a space, then*

$$\cdots\xrightarrow{\partial}[\Sigma Z;A]\xrightarrow{i_*}[\Sigma Z;X]\xrightarrow{j_*}[C\Sigma Z,\Sigma Z;X,A]\xrightarrow{\partial}[\Sigma Z;A]\xrightarrow{i_*}$$

$$[\Sigma Z;X]\xrightarrow{j_*}[CZ,Z;X,A]\xrightarrow{\partial}[Z;A]\xrightarrow{i_*}[Z;X]$$

is exact for any pair (X, A), *where* i: $A\subset X$ *and* j: $(X, x_0)\subset(X, A)$.

The set $[\Sigma Z; \Omega(X, A)]$ has a group structure. Therefore a group structure can bc defined on the set $[C\Sigma Z, \Sigma Z, X, A]$ by Lemma 2.1.

LEMMA. 2.4. *We have the following bijection*

$$[C\Sigma Z_1\vee C\Sigma Z_2, \Sigma Z_1\vee\Sigma Z_2: X, A]\approx[C\Sigma Z_1, \Sigma Z_1; X, A]\times[C\Sigma Z_2, \Sigma Z_2; X, A].$$

$[C\Sigma Z_1, \Sigma Z_1; X, A]$ and $[C\Sigma Z_2, \Sigma Z_2; X, A]$ are groups, therefore a group structure can be defined on $[C\Sigma Z_1\vee C\Sigma Z_2,\Sigma Z_1\vee\Sigma Z_2; X,A]$ by Lemma 2.4. Therefore we can replace in Lemma 2.4 bijection by isomorphism and cartesian product by direct product of groups. (If $[C\Sigma Z_1, \Sigma Z_1; X, A]$ and $[C\Sigma Z_2, \Sigma Z_2; X, A]$ are baelian, direct product coincides with direct sum.)

LEMMA. 2.5. *Let* $Z_1\xrightarrow{f}Z_2\xrightarrow{g}Z_3$, *and* $[Z_3;X]\xrightarrow{g^*}[Z_2;X]\xrightarrow{f^*}[Z_1;X]$ *be exact for any* X, *then*

$$[CZ_3, Z_3; X, A]\xrightarrow{(Cg)^*}[CZ_2, Z_2; X,A]\xrightarrow{(Cf)^*}[CZ_1,Z_1; X,A]$$

is also exact for any (X, A).

PROOF. Concider the following commutative diagram

$$\begin{array}{ccccc} [Z_3;\Omega(X,A)] & \xrightarrow{g^*} & [Z_2;\Omega(X,A)] & \xrightarrow{f^*} & [Z_1;\Omega(X,A)] \\ \wr\,\tau & & \wr\,\tau & & \wr\,\tau \\ [CZ_3,Z_3;X,A] & \xrightarrow{(Cg)^*} & [CZ_2,Z_2;X,A] & \xrightarrow{(Cf)^*} & [CZ_1,Z_1;X,A], \end{array}$$

the upper row is exact and $\tau[0]=[0]$, and so the lower is exact.

THEOREM. 2.6. *Let* $f:Y\to Z$, *then*

$$\cdots\xrightarrow{(\Sigma^2 f)^*}[\Sigma^2 Y;X]\xrightarrow{(\Sigma Qf)^*}[\Sigma C_f;X]\xrightarrow{(\Sigma Pf)^*}[\Sigma Z;X]\xrightarrow{(\Sigma f)^*}[\Sigma Y;X]\xrightarrow{(Qf)^*}$$

$$[C_f;X]\xrightarrow{(Pf)^*}[Z;X]\xrightarrow{f^*}[Y;X]$$

is exact for any X. Moreover, to the right of and including $[\Sigma Y;X]$, *the sets are groups and the functions are homomorphic, where* (Pf; *is the inclusion map and* Qf *the collapsing.*)

$$\text{(ii)}\cdots\longrightarrow[C\Sigma C_f,\Sigma C_f;X,A]\xrightarrow{(C\Sigma Pf)^*}[C\Sigma Z,\Sigma Z;X,A]\xrightarrow{(C\Sigma f)^*}[C\Sigma Y,\Sigma Y;X,A]\xrightarrow{(CQf)^*}$$
$$[CC_f,C_f;X,A]\xrightarrow{(CPf)^*}[CZ,Z;X,A]\xrightarrow{(Cf)^*}[CY,Y;X,A]$$

is exact for any (X, A), *moreover to the right of and including* $[C\Sigma Y,\Sigma Y;X,A]$, *the sets are groups and the functions are homomorphic.*

Proof. (i) It is well known as the Puppe sequence (cf. [4]).

(ii) Concider the commutative diagram

$$\begin{array}{ccccccc}\cdots\longrightarrow[\Sigma Z;\Omega(X,A)] & \xrightarrow{(\Sigma f)^*} & [\Sigma Y;\Omega(X,A)] & \xrightarrow{(Qf)^*} & [C_f;\Omega(X,A)] & \xrightarrow{(Pf)^*} \\ \wr\,\tau & & \wr\,\tau & & \wr\,\tau & \\ \cdots\longrightarrow[C\Sigma Z,\Sigma Z;X,A] & \xrightarrow{(C\Sigma f)^*} & [C\Sigma Y,\Sigma Y;X,A] & \xrightarrow{(CQf)^*} & [CC_f,C_f;X,A] & \xrightarrow{(CPf)^*}\end{array}$$

$$\begin{array}{ccc}[Z;\Omega(X,A)] & \xrightarrow{f^*} & [Y;\Omega(X,A)] \\ \wr\,\tau & & \wr\,\tau \\ [CZ,Z;X,A] & \xrightarrow{(Cf)^*} & [CY,Y;X,A].\end{array}$$

From the forth place on, the sequences consist of groups and homomorphisms. Applying (i) to the upper sequence, we get the theorem.

§ 3. Definition of $\Pi_n(X,A;G)$

Let G be a finitely generated abelian group, then we have the well known fact.

Lemma 3.1. *G is decomposed into a direct sum of cyclic groups in the form*

$$G\approx\underbrace{Z+Z+\cdots+Z}_{r\text{-fold}}+Z_{m_1}+Z_{m_2}+\cdots+Z_{m_s}.$$

The torsion coefficients m_i *are powers of primes. Furthermore this decomposition is unique up to order of the factors.*

Let f_m: $S^1\to S^1$, and denote $B^n_m=\Sigma^{n-2}C_{f_m}$ for $n\geqq 3$ and $B_m{}^2=C_{f_m}$. Define the space P $(G\ n)$ for $n\geqq 2(n\geqq 0$, where G is free) as follows

$$P(G,n)=\underbrace{S^n\vee S^n\vee\cdots\vee S^n}_{r\text{-fold}}\vee B^n_{m_1}\vee B^n_{m_2}\vee\cdots\vee B^n_{m_s}.$$

(cf. [2], [3])

From this definition, we get that

$$P(G,n)=\Sigma^{n-2}P(G,2).$$

$(P(G,n)=\Sigma^n P(G,0)$, where G is free)

DEFINITION 3.2. We define the homotopy set $\Pi_2(X,A;\ G)$ as follows,

$$\Pi_n(X,A;G)=[CP(G,n-1),P(G,n-1);X,A] \text{ for } n\geqq 3.$$

(for $n\geqq 1$, where G is free)

Clearly $\Pi_n\ (X,\ A;\ G)$ is a group for $n\geqq 4$ (for $n\geqq 3$, where G is free), and abelian for $n\geqq 5$ (for $b\geqq 3$, where G is free). And $\Pi_n(X,\ A;\ Z)=\Pi_n(X,\ A)$ and $\Pi_n\ (X,\ x_0;\ G)=\Pi_n\ (X;\ G)$ (Katuta's homotopy group).

§ 4. Some properties

THEOREM 4.1. (i) $\tau:\Pi_n\ (X,A)\approx\Pi_{n-1}(\Omega(X,A);G)$

(ii) *Let* φ: $(X,\ A)\longrightarrow(X',\ A')$, *then the following diagram is commutative*

$$\begin{array}{ccc}\Pi_n(X,A;G) & \approx & \Pi_{n-1}(\Omega(X,A);\ G)\\ \downarrow\varphi_* & & \downarrow\Omega\varphi_*\\ \Pi_n(X',A';G) & \overset{\tau}{\approx} & \Pi_{n-1}(\Omega(X',A');G)\ .\end{array}$$

BY $\Pi_n\ (\Omega\ (X,A);G)$ we mean the Katuta's homotopy group.

PROOF. It is easily verified by using 2,2. (ii).

THEOREM 4.2. (i) *Let* $\varphi_0\simeq\varphi_1$: $(X,\ A)\longrightarrow(X',\ A')$, *then* $\varphi_{0*}=\varphi_{1*}$.

(ii) *If* P *is a space consisting of a single point, then* $\Pi_n\ (P;\ G)=0$ *for* $n\geqq 2$. (*for* $n\geqq o$, *when* G *is free*)

(iii) *Let* φ: $(X,\ A)\longrightarrow(X',\ A')$, *then the square*

$$\begin{array}{ccc}\Pi_n(X,\ A;\ G) & \overset{\partial}{\longrightarrow} & \Pi_{n-1}(A;\ G)\\ \downarrow\varphi_* & & \downarrow(\varphi|A)_*\\ \Pi_n(X',A';G) & \overset{\partial}{\longrightarrow} & \Pi_{n-1}(A';G)\end{array}$$

is commutative.

(iv) *There is an exact sequence as follows,*

$$\cdots\overset{j_*}{\longrightarrow}\Pi_{n+1}(X,A;G)\overset{\partial}{\longrightarrow}\Pi_n(A;G)\overset{i_*}{\longrightarrow}\Pi_n(X;G)\overset{j_*}{\longrightarrow}$$

$$\Pi_n(X,A;G)\overset{\partial}{\longrightarrow}\cdots\overset{j_*}{\longrightarrow}\Pi_3(X,A;G)\overset{\partial}{\longrightarrow}\Pi_2(A;G)\overset{i_*}{\longrightarrow}\Pi_2(X;G)\ .$$

Moreover from the forth place on, the sets are groups and functions are homomorphic.

THEOREM 4.3. *Let* φ: $(X,\ A)\longrightarrow(X',A')$, *then* φ_* *is a homomorphism for* $n\geqq 4$ (*for* $n\geqq 2$ *where* G *is free*).

THEOREM 4.4. *If $G\approx G_1+G_2$, then*

$$\Pi_n(X,A;G)\approx\Pi_n(X,A;G_1)\times\Pi_n(X,A;G_2) \text{ for } n\geqq 4 .$$

The proofs of 4.2.～4.4. are just the same that of usual homotopy groups, and so we omit.

COROLLARY 4.5. *Let $G\approx Z+Z\cdots+Z+Z_{m_1}+Z_{m_2}+\cdots+Z_{m_s}$, then*

$$\Pi_n(X,A;G)\approx\Pi_n(X,A;Z)\times\cdots\times\Pi_n(X,A;Z)\times\Pi_n(X,A;Z_{m_1})\times\cdots\times\Pi_n(X,A;Z_{m_s}) \quad \text{for } n\geqq 4 .$$

THEOREM 4.6. (Universal coefficient theorem)

The following sequence is exact for $n\geqq 4$.

$$0\longrightarrow\Pi_n(X,A)\otimes G\longrightarrow\Pi_n(X,A;G)\longrightarrow Tor(\Pi_{n-1}(X,A),G)\longrightarrow 0 .$$

PROOF. By 3.5 it is sufficient to prove the theorem for the case where $G=Z$ or $G=Z_m$. The theorem is obvious when $G=Z$. In case $G=Z_m$, left $f_m:S^1\to S^1$ then we have the following fact,

$$\begin{array}{ccccccc}
[CS^{n-1},S^{n-1};X,A] & \xrightarrow{(C\Sigma^{n-2}f_m)^*} & [CS^{n-1},S^{n-1};X,A] & \xrightarrow{(C\Sigma^{n-3}Qf_m)^*} & [CB_m^{n-1},B_m^{n-1};X,A] & \xrightarrow{(C\Sigma^{n-3}Pf_m)^*} & \\
\wr\, \tau & & \wr\, \tau & & \wr\, \tau & & \\
[S^{n-1};\Omega(X,A)] & \xrightarrow{(\Sigma^{n-2}f_m)^*} & [S^{n-1};\Omega(X,A)] & \xrightarrow{(\Sigma^{n-3}Qf_m)^*} & [B_m^{n-1};\Omega(X,A)] & \xrightarrow{(\Sigma^{n-3}Pf_m)^*} &
\end{array}$$

$$\begin{array}{ccc}
[CS^{n-2},S^{n-2};X,A] & \xrightarrow{(C\Sigma^{n-3}f_m)^*} & [CS^{n-2},S^{n-2};X,A] \\
\wr\, \tau & & \wr\, \tau \\
[S^{n-2};\Omega(X,A)] & \xrightarrow{(\Sigma^{n-3}f_m)^*} & [S^{n-2};\Omega(X,A)]
\end{array}$$

is a commutative diagram of groups and homomorphisms for $n\geqq 4$, moreover the upper and the lower sequences are exact. We get it by 2.5. and 2.6. Therefore the following sequence is exact,

$$0\to Coker\ (C\Sigma^{n-2}f_m)^*\to\Pi_n(X,A;Z_m)\to Im(C\Sigma^{n-3}Pf_m)^*\to 0$$

$Coker\ (C\Sigma^{n-2}f_m)^*\approx Coker\ (\Sigma^{n-2}f_m)^*$ and $Im(C\Sigma^{n-3}Pf_m)^*\approx Im(\Sigma^{n-3}f_m)^*$. From the property of f_m and from the free resolution of Z_m:

$$0\to Z\to Z\to Z_m\to 0 ,$$

we get that

$Coker\ (\Sigma^{n-2}f_m)^*\approx\Pi_{n-1}(\Omega(X,A))\otimes Z_m$ and $Im(\Sigma^{n-3}f_m)^*\approx Tor(\Pi_{n-2}(\Omega(X,A)),Z_m)$.

And the proof is complete.

COROLLARY 4.7. *If $\Pi_3\ (X, A)$ is 0 or free, then $\Pi_4\ (X, A;\ G)$ is abelian.*

LEMMA 4.8. *Let $G_1\xrightarrow{f_1}G_2\xrightarrow{f_2}G_3\xrightarrow{f_3}G_4\xrightarrow{f_4}G_5$ be an exact sequence of gropus and* *homomorphisms, and* G_4 *be abelian. If there are two homomorphisms* h_2: $G_2 \to G_1$ *and* h_4: $G_5 \to G_4$ *such that* $f_1 h_1 = 1_{G_2}$ *and* $f_4 g_4 = 1_{G_5}$, *then* $G_4 \approx G_3 + G_5$.

PROOF. It is obvious that f_1 and f_4 are surjective. So $f_2=0$. This means that f_3 is injective. Therefore we get the following exact sequence,

$$0 \to G_3 \xrightarrow{f_3} G_4 \xrightarrow{f_4} G_5 \to 0 .$$

From this exact sequence we get the lemma.

LEMMA 4.9. *If* $i \simeq 0 : A \subset X$, *then* $\Pi_n(X, A; G) \approx \Pi_n(X; G) + \Pi_{n-1}(A; G)$ *for* $n \geqq 5$ (*for* $n \geqq 3$, *where G is free*).

PROOF. There is a homotopy $H: A \times I \to X$ such that $H(a, 0)=a$, $H(a, 1)=x_0$ and $H(x_0, t)=x_0$ for any $t \in I$. Let f: $P(G, n-1) \to A$ and define a map $kf:(CP(G, n-1), P(G, n-1)) \to (X, A)$ by taking $kf[z, t]= H[f(z), t]$ where $z \in P(G, n-1)$. It follows that if $f_0 \simeq f_1$ then $kf_0 \simeq kf_1$ and $k(f+g)=kf+kg$. Therefore, if we define a function $\kappa: \Pi_{n-1}(A; G) \to \Pi_n(X, A; G)$ by taking $\kappa a = [kf]$ where $[f] = a \in \Pi_{n-1}(A; G)$, then κ is a homomorphism for n$\geqq$**4** (for n$\geqq$**3**, where G is free). Clearly $kf/P(G, n-1)=f$, and so $\partial\kappa = 1_{\Pi_{n-1}(A;G)}$. On the other hand we have an exact sequence of groups and homorphisms for $n \geqq 4$ ($n \geqq 3$, G free),

$$\Pi_{n+1}(X,A;G) \xrightarrow{\partial} \Pi_n(A;G) \xrightarrow{i_*} \Pi_n(X;G) \xrightarrow{j_*} \Pi_n(X,A;G) \xrightarrow{\partial} \Pi_{n-1}(A;G) .$$

Applying 3.8. to this exact sequence, we get the lemma.

THEOREM 4.10. *Let* $i \simeq 0$: $A \subset X$ *and* $G = Z_p$ *where p is an odd prime, then the exact sequence of* **4.6.** *splits for* $n \geqq$**5**.

PROOF. By 4.8. $\Pi_n(X,A; Z_p) \approx \Pi_n(X; Z_p) + \Pi_{n-1}(A; Z_p)$ for $n \geqq 5$. On the other hand, the exact sequence

$$0 \longrightarrow \Pi_n(X) \otimes Z_p \longrightarrow \Pi_n(X; Z_p) \longrightarrow \mathrm{Tor}(\Pi_{n-1}(X), Z_p) \longrightarrow 0$$

splits for $n \geqq 4$ ([2] 3.11). Therefore the theorem is obtained.

THEOREM 4.11. *Let* $0 \to G_1 \xrightarrow{f} G_2 \xrightarrow{g} G_3 \to 0$ *be an exact sequence of finitely generated abelian groups and homomorphisms, then there is an exact sequence as follows,*

$$\cdots \xrightarrow{*\partial} \Pi_n(X,A;G_1) \xrightarrow{*f} \Pi_n(X,A;G_2) \xrightarrow{*g} \Pi_n(X,A;G_3) \xrightarrow{*\partial}$$

$$\Pi_{n-1}(X,A;G_1) \xrightarrow{*f} \cdots \xrightarrow{*f} \Pi_4(X,A;G_2) .$$

PROOF. We have two maps $\tilde{f}: P(G_2, \mathbf{3}) \to P(G_1, \mathbf{3})$ and

$\tilde{g}: P(G_3, \mathbf{3}) \to P(G_2, 3)$ (cf. [2] [5]) such that the sequence

$$\cdots \xrightarrow{\partial} \Pi_n(\Omega(X,A);G_1) \xrightarrow{(\Sigma^{n-3}\tilde{f})^*} \Pi_n(\Omega(X,A);G_2) \xrightarrow{(\Sigma^{n-3}\tilde{g})^*} \Pi_n(\Omega(X,A);G_3) \xrightarrow{\partial}$$
$$\Pi_{n-1}(\Omega(X,A);G_1) \xrightarrow{(\Sigma^{n-4}\tilde{f})^*} \cdots \xrightarrow{\partial} \Pi_3(\Omega(X,A);G_1) \xrightarrow{\tilde{f}^*} \Pi_3(\Omega(X,A);G_2)$$

is exact. (cf. [2] 3.16)

Define a function ${}_*\partial:\Pi_n(X,A;G_3)\to\Pi_{n-1}(X,A;G_1)$ by taking ${}_*\partial=\tau^{-1}\partial\tau$, then the following diagram is commutative for $n\geqq 4$.

$$\begin{array}{ccc}
\Pi_{n-1}(\Omega(X,A);G_3) & \xrightarrow{\partial} & \Pi_{n-2}(\Omega(X,A);G_1) \\
\uparrow \tau & & \uparrow \tau \\
\Pi_n(X,A;G_3) & \xrightarrow{{}_*\partial} & \Pi_{n-1}(X,A;G_1)\,.
\end{array}$$

Therefore we get a commutative diagram

$$\begin{array}{ccccccccc}
\cdots \xrightarrow{\partial} & \Pi_n(\Omega(X,A);G_1) & \xrightarrow{(\Sigma^{n-3}\tilde{f})^*} & \Pi_n(\Omega(X,A);G_2) & \xrightarrow{(\Sigma^{n-3}\tilde{g})^*} & \Pi_n(\Omega(X,A);G_3) & \xrightarrow{\partial} \\
 & \wr\,\tau & & \wr\,\tau & & \wr\,\tau & \\
\cdots \xrightarrow{{}_*\partial} & \Pi_{n+1}(X,A;G_1) & \xrightarrow{(C\Sigma^{n-3}\tilde{f})^*} & \Pi_{n+1}(X,A;G_2) & \xrightarrow{(C\Sigma^{n-3}\tilde{g})^*} & \Pi_{n+1}(X,A;G_3) & \xrightarrow{{}_*\partial}
\end{array}$$

$$\begin{array}{ccccc}
\Pi_{n-1}(\Omega(X,A);G_1) & \xrightarrow{(\Sigma^{n-4}\tilde{f})^*} \cdots & \xrightarrow{(\tilde{g})^*} & \Pi_3(\Omega(X,A);G_2) \\
\wr\,\tau & & & \wr\,\tau \\
\Pi_n(X,A;G_1) & \xrightarrow{(C\Sigma^{n-4}\tilde{f})^*} \cdots & \xrightarrow{(C\tilde{g})^*} & \Pi_4(X,A;G_2)\,.
\end{array}$$

The upper sequence is exact and also the lower. Replacing $(C\Sigma^i\tilde{f})^*$ and $(C\Sigma^i\tilde{g})^*$ by ${}_*f$ and ${}_*g$, we get the theorem.

THEOREM 4.12. *Let φ: $(X, A)\to(X', A')$ and $f:G_1\to G_2$, then the following diagram is commutative,*

$$\begin{array}{ccc}
\Pi_n(X,A;G_1) & \xrightarrow{{}_*f} & \Pi_n(X,A;G_2) \\
\downarrow \varphi_* & & \downarrow \varphi_* \\
\Pi_n(X',A';G_1) & \xrightarrow{{}_*f} & \Pi_n(X',A';G_2)\,.
\end{array}$$

PROOF. This can be obtained immediately from 2.2. and ([3] 3.17).

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