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By

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1. Summary and Introduction

A Steiner system $S(t, k, v)$ is a set of v points and some blocks, with an incidence relation, such that (i) every block has exactly k points, (ii) every set of t distinct points is on exactly one block, and (iii) two blocks with the same points are equal.

In this paper we prove the following theorems.

THEOREM 1. *Let B be a block of S $(2, k, v)$. If there are exactly n blocks distinct from B each of which has no point in common with B, then* $4n\equiv 0$ (*mod* $(k-1)$).

By this theorem we have the following theorem, which gives a characterization of the Witt system $S(4, 5, 11)$.

THEOREM 2. *In an $S(4, k, v)$, let B be a block of S. If there are exactly* 20 *blocks distinct from B each of which has exactly* 2 *points in common with B, then $S=S(4, 5, 11)$.*

2. Definitions and Lemmas

A Steiner system $S(t, k, v)$ reduces to the trivial case if $t=k$. In what follows we consider non-trivial Steiner systems which consist of more than one block. For a Steiner system $S=S(t, k, v)$ we use b to denote the number of blocks and $\lambda_i(1\leqq i\leqq t)$ the number of blocks which contain the given i points of S. Then, with the convention $b=\lambda_0$, we have

$$\lambda_i=\frac{(v-i)(v-i-1)\cdots\cdots(v-t+1)}{(k-i)(k-i-1)\cdots\cdots(k-t+1)}\qquad(0\leqq i\leqq t)\,. \tag{1}$$

For a block B of S we use $x_i(0\leqq i\leqq t-1)$ to denote the number of blocks distinct from B each of which has exactly i points in common with B. The number x_i depends on S, but as Mendelsohn has shown, is independent of the choice of a block B in S (see Lemma 2). These numbers are called intersection numbers of S. If S is a Steiner system $S(t, k, v)$ with $t\geqq 2$, and a is a point of S, then S_a is defined to be the set of all points of S except a and of all blocks of S which contain a. Then S_a is a system S $(t-1, k-1, v-1)$ and is called a contraction of S.

LEMMA 1 (Witt [1]). *There exists a unique Steiner system $S(t, k, v)$ with the following parameters*; $(t, k, v)=(2, 3, 9)$ *or* $(4, 5, 11)$.

LEMMA 2 (Mendelsohn [2]). *Let S be a Steiner system $S(t, k, v)$ and B a block of S. Let $x_i (0 \leqq i \leqq t-1)$ denote the number of blocks distinct from B each of which has exactly i points in common with B. Then the following equations hold*:

$$x_0+x_1+ \cdots\cdots\cdots\cdots +x_{t-1} = \lambda_0 - 1\,,$$

$$x_1+2x_2+ \cdots\cdots\cdots +(t-1)\, x_{t-1} = (\lambda_1 - 1)\binom{k}{1},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$x_i + \binom{i+1}{i} x_{i+1} + \cdots + \binom{t-1}{i} x_{t-1} = (\lambda_i - 1)\binom{k}{i},$$

$$\cdots\cdots\cdots$$

$$x_{t-1} = (\lambda_t - 1)\binom{k}{t-1}.$$

Therefore x_i's are independent of the choice of a block B in S.

3. Proofs of Theorem 1 and Theorem 2

First of all a proof of Theorem 1 will be given. Let S be a Steiner system $S(2, k, v)$. Let x_i's $(0 \leqq i \leqq 1)$ be intersection numbers of S. By Lemma 2 we have

$$x_0+x_1 = b-1\,,$$

$$x_1 = (\lambda_1 - 1)\, k\,.$$

From these equations it follows that

$$x_0 = \frac{(v-k)(v+k-1-k^2)}{k(k-1)}\,.$$

We put $n = x_0$, then by a simple calculation

$$v^2 - (k^2+1)\, v + k^3 - (n+1)\, k^2 + (n+1)\, k = 0\,.$$

Hence

$$v = \frac{(k^2+1) \pm \sqrt{k^4 - 4k^3 + (4n+6)k^2 - 4(n+1)\, k+1}}{2}\,.$$

Since λ_1 is an integer, by (1) we have

$$\frac{v-1}{k-1} = \frac{(k^2-1) \pm \sqrt{k^4 - 4k^3 + (4n+6)\, k^2 - 4(n+1)\, k+1}}{2(k-1)}$$

is an integer.
Therefore

$$\frac{k^4 - 4k^3 + (4n+6)\, k^2 - 4\,(n+1)\, k+1}{(k-1)^2}$$

is an integer.

Since

$$k^4-4k^3+(4n+6)k^2-4(n+1)k+1$$
$$=(k-1)^2(k^2-2k+4n+1)+4n(k-1)\,,$$

we deduce that

$$\frac{4n}{(k-1)} \quad \text{is an integer.}$$

So $4n \equiv 0 \pmod{(k-1)}$.

To prove the Theorem 2, we need the following lemma.

LEMMA 3. *In Theorem* 1, *if* $n=2$, *then* $S=S(2, 3, 9)$.

PROOF. Since $S(2, k, v)$ is non-trivial and by Theorem 1, k–1 divides 8, so $k=3$, 5, or 9. On the other hand

$$v=\frac{(k^2+1)\pm\sqrt{k^4-4k^3+14k^2-12k+1}}{2}\,. \tag{2}$$

If $k=5$ or 9, then $k^4-4k^3+14k^2-12k+1$ is not a square number. This is a contradiction.

By Lemma 1 and (2), $S=S(2, 3, 9)$.

We prove the Theorem 2. Let S be a Steiner system $S(4, k, v)$.

Let x_i's $(0\leqq i\leqq 3)$, y_i's $(0\leqq i\leqq 2)$, and z_i's $(0\leqq i\leqq 1)$ be intersection number of S, S_α and $S_{\alpha\beta}$, respectively. Assume that $x_2=20$. By Lemma 2, we have

$$\left\{\begin{aligned} x_0+x_1+x_2+x_3 &= b-1 \\ x_1+2x_2+3x_3 &= (\lambda_1-1)\binom{k}{1}, \\ x_2+3x_3 &= (\lambda_2-1)\binom{k}{2}, \\ x_3 &= (\lambda_3-1)\binom{k}{3}, \end{aligned}\right. \tag{3}$$

$$\left\{\begin{aligned} y_0+y_1+y_2 &= \lambda_1-1\,, \\ y_1+2y_2 &= (\lambda_2-1)\binom{k-1}{1}, \\ y_2 &= (\lambda_3-1)\binom{k-1}{2}, \end{aligned}\right. \tag{4}$$

and

$$\left\{\begin{aligned} z_0+z_1 &= \lambda_2-1\,, \\ z_2 &= (\lambda_3-1)\binom{k-2}{1}. \end{aligned}\right. \tag{5}$$

By (3), (4) and (5) we have

$$x_2 = \frac{1}{2} k y_1 , \qquad y_1 = (k-1) z_0 .$$

So
$$z_0 = \frac{40}{k(k-1)} .$$

Since $k>4$ and z_0 is an integer, $k=5$. Hence $z_0=2$. $S_{\alpha\beta}=S(2, 3, 9)$ by Lemma 3. So $S=S(4, 5, 11)$.

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References

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[2] Mendelsohn, N.S., A theorem on Steiner systems, Canadian J. Math. **22** (1970), 1010–1015.