DOUBLY TRANSITIVE PERMUTATION GROUPS OF DEGREE 5q+1, WHERE q IS a PRIME

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DOUBLY TRANSITIVE PERMUTATION GROUPS OF DEGREE $5q+1$, WHERE q IS a PRIME

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1. Introduction and Summary

M.D. Atkinson [4] and T. Tsuzuku [9] proved the following theorem independently.

THEOREM. *Let G be an insoluble transitive permutation group of degree $p=4q+1$ where p and q are primes, which is not doubly primitive. Then $G=PSL$ (3,3) and $p=$* 13.

Furthermore Atkinson [4] proved the following theorems.

THEOREM. *Let G be a doubly transitive group of degree $2q+1$, where q is a prime, which is not doubly primitive. Then G is either sharply doubly transitive or a group of automorphisms of a block design with $\lambda=1$ and $k=3$.*

THEOREM. *Let G be a doubly transitive permutation group on Ω of degree $3q+1$, where q is a prime. Then one of the following statements is true.*

(1) *G is doubly primitive.*
(2) *G is sharply doubly transitive.*
(3) *G is a group of automorphisms of a block design on Ω with $\lambda=1$ and $k=4$.*
(4) *$G=PSL$ (3,2) and $q=2$.*

In this paper we shall prove the following theorem.

THEOREM. *Let G be a doubly transitive permutation group on Ω of degree $5q+1$, where q is a prime and greater than* 11, *Then one of the following statements is true.*

(1) *G is doubly primitive.*
(2) *G is a group of automorphisms of a block design on Ω with $\lambda=1$ and $k=6$.*
(3) $|G_{\alpha\beta}|\,|\,24$.
(4) *G has a regular normal subgroup.*

Our notation for the parameters of a block design, v, k, r, λ, is standard; see [8]. Throughout this paper the term "block" is used only in the block design sense; however, a term such as "K-block" refers to a set of imprimitivity for a group K. In order to prove Theorem we need the several lemmas.

Lemma 1 (E. Witt [12]). *Let X be a doubly transitive group on a set Ω, let α, $\beta\in\Omega$ with $\alpha\neq\beta$ and let K be a weakly closed subgroup of $X_{\alpha\beta}$. Then, if $\Delta=fix(K)$, in the block design whose blocks are the images under X of Δ we have $\lambda=1$.*

Proof. We omit the proof of the lemma. (See [4]).

Lemma 2. (Atkinson [4]). *Let X be a doubly transitive group on a set Ω, let $\alpha\in\Omega$ and let Δ be a set of imprimitivity for the action of X_α on $\Omega-\{\alpha\}$. Let $\beta\in\Delta$ and suppose that $\Delta-\{\beta\}$ is invariant under $X_{\{\alpha,\beta\}}$ Then, in the block design whose blocks are the images under X of $\Gamma=\Delta\cup\{\alpha\}$ we have $\lambda=1$.*

Proof. (See [4]).

Lemma 3 (Atkinson [4]). *Let X be a doubly transitive group on a set Ω. Let $\alpha\in\Omega$ and let Δ be a set of imprimitivity of size m for the action of X_α on Ω-$\{\alpha\}$. Then, if $\beta\in\Delta$, $X_{\{\alpha,\beta\}}$ has an invariant set Γ of size m-1 on $\Omega-\{\alpha,\beta\}$. Furthermore, if $X_{\alpha\beta}$ is transitive on $\Delta-\{\beta\}$, $X_{\alpha\beta}$ and $X_{\{\alpha,\beta\}}$ are transitive on Γ.*

Proof. (See [4]).

Lemma 4. *Let Ω be a set on which there is a non-trivial block design with $\lambda=1$. Then if $|\Omega|=5q+1$, where q is a prime, then $q=3$ or 19 or $k=6$.*

Proof. We prove this lemma by considering the incidence equations of a block design.

Lemma 5 (E. Bannai [5]). *Let G be a transitive permutation group on Ω and $\alpha\in\Omega$. Let $H=G_\alpha$ and $x\in G$. Then we have the following equation,*

$$\frac{|\Omega|}{|I(x)|}\,|\{h\in H|\ h \text{ is } H\text{-conjugate to } x\}|$$
$$=|\{g\in G|g \text{ is } G\text{-conjugate to } x\}|.$$

Proof. We count the pairs $\{(\delta, g)\,|\,\delta\in\Omega,\ g\in G\ \delta^g=\delta.\ g$ *is G-conjugate to* $x\}$ in two ways. We get the above equation.

We shall frequently use the well-known theorem of Burnside that a transitive group of prime degree is either doubly transitive or is a metacyclic Frobenius group.

2. Proof of the theorem

Let G be a doubly transitive group on a set Ω of size $5q+1$, where q is a prime. If G is a counterexample to theorem. By a theorem of [1] we have that q divides $|G|$ to the first power only. Let Q be a Sylow q-subgroup of G_α where $\alpha\in\Omega$. Let $\Delta_1, \Delta_2, \Delta_3, \ldots.$ be a non-trivial system of imprimitivity for the action of G_α on $\Omega-\{\alpha\}$. Let $H=\{x|x\in G_\alpha.\ \Delta_1 x=\Delta_1\}$, $K=\{x\in G_\alpha|\Delta_i x=\Delta_i,\ i=1, 2, 3, \ldots.\}$ and $\beta\in\Delta_1$. Then $G_{\alpha\beta}\subseteqq H$

and $K \lhd G_\alpha$. Furthermore we can consider G_α to act on Δ, where $\Delta=\{\Delta_1, \Delta_2, \dots\}$. There are two cases to consider depending on the size of the G_α-blocks.

Case 1. q G_α-blocks of size 5

Clearly H acts transitively on Δ_1. At first we assume that G_α acts on Δ as an insoluble group and H acts on Δ_1 as a soluble group. If H acts on Δ_1 as a regular group of order 5, then $G_{\alpha\beta}=1$ on Δ_1. Consequently $G_{\alpha\beta}$ fixes the points of Δ_1. So we get a contradiction by using Lemma 1. If H acts on Δ_1 as a Frobenius group of of order 10, then we can assume that $H=<(\beta\gamma\delta\varepsilon\eta), (\beta)(\gamma\eta)(\delta\varepsilon)>$, where $\{\beta, \gamma, \delta, \varepsilon, \eta\}=\Delta_1$. $G_{\alpha\beta}$ acts on $\Delta_1-\{\beta\}$ semi-regularly and $\{\delta, \varepsilon\}$, $\{\gamma, \eta\}$ are $G_{\alpha\beta}$-orbits. So $\{\delta, \varepsilon\}$ and $\{\gamma, \eta\}$ are $G_{\alpha\beta}$-invariant. By Lemma 1 we can assume that $G_{\alpha\beta}$ fixes no points of Ω except α and β. So $N(G_{\alpha\beta})=G_{\{\alpha,\beta\}}$. $|G_{\{\delta,\varepsilon\}}: G_{\alpha\beta}|=|G_{\{\gamma,\eta\}}: G_{\alpha\beta}|=2$. Consequently $N(G_{\alpha\beta})=G_{\{\alpha,\beta\}}\supset G_{\{\delta,\varepsilon\}}, G_{\{\gamma,\eta\}}$. Therefore $\Delta_1-\{\beta\}$ is $G_{\{\alpha,\beta\}}$-invariant. This fact contradicts Lemma 2.

If H acts on Δ_1 as a doubly transitive group, then $\Delta_1-\{\beta\}$ is an orbit of $G_{\alpha\beta}(=H_\beta)$. Since G_α acts on Δ as a doubly transitive group, H acts transitively on $\{\Delta_2, \dots \Delta_q\}$. As $|H: G_{\alpha\beta}|=|H: H_\beta|=5$, all the orbits of $G_{\alpha\beta}$ on $\{\Delta_2, \dots \Delta_q\}$ have size at least $(q-1)/5$ (>4) when $q>19$, and if $q\leqq 19$, then all the orbits of $G_{\alpha\beta}$ on $\{\Delta_2 \dots \Delta_q\}$ have size at least $(q-1)$ (>4) by Lemma 17. 1 [10]. It follows that $\Delta_1-\{\beta\}$ is the unique orbit of $G_{\alpha\beta}$ of size 4 and therefore $\Delta_1-\{\beta\}$ is $G_{\{\alpha,\beta\}}$-invariant. We may now obtain a contradiction from Lemma 2.

Secondly if G_α acts on Δ as a soluble group and K is insoluble, then we may assume that G_α is not local in O'Nan's sense [7]. For if G_α is local, then $G_\alpha=N(P)$, where P is an abelian p-group (p: prime), and P is half-transitive on $\Omega-\{\alpha\}$. So P is a 5-group or a q-group. If P is a 5-group and P does not act on $\Omega-\{\alpha\}$ as semi-regular group, then we have a block design with $\lambda=1$ by Corollary B1 [6]. So this is a contradiction. So P acts semi-regularly. Consequently $|P|=|P_\beta||\beta^P|=5$. P is cyclic. Thus G is known by a theorem of Aschbacher [2]. We have a contradiction by considering the degree of G. Similarly we get a contradiction when P is a q-group. Therefore from now on in this particular case we can assume that G_α has a unique minimal normal simple subgroup N by the result of O'Nan [7]. Consequently H acts on Δ_i as A_5 or S_5 for any i. Now let be a element x of N of order 3. Then x fixes $1+2q$ points on Ω because N acts faithfully on Δ_i for any i. The number of the conjugate elements of x in G_α is 20. For G_α acts on Δ as a Frobenius group and so any element of G_α-K does not fix $1+2q$ points.

Therefore the number of the conjugate elements of x is $(5q+1)\ 20/(2q+1)$ by Lemma 5 and this number must be integer. $(5q+1)\ 20/(2q+1)=50-30/(2q+1)\neq$ an integer $(q>11)$. This is a contradiction.

Finally if G_α acts on Δ as a soluble group and K is soluble and $K\neq 1$, then K has an abelian characteristic subgroup $M\neq 1$. Clearly $\pi(M)\subseteqq\{2, 3, 5\}$. Let S be a S_2-subgroup of M. If $S\neq 1$, then S is weakly closed in G_α. For G_α acts on Δ as a Frobenius group and so any element of order 2^i in G_α-K fixes at most one Δ_i as a set, but every element of order 2^i in S fixes at least q points on $\Omega-\{\alpha\}$. So any element of S is not conjugate to any element of $G_\alpha-K$ in G_α. If $S^g\subseteqq G_\alpha$ for any $g\in G$, then by the above argument $S^g\subseteqq K$ and so $S^g=S^k$ for some $k\in K$ because S is a S_2-subgroup of K and S is normal in K. Thus $S^g=S$. Clearly $S\subseteqq G_{\alpha\gamma}$ for some $\gamma\in\Omega$ and S is weakly closed in $G_{\alpha\gamma}$ and S fixes at least q points on $\Omega-\{\alpha\}$. This result contradicts our assumption by Lemma 1. If $S=1$, then we consider S_3-subgroup of M. Similarly we get a contradiction. So we assume that M is a 5-group. If M does not act on $\Omega-\{\alpha\}$ as semi-regular group, then we can construct a block design with $\lambda=1$ by Corollary B1 [6]. This is not our case. So M acts on $\Omega-\{\alpha\}$ semi-regularly. Thus $|M|=5$, M is cyclic. In this case we have a contradiction by Aschbacher' result [2]. If $K=1$, then it follows that the S_2-subgroup of G_α is cyclic. Consequently G is known by a result of Aschbacher [3]. We have a contradiction by considering the degree of G.

Case 2. 5 G_α-blocks of size q

Since $G_\alpha/K\subseteqq S_5$, $q\nmid|G_\alpha: K|$. Therefore $Q\subseteqq K$ and K is transitive on each Δ_1, Δ_2, Δ_3, Δ_4 and Δ_5. If N is the kernel of the action of K on Δ_1 and $N\neq1$, then N acts transitively on some Δ_i which contradicts the fact that $q^2\nmid|G|$. Thus K acts faithfully on each of Δ_1, Δ_2, Δ_3, Δ_4 and Δ_5. If K is soluble we shall show that $K_\beta=1$. If $K_\beta\neq1$, then K_β fixes precisely one point from each of Δ_1, Δ_2, Δ_3, Δ_4 and Δ_5 because K has a unique conjugacy class of subgroups of index q. Thus K_β and any conjugate of K_β fix exactly 6 points. Consider some conjugate $K_\beta{}^g$ of K contained in $G_{\alpha\beta}$. If $K_\beta{}^g\subsetneqq K$ then some Δ_i contains none of the fixed points of $K_\beta{}^g$ and hence there is some Δ_j which contains at least two of these fixed points; but then $K_\beta{}^g$ must fix pointwise the whole of Δ_j and so has more than 5 fixed points. Thus $K_\beta{}^g\subseteqq K$. and, as K has a unique subgroup of index q which fixes β, we have $K_\beta{}^g=K_\beta$. Therefore K_β is weakly closed in $G_{\alpha\beta}$ and Lemma 1 gives us a contradiction. This means that $K_\beta=1$ as we asserted. $|G_\alpha|=5q|G_{\alpha\beta}|=vq(v|120)$. Consequently $|G_{\alpha\beta}|\,|24$. This is a contradiction. If K is insoluble and G_α is local, then there is a normal q-subgroup Q' of G_α. $|Q'|=q$ by Theorem [1]. So G is known by Aschbacher's Theorem [2]. Again we have a contradiction by considering the degree of G.

From now on we can assume that G_α has a unique minimal normal subgroup N which is simple. If $C_{G_\alpha}(N)\neq1$, then $C_{G_\alpha}(N)\supseteqq N$ because $C_{G_\alpha}(N)\lhd G$ and N is a unique minimal subgroup. Therefore $Z(N)=C_N(N)\neq1$. So N is a cyclic group of order q. G is local. This is not our case. So $C_{G_\alpha}(N)=1$. $G_\alpha=N_{G_\alpha}(N)/C_{G_\alpha}(N)$ is considered to be included in Aut N, where Aut N is the group of the automorphisms of N. Since $N\cong$Inn N, where Inn N is the group of the inner automorphisms of N, we can consider G_α/N to be included in Aut N/Inn N. By a theorem of Wielandt [11] Aut N/Inn N is cyclic. So it follows that G_α/N is cyclic. Since G_α/K is a homomorphic image of G_α/N and $G_\alpha/K\subseteqq S_5$. Thus G_α/K is cyclic and G_α acts on Δ regularly.....(1)

As $K\subseteqq G_{\alpha\beta}$, $\Gamma_1=\Delta_1-\{\beta\}$ is a $G_{\alpha\beta}$-orbit of size q-1. If Γ_1 is $G_{\{\alpha,\beta\}}$-invariant, then we have a contradiction from Lemma 2. If Γ_1 is not $G_{\{\alpha,\beta\}}$-invariant, then there exists another $G_{\alpha\beta}$-orbit Γ_2 of size $q-1$ such that $\Gamma_1 \cup \Gamma_2$ is an orbit of $G_{\{\alpha,\beta\}}$. By Lemma 3 there is yet another $G_{\alpha\beta}$-orbit Γ_3 of size $q-1$ and since it is a $G_{\{\alpha,\beta\}}$-orbit it is distinct form Γ_1 and Γ_2. If either of Γ_2 or Γ_3 is contained in any Δ_i then $G_{\alpha\beta}$ leaves Δ_i invariant and fixes the remaining point of Δ_i; using Lemma 1, this leads to a contradiction. There are two cases to remain. In first case there is Δ_k $(2 \leqq k \leqq 5)$ such that $\Delta_k \cap \Gamma_2 \neq \emptyset$ and $\Delta_k \cap \Gamma_3 \neq \emptyset$. But $\Gamma_2 \cap \Delta_k$ and $\Gamma_3 \cap \Delta_k$ are invariant under K_β and, as they are set of imprimitivity for the action of $G_{\alpha\beta}$ on Γ_2 and Γ_3, we have $|\Delta_k \cap \Gamma_2| \leqq (q-1)/2$ and $|\Delta_k \cap \Gamma_3| \leqq (q-1)/2$. Consequently, K has at least 3 orbits on Δ_k. Now K acts doubly transitively on each of Δ_1 and Δ_k with characters $1+x_1$ and $1+x_2$, say, and the number of orbits of K_β on Δ_k is $(1+x_1, 1+x_2) \leqq 2$ and this is a contradiction. In final case we have $\Gamma_2 \subseteqq \Delta_i \cup \Delta_j$, $\Gamma_3 \subseteqq \Delta_k \cup \Delta_l$, where $\{i, j\} \cap \{k, l\} = \emptyset$. So $G_{\alpha\beta}$ has a element of order 2 on $\{\Delta_2, \Delta_3, \Delta_4, \Delta_5\}$. This fact contradicts (1). Thus we complete the proof of the Theorem.

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