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A CORRECTION TO : ON THE PROPAGATION OF ERROR IN NUMERICAL INTEGRATIONS

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In Proc. Japan Acad. Vol. 48, No. 7 (1972), pp. 484–488, Theorems 1,2 and 3 are not valid as they stand. From Lemmas 1.2 and 1.3 we can not derive these Theorems, and we shall give the corrected forms of them in this paper. And we shall give the detailed proofs of all results, which are stated without proofs in the paper mentioned above.

First we consider the first order differential equation;

(1.1) $$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 . \end{cases}$$

According to Euler's method, we shall try to approximate the equation (1.1) by the following difference equation;

(1.2) $$y_{n+1} = y_n + hf(x_n,y_n),$$

where y_n is an approximation of the solution of (1.1) at $x=x_n=x_0+nh$.

In actual calculation, the calculated value of y_{n+1} is given by the formula;

(1.3) $$y_{n+1} = y_n + hf(x_n,y_n) - R_n \qquad (R_n \text{ ; round-off error}).$$

On the other hand, if we denote the true value of the solution of (1.1) at the point $x=x_n$ by $y(x_n)$, we have also the relation;

(1.4) $$y(x_{n+1}) = y(x_n) + hf(x_n,y(x_n)) + (1/2)h^2y''(\xi_n)$$

where $\xi_n = x_n + \theta_n h \quad (0 \leq \theta_n \leq 1)$.

If we subtract (1.3) from (1.4) and write

(1.5) $$E_n = R_n + (1/2)h^2y''(\xi_n)$$
$$e_n = y(x_n) - y_n, \qquad e_0 = 0,$$

we find the difference equation;

(1.6) $$e_{n+1} = e_n + h(f(x_n,y(x_n)) - f(x_n,y_n)) + E_n .$$

We notice first that we may write

$$f(x_n,y(x_n)) - f(x_n,y_n) = f_y(x_n,\eta_n)(y(x_n)-y_n)$$

if f_y exists, where η_n is a number between y_n and $y(x_n)$, so that (1.6) may be written in the form;

(1.7) $$e_{n+1} = e_n + he_nf_y(x_n,\eta_n) + E_n ,$$

or

(1.8) $$\nabla e_n = \rho_n e_{n-1} + (hf_y(x_{n-1},\eta_{n-1}) - \rho_n)e_{n-1} + E_{n-1}.$$

Here we discuss respectively the asymptotic behavior of the solution of (1.7) and (1.8).

Theorem [1]. *Consider the difference equation* (1.7) *under the assumptions*;

(1) $$\left|\frac{d}{dx}f(x, y(x))\right| \leq \psi(x) \qquad (-\infty<y<+\infty)$$

$$\sum_{\nu=0}^{\infty}\{h^2\psi(x_0+(\nu+\theta_\nu)h)+|R_\nu|\}\leq L \qquad \text{for } 0<h\leq h_0,$$

(2) $$|f_y(x, y)|\leq\Phi(x) \qquad (-\infty<y<+\infty)$$

where $\exists M$, $\exists h_1$, >0

$$\sum_{\nu=0}^{\infty}h\Phi(x_0+\nu h)\leqq\frac{C-L}{C} \quad \text{for} \quad 0<h\leq h_1,$$

(3) $e_0=0$,

then we have

$$|e_n| \leq C \ (L<C)$$

for $0<h\leq min\{h_0, h_1\}$.

Proof. The proof is derived by mathematical induction. Let us assume

$$|e_\nu|\leq C \qquad (\nu=0, 1, 2,\cdots,n-1),$$

and we shall show

$$|e_n|\leq C.$$

From (1.7) and the hypothesis of the induction, we have

$$|e_n|\leq\sum_{\nu=0}^{n-1}|hf_y(x_0+\nu h, \eta_\nu)e_\nu|+\sum_{\nu=0}^{n-1}|E(x_0+\nu h)|$$

$$\leq hC\sum_{\nu=0}^{n-1}|f_y(x_0+\nu h, \eta_\nu)|+L,$$

and taking h so small that conditions (1) and (2) are satisfied,

$$|e_n|\leq C. \qquad \text{Q. E. D.}$$

Next we shall show that, under certain conditions, the solution of difference equation (1.8) monotonically decreases as $n\to\infty$. Before stating the Theorem, we shall give a Lemma.

Lemma [1.1]. *The solution of the equation*;

$$\nabla z(x_0+nh)=A_nz(x_0+(n-1)h)$$
$$+B(x_0+(n-1)h)z(x_0+(n-1)h)+w(x_0+(n-1)h)$$

is given by

$$z(x_0+nh)=z(x_0)Y(x_0+nh)$$
$$+\sum_{\nu=0}^{n-1}Y(x_0+nh)Y^{-1}(x_0+(\nu+1)h)B(x_0+\nu h)z(x_0+\nu h)$$
$$+\sum_{\nu=0}^{n-1}Y(x_0+nh)Y^{-1}(x_0+(\nu+1)h)w(x_0+\nu h),$$

where $Y(x)$ is a solution of the following equation;

$$\begin{cases}\nabla Y(x_0+nh)=A_n Y(x_0+(n-1)h)\\ Y(x_0)=1.\end{cases}$$

Proof. The proof proceeds by the well known method, namely the variation of parameters.

Let

$$z(x_0+nh)=Y(x_0+nh)u(x_0+nh),$$

then

$$\begin{aligned}\nabla z(x_0+nh)&=\nabla Y(x_0+nh)u(x_0+nh)\\ &=u(x_0+(n-1)h)\nabla Y(x_0+nh)+Y(x_0+nh)\nabla u(x_0+nh),\end{aligned}$$

and

$$\nabla Y(x_0+nh)=A_n Y(x_0+(n-1)h).$$

Thus

$$\begin{aligned}&A_n Y(x_0+(n-1)h)u(x_0+(n-1)h)+Y(x_0+nh)\nabla u(x_0+nh)\\ &=A_n Y(x_0+(n-1)h)u(x_0+(n-1)h)+B(x_0+(n-1)h)Y(x_0+(n-1)h)u(x_0+(n-1)h)\\ &\quad+w(x_0+(n-1)h),\end{aligned}$$

and hence

$$\begin{aligned}\nabla u(x_0+nh)&=Y^{-1}(x_0+nh)B(x_0+(n-1)h)z(x_0+(n-1)h)\\ &\quad+Y^{-1}(x_0+nh)w(x_0+(n-1)h).\end{aligned}$$

From the above equation, we have

$$\begin{aligned}u(x_0+nh)&=u(x_0)+\sum_{\nu=0}^{n-1}Y^{-1}(x_0+(\nu+1)h)B(x_0+\nu h)z(x_0+\nu h)\\ &\quad+\sum_{\nu=0}^{n-1}Y^{-1}(x_0+(\nu+1)h)w(x_0+\nu h),\end{aligned}$$

where

$$z(x_0)=Y(x_0)u(x_0)=u(x_0).$$

Thus, we have the solution;

$$\begin{aligned}z(x_0+nh)&=Y(x_0+nh)u(x_0+nh)\\ &=z(x_0)Y(x_0+nh)+Y(x_0+nh)\sum_{\nu=0}^{n-1}Y^{-1}(x_0+(\nu+1)h)B(x_0+\nu h)z(x_0+\nu h)\\ &\quad+Y(x_0+nh)\sum_{\nu=0}^{n-1}Y^{-1}(x_0+(\nu+1)h)w(x_0+\nu h).\end{aligned}$$

Q. E. D.

Theorem [2]. *Suppose that there exist constants* $\exists M>0$, $\exists h_0$, $\exists h_1>0$, $\exists\{\lambda_\nu\}$, $\exists\{a_\nu\}$, $\exists L_1>0$, $\exists L_2>0$, $\exists L_3>0$, $(L_1+L_2<1)$, *which satisfy the following conditions;*

(1) $\quad 0<|1+A_{\nu+1}|\leq e^{\lambda}\nu \qquad (\nu=1,2,\cdots)$

$$\sum_{\nu=1}^{n-1}|A_{\nu+1}|\leq L_1 e^{\lambda}n,$$

(2) $\quad |f_y(x,y)| \leq \Phi(x) \qquad (-\infty < y < \infty)$

$$\sum_{\nu=0}^{n-1} h\Phi(x_0+\nu h) \leq (1-L_1-L_2)e^{\lambda_n} \qquad \text{for } 0<h\leq h_0,$$

(3) $$\left|\frac{d}{dx}f(x,y))\right| \leq \psi(x) \qquad (-\infty<y<\infty)$$

where

$$(1/2)h^2\psi(x_0+(\nu+\theta_\nu)h)+R_\nu \leq a_\nu L_3 e^{\lambda_0+\lambda_1+\lambda_2+\cdots+\lambda_\nu} \text{ for } 0<h\leq h_1$$

$$\sum_{\nu=0}^{n-1} a_\nu \leq L_2 e^{\lambda_n} \qquad (\lambda_0=0),$$

(4) $\quad e_0 = 0,$

then we have

$$|e_n| \leq L_3 e^{\lambda_1+\lambda_2+\cdots+\lambda_n} \qquad (n=1,2,\cdots)$$

for $0<h\leq min\{h_0, h_1\}$.

Proof. The proof is carried out by mathematical induction. For the case $n=1$, the proposition is clearly true, and for the case $n=2$, we have

$$\begin{aligned} |e_2| &\leq h|e_1 f_y(x_1,\eta_1)| + |E_0| + |E_1| \\ &\leq h|e_1|\Phi(x_1)+(a_0+a_1)e^{\lambda_1} \\ &\leq L_3 e^{\lambda_2}. \end{aligned}$$

Let us assume

$$|e_m| \leq L_3 e^{\lambda_1+\lambda_2+\cdots+\lambda_m} \qquad (m=1,2,\cdots,n-1),$$

and we shall show that the above inequality holds for $m=n$ as well. From Lemma [1.1], we have

$$|e_n| \leq |Y(x_0+nh)\sum_{\nu=0}^{n-1} Y^{-1}(x_0+(\nu+1)h)(hf_y(x_0+\nu h,\eta_\nu)-A_{\nu+1})e_\nu|$$

$$+|Y(x_0+nh)\sum_{\nu=0}^{n-1} Y^{-1}(x_0+(\nu+1)h)E_\nu|.$$

Thus from the conditions (1), (2), and the hypothesis of induction

$$|Y(x_0+nh)\sum_{\nu=0}^{n-1} Y^{-1}(x_0+(\nu+1)h)(hf_y(x_0+\nu h, \eta_\nu)-A_{\nu+1})e_\nu|$$

$$\leq |hY(x_0+nh)\sum_{\nu=0}^{n-1} Y^{-1}(x_0+(\nu+1)h)e_\nu\Phi(x_0+\nu h)|$$

$$+|Y(x_0+nh)\sum_{\nu=0}^{n-1} Y^{-1}(x_0+(\nu+1)h)| \quad |A_{\nu+1}||e_\nu|$$

$$\leq L_3(1-L_2)e^{\lambda_1+\lambda_2+\cdots+\lambda_n}.$$

And by making (2) and (3),

$$|Y(x_0+nh)\sum_{\nu=0}^{n-1} Y^{-1}(x_0+(\nu+1)h)E_\nu|$$

$$\leq |E_0 \prod_{\nu=2}^{n} (1+A_\nu)| + |E_1 \prod_{\nu=3}^{n} (1+A_\nu)| + \cdots + |E_{n-1}|$$

$$\leq L_2 L_3 e^{\lambda_1+\lambda_2+\cdots+\lambda_n}.$$

Hence we have the inequality

$$|e_n| \leq L_3{}^{\lambda_1+\lambda_2+\cdots+\lambda_n}. \qquad \text{Q. E. D.}$$

Remark. The following example satisfies the conditions of Theorem [2],

$$y' = \frac{e^{-y}}{(x^2+1)}.$$

In Theorem [2], if we take the constants A_n as

$$-1 < A_n < A_{n+1} < 0$$

and

$$|1+A_n| = e^{-\lambda_{n-1}},$$

then $\lambda_{n-1} > \lambda_n > 0$, and we have the following result.

Corollary. *Under the same assumptions as in Theorem* [2] *on the constants* $\exists M, \exists h_0, \exists h_1, \exists \{a_\nu\}_{\nu=0}^{\infty}, \exists L_1, \exists L_2, \exists L_3$, *if the following conditions are satisfied*

(1) $$|1+A_{\nu+1}| = e^{-\lambda_\nu}, \qquad \sum_{\nu=1}^{n-1} |A_{\nu+1}| \leq L_1{}^{-\lambda_n},$$

(2) $$|f_y(x,y)| \leq \Phi(x) \qquad (-\infty < y < \infty)$$

$$\sum_{\nu=1}^{n-1} h\Phi(x_0+\nu h) \leq (1-L_1-L_2)e^{-\lambda_n} \quad \text{for } 0<h\leq h_0,$$

(3) $$\left|\frac{d}{dx} f(x,y)\right| \leq \psi(x) \qquad (-\infty < y < \infty),$$

where

$$h^2\psi(x_0+(\nu+\theta_\nu)h) + R_\nu \leq a_\nu L_3 e^{-\lambda_1-\lambda_2-\cdots-\lambda_\nu} \quad \text{for } 0<h\leq h_1$$

$$\sum_{\nu=0}^{n-1} a_\nu \leq L_2 e^{-\lambda_n} \qquad (\lambda_\nu > 0.\nu=0,1,2,\cdots),$$

then

$$|e_n| \leq L_3 e^{-\lambda_1-\lambda_2-\cdots-\lambda_n}$$

for $0<h\leq min\{h_0,h_1\}$.

In the above corollary, for instance, if we take the constants A_n as

$$A_m = \frac{-1}{(\rho+m-1)^\alpha} \qquad (m=1,2,\cdots) \quad (\rho>1 \ : \ \text{constant}).$$

and $\alpha > \dfrac{e^{\lambda_1}}{L_1}\rho+1$,

then the conditions (1) is satisfied.

Moreover, using the same idea, we may have the similar result for the propagation of error of general one step methods.

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