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A NOTE ON THE RELATION BETWEEN THE GRAPH OF THE DEGREES OF THE GROUP CHARACTERS AND NON-SIMPLICITY

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1. Introduction and Summary

Let G be a finite group, $D(G)$ the set of degrees of the irreducible complex non-principal characters of G. We introduce an ordering in $D(G)$ as follows: let a and b be two elements of $D(G)$. Then $a>b$ if and only if a divides b. Let k be the number of maximal elements in $D(G)$. Then G is k-headed. We form a graph $D(G)$ of G as follows: the points of $D(G)$ are the elements of $D(G)$. The (oriented) edge ab of $D(G)$ exists, where a and b are points of $D(G)$, if and only if $a>b$. Now we shall have the following conjecture.

Conjecture. *If $D(G)$ is a 2-headed graph then G is non-simple.*

In special cases the above problem and the related problems were solved by I.M. Isaacs and D.S. Passman in [2], [3], [4] and [5]. In this note we shall prove the following theorem.

Theorem. *Let G be a finite group with the following properties, the set of degrees of the irreducible complex characters of G is $\{1, m, n, k_1, k_2, \cdots\cdots, k_l\}$ and $mn|k_i$ for all i. Then G is not a simple group.*

2. Proof of the theorem

Suppose the statement is false and let G be a counter example to the theorem. We can assume that $m<n$ and by a result of Thompson [6] $(m, n)=1$. Let χ be an irreducible non-linear character of G with $\chi(1)=m$. Since G has the irreducible characters of degree n it follows from a theorem of Burnside and Brauer (see Satz 10.8 on p. 519 of [1]) that some power χ^r has an irreducible constituent of degree n. Choose r minimal with this property. Similarly χ^{s_i} has an irreducible constituent of degree k_i and s_i is minimal with this property.

Let $\phi_i \in \mathrm{Irr(G)}$, $\phi_i(1)=k_i$ with ϕ_i a constituent of χ^{s_i}.

If there exists $i \in Z$ such that $s_i<r$, then let the minimal number of $\{s_i | s_i<r\}$ be s_i. For some irreducible constituent ψ of χ^{s_i-1} we must have

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$$0 \neq [\psi\chi, \phi_i]$$

and by the minimality of s_i we have $\psi(1)=m$ or $\psi(1)=1$.

Then $\psi(1)\chi(1)\leq m^2<\phi_i(1)=k_i$. This is a contradiction.

So from now on we assume that for all i $s_i\geq r$. As above let $\phi\in$ Irr (G), $\phi(1)=n$ with ϕ a constitutent of χ^r.

For some irreducible contitnent ψ of χ^{r-1} we must have

$$0 \neq [\psi\chi,\phi] = \frac{1}{|G|} \sum_{x\in G} \psi(x)\chi(x)\overline{\phi(x)} = [\bar{\psi}, \chi\bar{\phi}]$$

and by the minimality of r we have $\psi(1)=m$. (The case that $\psi=1$ is impossible since then χ is irreducible of degree m.)

Thus $\chi\bar{\phi}$ has an irreducible constituent of degree m and has no linear constituent (in this case this is 1) since otherwise

$$0 \neq [\chi\bar{\phi},1] = [\bar{\phi},\bar{\chi}]\,,$$

contradicting $\bar{\phi}(1)=n>m=\bar{\chi}(1)$. Thus all irreducible constituents of $\chi\bar{\phi}$ have degree m, n or k_i and at least one has degree m. Let a be the number of constituents of degree m, b the number of those of degree n and c_i the number of those of degree k_i. We obtain $mn=am+bn+\sum_{i=1}^{l} c_ik_i$. Now $n\,|\,am$ and since $(m, n)=1$, we have $n\,|\,a$. However $a>0$ and thus $a\geqq n$. It follows that $a=n$, $b=0$, $c_i=0$ for all i. So every irreducible constituent of $\chi\bar{\phi}$ has degree m. We may write

$$\chi\bar{\phi} = \sum_{i=1}^{n} \theta$$

where the $\theta_i\in$ Irr(G) all have degree m and not necessarily all distinct. Suppose some θ_i is not χ. Then we have

$$0 = [\chi,\theta_i] = [1,\bar{\chi}\theta_i] \text{ and } \bar{\chi}\theta_i \text{ has not } 1.$$

However $0\neq[\chi\bar{\phi},\theta_i]=[\bar{\phi},\bar{\chi}\theta_i]$ so $\bar{\chi}\theta_i$ has a constituent $\bar{\phi}$ of degree n. Let c be the number of the irreducible constituents of $\bar{\chi}\theta_i$ of degree m, d the number of degree n and e_i the number of degree k_i. Then as above we have

$$m = cm+dn+ \sum_{i=1}^{l} e_ik_i\,.$$

Thus $m\,|\,d$ and $d>0$ so $d\geqq m$ and we have $m^2\geqq dn\geqq mn$ which contradicts $n>m$. It follows that each θ_i is χ.

This yields

$$\chi\bar{\phi} = n\chi\,.$$

Since $\bar{\phi}$ is faithful. $\chi(x) = 0$ for $x\in G$, $x\neq 1$.

This yields $[\chi,1]\neq 0$. This is a contradiction. So we complete the proof of the theorem.

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