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ON THE GENERALIZED HUREWICZ THEOREM

By

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1. Introduction

There are known two definitions of $\pi_n(X; G)$——n-th homotopy groups of the space X with coefficients in the finitely generated abelian group G (cf. [1] and [3]). In [5], Y. Katuta proved the following absolute generalized Hurewicz theorem.

Theorem. *If X is a space such that $\pi_i(X)=0$ for $i\leqq 2$, $\pi_i(X)$ is finitely generated for $i\leqq n$ and $\pi_i(X; G)=0$ for $2<i<n$, then*

$$\Theta: \pi_n(X; G) \cong H_n(X; G).$$

And K. Gęba proved in [2] this theorem without assuming that $\pi_i(X)$ are finitely generated. On the other hand we have the definition and some properties of $\pi_n(X,A;G)$ (cf. [3]). In this note we shall define the relative Hurewicz homomorphism

$$\Theta: \pi_n(X,A;G) \longrightarrow H_n(X,A;G)$$

and show the generalized relative Hurewicz theorem. This theorem is essentially the same as that in [6]. Throughout this note, we work in the pointed categories. Finally the author is deeply grateful to Prof. T. Kudo and Prof. M. Shiraki for their kindness.

2. Definition

Let $f_m: S^1 \longrightarrow S^1$ be a map of degree m. Let $B^n_m = \Sigma^{n-2} C_{f_m}$——$(n$-$1)$-fold suspension of the mapping cone. Then,

$$H_{n-1}(B^n_m) \cong Z_m.$$

From the ordinal universal coefficient theorem, we obtain

$$H_{n-1}(B^n_m; Z_m) \cong Z_m.$$

We denote by α_n the generator of $H_{n-1}(B^n_m; Z_m)$. And from the homology exact sequence of the pair (CB^n_m, B^n_m), it follows that

$$\partial_* ; H_n(CB^n_m, B^n_m; Z_m) \longrightarrow H_{n-1}(B^n_m; Z_m)$$

is isomorphic for $n\geqq 2$. Let $\beta_n = \partial_*^{-1}(\alpha_{n-1})$, then β_n is a generator of $H_n(CB^n_m, B^n_m; Z)$.

On the other hand we know the fact

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$$H_n(CS^{n-1}, S^{n-1}; Z_m) \cong Z_m .$$

We denote by ι_n the generator of $H_n(CS^{n-1}, S^{n-1}; Z_m)$. Now, we define the generalized Hurewicz homomorphism for an arbitrary finitely generated abelian group as follows.

i) $\Theta_Z: \pi_n(X,A;Z) \longrightarrow H_n(X,A;Z)$

is the ordinal one. That is, $\Theta_Z[f] = f_*(\iota_n)$ where $f: (CS^{n-1}, S^{n-1}) \longrightarrow (X,A)$.

ii) $\Theta_{Z_m}: \pi_n(X,A;Z_m) \longrightarrow H_n(X,A;Z_m)$

is difined by $\Theta_{Z_m}[g] = g_*(\beta_n)$, where

$g: (CB^n_m, B^n_m) \longrightarrow (X,A)$.

iii) $\Theta_{G_1 \oplus G_2} = \Theta_{G_1} \oplus \Theta_{G_2}$. That is,

$\Theta_{G_1 \oplus G_2}([f_1] \oplus [f_2]) = \Theta_{G_1}[f_1] \oplus \Theta_{G_2}[f_2]$,

where $[f_i] \in \pi_n(X,A;G_i)$.

remark. $\pi_n(X,A;G_1 \oplus G_2) = \pi_n(X,A:G_1) \oplus \pi_n(X,A;G_2)$ (cf. [3]).

And, $\Theta_{G_i}: \pi_n(X,A;G_i) \longrightarrow H_n(X,A;G_i)$.

Then we get the homomorphism Θ_G for an arbitrary finitely generated abelian group G, because G is decomposed into a finite direct sum of free groups and torsion groups.

And if $A = x_0$ (base point), then Θ_G coincides with the absolute Hurewicz homomorphism in [5].

3. Preliminaries

In order prove the main theorem, we need some properties about $\pi_n(X,A;G)$ and Θ.

PROPOSITION 1. *Let X be 2-connected and $\pi_i(XG) = 0$ for $3 \leqq i < n$, then*

$$\Theta: \pi_n(X;G) \longrightarrow H_n(X;G)$$

is isomorphic (*cf.* [2]).

PROPOSITION 2. *The squares* (i) *and* (ii) *are commutative*

(i)
$$\begin{array}{ccc} \pi_n(X,A;G) & \xrightarrow{\partial_\#} & \pi_{n-1}(A;G) \\ \Theta \downarrow & & \downarrow \Theta \\ H_n(X,A;G) & \xrightarrow[\partial_*]{} & H_{n-1}(A;G) \end{array}$$

(ii)
$$\begin{array}{ccc} \pi_n(X,A;G) & \xrightarrow{f_\#} & \pi_n(Y,B;G) \\ \Theta \downarrow & & \downarrow \Theta \\ H_n(X,A;G) & \xrightarrow[f_*]{} & H_n(Y,B;G) \end{array}$$

where $\quad f\colon (X,A) \longrightarrow (Y,B)$.

Hence the proof is similar to the ordinal one, we omit it.

PROPOSITION 3. *Let $\Omega(X,A)$ be the function space $F(I,1;X,A)$ with compact open topology. Let*

$$\bar{\tau}\colon F(CP(G,n-1),P(G,n-1);\ X,A) \longrightarrow F(P(G,n-1);\ \Omega(X,A))$$

be a map such that $\bar{\tau}(f)(a)(t)=f[a,t]$, where $a\in P(G,n-1)$, $t\in I$ and $f\in F(CP(G,n\text{-}1), P(G,n\text{-}1);X,A)$. Then $\bar{\tau}$ induces an isomorphism

$$\tau\colon \pi_n(X,A;G) \longrightarrow \pi_{n-1}(\Omega(X,A);G)$$

such that $\tau[f]=[\bar{\tau}(f)]$. (*cf.* [3])

4. Main theorem

THEOREM. *If X and A are 1-connected, (X, A) is 3-connected and $\pi_i(X,A;G)=0$ for $4\leqq i<n$, then*

$$\Theta\colon \pi_n(X,A;G) \longrightarrow H_n(X,A;G)$$

is isomorphic.

PROOF. From the ordinal Hurewicz theorem and the homology exact sequence of the pair (X, A), we get that $H_i(X,A;G)=0$ for $i\leqq 3$. On the other hand there is a commutative diagram as follows,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_4(X,A)\otimes G & \longrightarrow & \pi_4(X,A;G) & \longrightarrow & \pi_3(X,A)*G & \longrightarrow & 0\\
 & & \Theta_1\otimes 1_G\downarrow & & \Theta_2\downarrow & & \Theta_3*1_G\downarrow & & \\
0 & \longrightarrow & H_4(X,A)\otimes G & \longrightarrow & H_4(X,A;G) & \longrightarrow & H_3(X,A)*G & \longrightarrow & 0
\end{array}$$

and the rows are exact. Since (X, A) is 3-connected, Θ_1 is isomorphic. It is obvious that

$$0=\pi_3(X,A)*G\cong H_3(X,A)*G\,.$$

This implies that Θ_2 is isomorphic. Therefore we obtain that $H_i(X,A;G)=0$ for $i\leqq 4$.

To prove that

$$H_5(X,A;G)\cong \pi_5(X,A;G)=0\,,$$

we consider the fibration $\omega\colon PX \longrightarrow X$, where PX is the path space with compact open topology.

Naturally $\omega^{-1}(A)=\Omega(X,A)$. It follows from Proposition 2 that the diagram

$$\begin{array}{ccccc}
\pi_4(\Omega(X,A);G) & \xleftarrow{\partial_\#} & \pi_5(PX,\Omega(XA);G) & \xrightarrow{\omega_\#} & \pi_5(X,A;G)\\
\Theta'''\downarrow & & \Theta''\downarrow & & \Theta'\downarrow\\
H_4(\Omega(X,A);G) & \xleftarrow[\partial_*]{} & G_5(PX,\Omega(X,A);G) & \xrightarrow[\omega_*]{} & H_5(X,A;G)
\end{array}$$

is commutative. (remark. $\partial_\#\circ\omega_\#{}^{-1}=\tau$)

From the homotopy and homology exact sequences, we have $\partial_\#$, $\omega_\#$ and ∂_* are isomorphic. It follows from Proposition 3 that $\Omega(X, A)$ is 2-connected, and so $\pi_i(\Omega(X, A); G)=0$ for $3 \leqq i < n-1$. Therefore Θ''' is isomorphic by Proposition 1.

It remains to prove that ω_* is isomorphic. We now concider the homology spectral sequence of the fibration ω: $(PX, \Omega(X,A)) \longrightarrow (X,A)$ with fibre ΩX. Then we have

$$E^2_{p,q} = H_p(X,A; H_q(\Omega X; G)) .$$

And, this fibration is orientable, because the base space X is 1-connected (cf. [7]). Therefore,

$$E^2_{p,q} = H_p(X,A; G) \otimes H_q(\Omega X; G) \oplus H_{p-1}(X,A; G) * H_q(\Omega X; G) \quad \text{(cf. [7])} .$$

It follows that

$$E^2_{p,q} \cong E^3_{p,q} \cong E^4_{p,q} \cong \cdots \cong E^\infty_{p,q} = 0 \text{ for } p \leqq 4 ,$$

and

$$E^2_{5,0} \cong E^3_{5,0} \cong E^4_{5,0} \cong \cdots \cong E^\infty_{5,0} .$$

On the other hand, since

$$H_5(PX, \Omega(X,A); G) = F_{5,0} \supset F_{4,1} \supset F_{3,2}, \supset \cdots \supset F_{-1,6} = 0$$

and

$$F_{p,q}/F_{p-1,q+1} = E^\infty_{p,q} ,$$

we get that

$$E^\infty_{5,0} = H_5(PX, \Omega(X,A); G) .$$

Therefore, ω induces an isomorphism

$$\omega_* : H_5(PX, \Omega(X,A); G) = E^\infty_{5,0} \longrightarrow E^2_{5,0} = H_5(X,A; G) .$$

Consequently it means that Θ' is isomorphic and $H_5(X,A; G)=0$.

Inductively, we can prove that

$$H_i(X,A; G) = 0 \text{ for } i < n$$

and

$$\Theta : \pi_n(X,A; G) \longrightarrow H_n(X,A; G)$$

is isomorphic.

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