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# AN APPLICATION OF NONSTANDARD ANALYSIS TO CHARACTERS OF GROUPS OF CONTINUOUS FUNCTIONS

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Let $K$ be a compact totally disconnected space, and let $S(K)$ be the multiplicative group of all continuous complex-valued functions $f$ on $K$ such that $|f|=1$. The group $S(K)$ is a topological group under the metric

$$d(f,g) = \sup_{k\in K} |f(k)-g(k)| \qquad (f, g \in S(K)).$$

In [3] and [4], Varopoulos has discussed continuous characters of $S(K)$ (see also [1]). Projective limit techniques have been used in [4]. Applying nonstandard methods, we give a simple and natural proof of Varopoulos's theorem.

We use a nonstandard set theory **NST** with an axiom system in [2]. Instead of the axiom schema of saturation ([A.5] in [2]), we may adopt the axiom schema of enlarging ([A.5E] in [2]), which is weaker. Whichever we choose, every standard infinite set has nonstandard elements. Lightface Latin letters denote standard sets, and Greek letters denote internal sets.

Theorem (Varopoulos). *Let $F$ be a continuous character of $S(K)$. Then there exist a finite number of points $k_1, \cdots, k_J \in K$ and integers $p(1), \cdots, p(J) \in Z$ such that*

$$F(g) = \prod_{j=1}^{J} [g(k_j)]^{p(j)} \qquad \text{for all} \quad g \in S(K).$$

Proof. Let $\mathscr{D}$ be the collection of all finite partitions of $K$ into non-empty open closed subsets. For $D_1, D_2 \in \mathscr{D}$, we write $D_1 \leq D_2$ if for every $A \in D_2$ there is $B \in D_1$ such that $A \subset B$. Since the relation "$\leq$" is concurrent, there is an internal partition $\Delta \in \mathscr{D}$ such that $D \leq \Delta$ for any standard $D \in \mathscr{D}$. For each $D \in \mathscr{D}$, let

$$K_D : [1, |D|] \to D$$

be a bijection, and let

$$x_D : [1, |D|] \to K$$

be a choice function such that $x_D(m) \in K_D(m)$ $(1 \leqq m \leqq |D|)$, where $|D|$ is the cardinal number of $D$, and $[1, |D|]$ is the interval in $Z$. Since $K$ is totally disconnected, the property of $\Delta$ shows that the standard part of $K_\Delta(\mu)$ is a singleton set for any $\mu \in [1, |\Delta|]$. For each $D \in \mathscr{D}$, we define

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$$A=\{g \in S(K):\ g \text{ is constant on each partition set of } D\}$$

and $S_D={}^*A$, where ${}^*A$ is the standard set having the same standard elements as the external set $A$. For $D \in \mathscr{D}$, the restriction $F|S_D$ is a continuous character of $S_D$. Since $S_D$ is topologically isomorphic to the $|D|$-dimensional torus group, there is a mapping

$$l:[1,\ |D|]\to Z$$

such that

$$F(f)=\prod_{m=1}^{|D|}[f(x_D(m))]^{l(m)} \qquad \text{for all } f \in S_D\,.$$

By the transfer principle, there is an internal mapping

$$\lambda:[1,\ |\varDelta|]\to Z$$

such that

$$F(\phi)=\prod_{\mu=1}^{|\varDelta|}[\phi(x_\varDelta(\mu))]^{\lambda(\mu)} \qquad \text{for all } \phi \in S_\varDelta\,. \tag{1}$$

Suppose that $\varOmega=\prod_{\mu=1}^{|\varDelta|}|\lambda(\mu)| \neq 0$. Define a function $\psi \in S_\varDelta$ by

$$\psi(x_\varDelta(\mu))=exp\,(i\ sgn\ \lambda(\mu)/\varOmega) \quad (\mu \in [1,\ |\varDelta|])\,.$$

Then (1) shows that $F(\psi)=e^i$. Since $F$ is continuous, $d(\psi,\ 1)$ is not infinitesimal. This implies that the positive integer $\varOmega$ is finite and is hence standard. Thus $\lambda(\mu)$ is a standard integer for each $\mu \in [1,\ |\varDelta|]$, and

$$\{\mu \in [1,\ |\varDelta|]:\lambda(\mu)\neq 0\}$$

is a finite set of internal positive integers. Therefore (1) can be rewritten as

$$F(\phi)=\prod_{j=1}^{J}[\phi(\kappa_j)]^{p(j)} \qquad \text{for all } \phi \in S_\varDelta\,, \tag{2}$$

where $J$ is a standard natural number, $p(j)$ are standard integers and $\kappa_j$ are internal elements of $K$. For each $j$, let $\{k_j\}$ be the singleton set which is the standard part of the partition set of $\varDelta$ containing $\kappa_j$. Then we have $k_j\approx\kappa_j$ $(1\leqq j\leqq J)$. If $g \in S(K)$, then there is a $\phi \in S_\varDelta$ such that $d(g,\ \phi)\approx 0$. This implies that

$$\phi(\kappa_j)\approx g(\kappa_j)\approx g(k_j)\,.$$

It follows from continuity of $F$ that $F(g)\approx F(\phi)$. Taking the standard part in (2), we have

$$F(g)=\prod_{j=1}^{J}[g(k_j)]^{p(j)}\,.$$

This completes the proof.

## References

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