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TWO-SIDED SPLINE APPROXIMATE METHODS FOR TWO POINT BOUNDARY VALUE PROBLEMS

By

Monabu SAKAI*

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Abstract

This paper deals with the two-sided approximations of the solution of the two point boundary value problem by use of cubic and quintic splines. These methods have been tested on several examples. Numerical results show that the accuracy predicated can be achieved.

1. Introduction

Splines are of much use for approximating solutions of two point boundary value problems of ordinary differential equations. In the present paper, we shall consider the two-sided approximations of the nonlinear problems:

(1) $$x''=f(t,x,x') \qquad (0\leq t\leq 1)$$

with the boundary conditions

(2) $$a_0x(0)-b_0x'(0)=c_0$$

(3) $$a_1x(1)+b_1x'(1)=c_1,$$

where $f(t, x, y)$ is sufficiently smooth in the region D of (t, x, y)-space intercepted by two hyperplanes $t=0$ and $t=1$. We assume that the problem (1)–(3) has an isolated solution $\hat{x}(t)$ satisfying the internality condition

$$U=\{(t,x,y)\,|\,|x-\hat{x}(t)|+|y-\hat{x}'(t)|\leq\delta, t\in[0,1]\}\subset D$$

for some $\delta>0$. The solution $\hat{x}(t)$ is isolated if and only if

$$G=\begin{bmatrix}a_0 & -b_0\\ a_1z_0(1)+b_1z_0'(1) & a_1z_1(1)+b_1z_1'(1)\end{bmatrix}$$

is nonsingular, where $z_k(t)$ $(k=0, 1)$ is the solution of the first variation equation of (1), that is,

$$z_k''=f_2(t,\hat{x},\hat{x}')z_k+f_3(t,\hat{x},\hat{x}')z_k' \qquad (0\leq t\leq 1)$$

subject to $z_k^{(j)}(0)=\delta_{j,k}$. Here $f_i(t, x, y)$ denotes the partial derivative with respect to the i-th variable.

* Department of Mathemates, Kagoshina University.

Now making use of B-spline $Q_{m+1}(t)=(1/m!)\sum(-1)^i\binom{m+1}{i}(t-i)_+^m$, we consider cubic spline of the form

$$x_k(t)=\Sigma\,\alpha_i Q_4(t/h-i)\qquad (nh=1)$$

with the undetermined coefficient $(\alpha_{-3}, \alpha_{-2},\cdots,\alpha_{n-1})$. The above x_k will be an approximate solution to the problem (1)–(3) if it satisfies

$$x_k''=P_k f(t, x_k, x_k')\qquad (0\leq t\leq 1)$$
$$a_0x_k(0)-b_0x_k'(0)=c_0$$
$$a_1x_k(1)+b_1x_k'(1)=c_1\,.$$

Here $P_k(k=1,\ 2)$ is an operator defined by

$$(P_1g)(t)=\Sigma\, g(t_i)\,L_i(t)\qquad (t_i=ih)$$
$$(P_2g)(t)=\Sigma\,\beta_i L_i(t)\qquad (r\geq 6)$$

such that

$$\Delta^r\beta_0=0\,,\ (1/6)(\beta_{i+1}+4\beta_i+\beta_{i-1})=g_i\qquad (i=1, 2,\cdots, n-1)\,,\ \nabla^r\beta_n=0\,,$$

where $L_i(t)$ is a piecewise linear function with the property $L_i(t_j)=\delta_{i,j}$. In case the first derivatives don't appear both in the differential equation and the boundary conditions, the approximate method ($k=2$) reduces to the classical $O(h^2)$ difference one. In [5], we have obtained the following

THEOREM 1. *In a sufficiently small neighbourhood of the isolated solution* $\hat{x}(t)$ *of the problem* (1)–(3), *there exists a cubic spline approximation* $x_k(t)$ ($k=1$, 2) *such that*

$$\|\hat{x}^{(m)}-x_k^{(m)}\|_\infty=\max|\hat{x}^{(m)}(t)-x_k^{(m)}(t)|=O(h^2)\qquad (m=0, 1, 2)\,.$$

In an analogous manner as in cubic spline, we shall consider the quintic one of the form

$$x_k(t)=\Sigma\,\alpha_i Q_6(t/h-i)\qquad \text{such that}$$
$$x_k''=P_k f(t, x_k, x_k')\qquad (0\leq t\leq 1)$$
$$a_0x_k(0)-b_0x_k'(0)=c_0$$
$$a_1x_k(1)+b_1x_k'(1)=c_1\,.$$

Here $(P_k\,g)\,(t)$ is a cubic spline with the node t_i such that

$$(P_3g)(t_i)=g_i\qquad (i=0, 1,\cdots, n)\,,\ \Delta^r(P_3g)_0''=\nabla^r(P_3g)_n''=0\qquad (r\geq 6)\,;$$
$$(P_4g, L_i)=(h/12)\,(g_{i+1}+10g_i+g_{i-1})\qquad (i=1, 2,\cdots, n-1)$$
$$\Delta^r(P_4g)_0''=\Delta^{r+1}(P_4g)_0''=\nabla^r(P_4g)_n''=\nabla^{r+1}(P_4g)_n''=0\,,$$

where for any $u(t)$ and $v(t)\in L^2[0, 1]$, we shall denote $\int u(t)v(t)dt$ by (u, v). In case the first derivatives don't appear both in the differential equation and the boundary conditions, the approximate method ($k=4$) reduces to the well-known Numerov difference one. In [5], we have have obtained

THEOREM 2. *In a sufficiently small neighbourhood of the isolated solution* $\hat{x}(t)$, *there exists a quintic spline* $x_k(t)$ ($k=3$, 4) *such that*

$$\|\hat{x}^{(m)} - x_k^{(m)}\|_\infty = O(h^4) \qquad (m = 0, 1, 2).$$

The object of this paper is to show the asymptotic expansion:

For k=1, 2:

$$e_k(t) = \hat{x}(t) - x_k(t) = (-1)^k (h^2/12)\psi(t) + h^4\phi(t) + h^6\phi_k(t) + O(h^{\min(8,r)});$$

For k=3, 4:

$$e_k(t) = d_k h^4\theta(t) + h^6\theta_k(t) + O(h^8), \ (d_3, d_4) = (1/720, -1/240)$$

where ψ and θ are the solutions of the auxiliary two point linear boundary value problems:

$$\psi'' = f_2(t, \hat{x}, \hat{x}')\psi + f_3(t, \hat{x}, \hat{x}')\psi' + \hat{x}^{(4)}(t)$$

$$a_0\psi(0) - b_0\psi'(0) = 0, \ a_1\psi(1) + b_1\psi'(1) = 0;$$

$$\theta'' = f_2(t, \hat{x}, \hat{x}')\theta + f_3(t, \hat{x}, \hat{x}')\theta' + \hat{x}^{(6)}(t)$$

$$a_0\theta(0) - b_0\theta'(0) = 0, \ a_1\theta(1) + b_1\theta'(1) = 0.$$

Using these asymptotic expansions, we obtain the following two-sided approximations of $\hat{x}(t)$ at the mesh point:

For $k = 1, 2$:

$$\hat{x}(t) - \{x_1(t; h) + x_2(t; h)\}/2 = h^4\phi(t) + O(h^6)$$

$$\hat{x}(t) - \{4x_1(t; h/2) - x_1(t; h) + 4x_2(t; h/2) - x_2(t; h)\}/6 = -(h^4/4)\phi(t) + O(h^6);$$

For $k = 3, 4$:

$$\hat{x}(t) - \{3x_3(t; h/2) + x_4(t; h/2)\}/4 = (h^6/256)\{3\theta_3(t) + \theta_4(t)\} + O(h^8)$$

$$\hat{x}(t) - [3\{16x_3(t; h/2) - x_3(t; h)\} + 16x_4(t; h/2) - x_4(t; h)]/60$$

$$= -(h^6/80)\{3\theta_3(t) + \theta_4(t)\} + O(h^8),$$

where $x_k(t; h)$ is the approximate solution using the operator P_k with the mesh size h.

Before we proceed with analysis, we shall define the Green function $\underline{H}(t, s) = (H_{i,j}(t, s))$ $(1 \le i, j \le 2)$ such that

$$\underline{H}(t, s) = \begin{cases} Z(t)\left[E - G^{-1}\begin{bmatrix} 0 & 0 \\ a_1 & b_1 \end{bmatrix} Z(1)\right] Z^{-1}(s) & (s \le t) \\ -Z(t)G^{-1}\begin{bmatrix} 0 & 0 \\ a_1 & b_1 \end{bmatrix} Z(1) Z^{-1}(s) & (t < s) \end{cases}$$

where E is the unit matrix and $Z(t) = (z_j^{(i)}(t))$ $(0 \le i, j \le 1)$.

2. Asymptotic expansion of error function $e_k(t)$ (k=1, 2)

In what follows, let g be sufficiently smooth and use the notation: $I_m(t; g) = \int H^{(m)}(t, s) g(s)\, ds$ and $I(t; g) = I_0(t; g)$, where $H^{(m)}(t, s)$ denotes the m-th partial derivative with respect to the second variable and $H(t, s) = H_{12}(t, s)$ or $H_{22}(t, s)$. First of all, P_2 is well defined. In fact, we have the following result:

LEMMA ([6]). *The $n\times n$ tridiagonal matrix A*

$$\begin{pmatrix} 1 & \lambda & & & & \\ 1 & 4 & 1 & & & \\ 0 & 1 & 4 & 1 & & \\ \cdot & \cdot & \cdot & \dot{1} & \dot{4} & \dot{1} \\ & & & & \lambda & 1 \end{pmatrix}$$

is nonsingular for sufficiently large n if $\lambda \neq 2+\sqrt{3}$.

LEMMA 2. *At the mesh point t_i, we have*

$$I(t;(I-P_k)g)=(-1)^k(h^2/12)\,I(t;g'')+h^4\mu(t)+h^6\mu_k(t)+O(h^{\min(8,r)})$$

for sufficiently smooth $\mu(t)$ and $\mu_k(t)$ ($k=1, 2$).

PROOF. From now on, we set $c_i=(t_i+t_{i+1})/2$. A simple Taylor series argument shows:

$$I(t;(I-P_1)\,g)=-(h^3/12)\,\Sigma\, H(t,c_i)\,g''(c_i)-(h^5/480)\,\Sigma\,\{H(t,c_i)\,g^{(4)}(c_i)$$
$$+(2/3)\,H'(t,c_i)\,g^{(3)}(c_i)+H''(t,c_i)\,g''(c_i)\}-(h^7/53760)\,\Sigma\Sigma\,\eta_j H^{(6-j)}(t,c_i)\,.$$
$$g^{(i)}(c_i)+O(h^8)\qquad (\eta_2,\eta_3,\eta_4,\eta_5,\eta_6)=(1,4/3,10/3,4/3,1)\,.$$

By use of the mid point rule:

$$\int g(t)dt=(b-a)\,g((b+a)/2)+(1/24)\,(b-a)^2\,[g']_a^b-(7/5760)\,(b-a)^4\,[g^{(3)}]_a^b$$
$+\cdots$, we obtain

$$I(t;(I-P_1)\,g)=-(h^2/12)\,I(t;g'')-(h^4/480)\,[I_2(t;g'')+(2/3)\,I_1(t;g^{(3)})$$
$$+I(t;g^{(4)})]+(h^4/288)\,\{[H(t,s)\,g''(s))']_0^{t-}+[(H(t,s)\,g''(s))']_{t+}^1\}$$
$$+h^6\mu_1(t)+O(h^6)\,.$$

Since $(P_2g)(t)=\Sigma\,\beta_iL_i(t)$ such that

$$\Delta^r\beta_0=0,\ (1/6)\,(\beta_{i+1}+4\beta_i+\beta_{i-1})=g_i\qquad (i=1,2\cdots,n-1)\,,\ \nabla^r\beta_n=0\,,$$

we have

$$(P_2g)\,(t_i)=g_i-(h^2/6)\,g_i''+(h^4/72)\,g_i^{(4)}-(h^6/2160)\,g_i^{(6)}+O(h^{\min(8,r)})\,.$$

Hence we obtain

$$I(t;(I-P_2)\,g)=I(t;(I-P_1)\,g)+I(t;(P_1-P_2)\,g)=I(t;(I-P_1)\,g)$$
$$+(h^2/6)\,\{I(t;g'')-I(t;(I-P_1)\,g'')\}-(h^4/72)\,\{I(t;g^{(4)})-I(t;(I-P_1)\,g^{(4)})\}$$
$$+(h^6/2160)\,I(t;g^{(6)})+O(h^{\min(8,r)})\,.$$

This completes the proof of Lemma 2.

Combining Theorem 1 and Lemma 2 yields:

THEOREM 3. *At the mesh point t_i, we have*

$$e_k^{(m)}(t)=(-1)^k\,(h^2/12)\,\psi^{(m)}(t)+O(h^4)\qquad (m=0,1) \tag{7}$$

(8) $$e_k''(t)=(h^2/12)\{(-1)^k\psi''(t)+\hat{x}^{(4)}(t)\}+O(h^4).$$

Proof. Using Taylor series expansion we obtain

$$e_k''=f_2e_k+f_3e_k'+(I-P_k)f-(I-P_k)(f_2e_k+f_3e_k')+(1/2)P_k[f_{22}(t,\hat{x}-\kappa e_k,$$
$$\hat{x}'-\kappa e_k')e_k^2+\cdots]\qquad(0\le t\le 1),$$

where $f(t)=f(t,\hat{x},\hat{x}')$ and $f_2(t)=f_2(t,\hat{x},\hat{x}'),\cdots$.
Thus we have

$$e_k^{(m)}(t)=\int H(t,s)(I-P_k)f(s)\,ds-\int H(t,s)(I-P_k)(f_2e_k+f_3e_k')(s)\,ds+O(h^4)$$
$$(H(t,s)=H_{m+1,2}(t,s)).$$

For $t=t_i$, we have

$$\int H(t,s)(I-P_k)(f_3e_k')(s)\,ds=-(h^3/12)\Sigma H(t,c_i)f_3(c_i)e_k^{(3)}(c_i)$$
$$+O(h^4)=-(h^3/12)\Sigma H(t,c_i)f_3(c_i)\{e_k''(t_{i+1})-e_k''(t_i)\}+O(h^4)=O(h^4)$$

The conclusion now follows.

Corollary. *At the mid point c_i, we obtain*

$$e_k^{(m)}(t)=(-1)^k(h^2/12)\psi^{(m)}(t)+O(h^4)\qquad(m=0,1).$$

Proof. Since $x_k(t)$ is cubic on $[0, h]$, it is represented as follows:

$$x_k(t)=x_k(0)p_0(t/h)+x_k(h)p_0((h-t)/h)+h\{x_k'(0)p_1(t/h)-x_k'(h)p_1((h-t)/h)\}$$
$$(0\le t\le h),$$

from which follows

$$e_k(t)=\hat{x}(t)-p_3(t;\hat{x})+[e_k(0)p_0(t/h)+e_k(h)p_0((h-t)/h)+h$$
$$\{e_k'(0)p_1(t/h)-e_k'(h)p_1((h-t)/h)\}]=\hat{x}(t)-p_3(t;\hat{x})+(-1)^k(h^2/12)p_3(t;\psi)+O(h^4),$$

where $p_k(t)$ $(k=0, 1)$ is a cubic polynomial with the property $p_k^{(j)}(0)=\delta_{k,j}$, and $p_k^{(j)}(1)=0$; $p_3(t;g)$ denotes the cubic Hermite polynomial interpolating to g. Since $x_k'(t)$ is considered to be cubic on [0, h], we have

$$e_k'(t)=\hat{x}'(t)-p_3(t;\hat{x}')+(-1)^k(h^2/12)p_3(t;\psi')+(h^3/12)\{\hat{x}^{(4)}(0)p_1(t/h)$$
$$-\hat{x}^{(4)}(h)p_1((h-t)/h)\}+O(h^4)=(-1)^k(h^2/12)\psi'(t)+(h^3/12)\{\hat{x}^{(4)}(0)p_1(t/h)$$
$$-\hat{x}^{(4)}(h)p_1((h-t)/h)\}+O(h^4)\qquad(0\le t\le h).$$

Thus we have the desired result.

Lemma 3. *At the mesh point t_i, we have*

(10) $$I(t;(I-P_k)ge_k)=(h^4/144)I(t;(g\psi)'')+(h^4/720)I((t;g\hat{x}^{(4)})+O(h^6)$$

(11) $$I(t;(I-P_k)ge_k')=(h^4/144)I(t;(g\psi')'')-(h^4/180)I(t;g\hat{x}^{(5)})$$
$$-(h^4/720)I(t;g'\hat{x}^{(4)})-(h^4/720)I_1(t;g\hat{x}^{(4)})+O(h^6)\qquad(k=1,2).$$

Proof. Since $x_k''(t)$ and $x_k^{(3)}(t)$ are piecewise linear and step functions respectively, we obtain

$$e_k'(c_i) = (h^2/24)\{-\hat{x}^{(4)}(c_i)+2(-1)^k\psi''(c_i)\} + O(h^4)$$

$$e_k^{(3)}(c_i) = (1/h)\{e_k''(t_{i+1})-e_k''(t_i)\} - (h^2/24)\hat{x}^{(5)}(c_i) + O(h^4).$$

Thus we have

$$\begin{aligned} I(t;(I-P_1)ge_k) &= -(h^3/12)\Sigma H(t,c_i)[(g''e_k)(c_i)+2(g'e_k')(c_i) \\ &\quad +(ge_k'')(c_i)] - (h^5/480)\Sigma H(t,c_i)(g\hat{x}^{(4)})(c_i)+O(h^6) \\ &= (-1)^{k+1}(h^4/144)I(t;(g\psi)'')+(h^4/720)I(t;g\hat{x}^{(4)})+O(h^6). \end{aligned}$$

For $k=2$, we have

$$I(t;(P_1-P_2)ge_2) = \Sigma\lambda_i \int H(t,s)L_i(s)\,ds,$$

where the parameter $\lambda_i(i=0, 1,\cdots, n)$ satisfies the following system of equations:

$$\begin{aligned} (1/6)(\lambda_{i+1}+4\lambda_i+\lambda_{i-1}) &= (1/6)[(ge_2)(t_{i+1})-2(ge_2)(t_i)+(ge_2)(t_{i-1})] \\ &= (h^2/6)(ge_2)''(t_i)+(h^3/36)g(t_i)\{e_2^{(3)}(t_i+)-e_j^{(3)}(t_i-)\} \\ &\quad +(h^4/72)g(t_i)\hat{x}^{(4)}(t_i)+O(h^6) = (h^4/72)(g\psi)''(t_i)+O(h^6), \\ \Delta^r\lambda_0 = \Delta^r(ge_2)_0 &= \Delta^{r-2}(\Delta^2 ge_2)_0 = \Delta^{r-2}[(ge_2)(t_2)-2(ge_2)(t_1)+(ge_2)(t_0)] \\ &= (h^4/12)\Delta^{r-2}(g\psi'')_1+O(h^6) = O(h^6), \quad \nabla^r\lambda_n = O(h^6). \end{aligned}$$

Therefore we have

$$\lambda_i = (h^4/72)(g\psi)''(t_i)+O(h^6)$$

from which follows

$$\begin{aligned} I(t;(P_1-P_2)ge_2) &= (h^4/72)\Sigma(g\psi)''(t_i)\int H(t,s)L_i(s)\,ds+O(h^6) \\ &= (h^4/72)I(t;(g\psi)'')+O(h^6). \end{aligned}$$

Next, using a simple series argument we see that

$$\begin{aligned} I(t;(I-P_1)ge_k') &= -(h^3/12)\Sigma H(t,c_i)(ge_k')''(c_i)-(h^5/480)\Sigma H(t,c_i)\cdot \\ &\quad [(g\hat{x}^{(5)}(c_i)+4(g'\hat{x}^{(4)})(c_i)]-(h^5/720)\Sigma H'(t,c_i)(g\hat{x}^{(4)})(c_i)+O(h^6) \\ &= (-1)^{k+1}(h^4/144)I(t;(g\psi')'')-(h^4/180)I(t;g\hat{x}^{(5)})-(h^4/720)I(t;g'\hat{x}^{(4)}) \\ &\quad -(h^4/720)I_1(t;g\hat{x}^{(4)})+O(h^6). \end{aligned}$$

Since $(P_1-P_2)(ge_2')(t) = \Sigma\xi_i L_i(t)$, we have

$$\int H(t,s)(P_1-P_2)(ge_2')(s)\,ds = \Sigma\xi_i\int H(t,s)L_i(s)\,ds,$$

where the parameter ξ_i $(i=0, 1,\cdots, n)$ satisfies the system of equations:

$$\begin{aligned} (1/6)(\xi_{i+1}+4\xi_i+\xi_{i-1}) &= (1/6)\{(ge_2')(t_{i+1})-2(ge_2')(t_i)+(ge_2')(t_{i-1})\} \\ &= (h^2/6)(g''e_2'+2g'e_2'')(t_i)+(h^2/12)g(t_i)\{e_2^{(3)}(t_i+)+e_2^{(3)}(t_i-)\} \\ &\quad +(h^3/12)g(t_i)\{e_2^{(3)}(t_i+)-e_2^{(3)}(t_i-)\}+(h^4/72)(g\hat{x}_i^{(5)}+4g'\hat{x}_i^{(4)})(t_i) \\ &\quad +O(h^6) = (h^4/72)(g\psi')''(t_i)+O(h^6), \end{aligned}$$

$$\Delta^r \xi_0 = \Delta^{r-2}(\Delta^2(ge')_0) = (h^4/12)\,\Delta^{r-2}(g\psi')_1 + O(h^6) = O(h^6)\,,\ \nabla^r \xi_n = O(h^6)\,.$$

Hence we have $\xi_i = (h^4/72)\,(g\psi')''\,(t_i) + O(h^6)$
from which follows

$$\int H(t,s)\,(P_1 - P_2)\,(ge_2')(s)\,ds = (h^4/72)\int H(t,s)\,(g\psi')''(s)\,ds + O(h^6)\,.$$

Combining Lemmas 2 and 3 yields:

THEOREM 4. *At the mesh point t_i, we have*

(12) $$e_k^{(m)}(t) = (-1)^k\,(h^2/12)\,\psi^{(m)}(t) + h^4\phi^{(m)}(t) + O(h^6) \qquad (m=0,1)$$

(13) $$e_k''(t) = (h^2/12)\,\{(-1)^k\psi''(t) + \hat{x}^{(4)}(t)\} + h^4\sigma_k(t) + O(h^6)$$

for sufficiently smooth $\phi(t)$ and $\sigma_k(t)$ $(k=1,\ 2)$.

COROLLARY. *At the mid point c_i, we have*

(14) $$e_k^{(m)}(t) = (-1)^k\,(h^2/12)\,\psi^{(m)}(t) + h^4\phi^{(m)}(t) + \rho_m h^4\hat{x}^{(4+m)}(t) + O(h^6)$$

$$(\rho_0, \rho_1) = (1/384, -1/128) \qquad (m=0,1)\,.$$

PROOF. Since $x_k(t)$ is a cubic polynomial on $[0,\ h]$, we haue

$$e_k(t) = \hat{x}(t) - p_3(t:\hat{x}) + (-1)^k\,(h^2/12)\,p_3(t:\psi) + h^4 p_3(t:\phi) + O(h^6)\,.$$

Since $g(t) - p_3(t;\,g) = (1/24)t^2(t-h)^2 g^{(4)}(h/2) + (1/120)\,t^2(t-h)^2\,(t-(h/2))$.
$g^{(5)}(h/2) + O(h^6)$ $\quad (0 \le t \le h)$, we have

$$e_k(h/2) = (-1)^k\,(h^2/12)\,\psi(h/2) + h^4\phi(h/2) + \rho_0 h^4\hat{x}^{(4)}(h/2) + O(h^6)\,.$$

Since $x_k'(t)$ is considered to be cubic on $[0,\ h]$, we have

$$e_k'(t) = \hat{x}'(t) - p_3(t;\,\hat{x}') + (-1)^k\,(h^2/12)\,p_3(t;\,\psi') + h^4 p_3(t;\,\phi')$$
$$+ (h^3/12)\,\{\hat{x}^{(4)}(0)\,p_1(t/h) - \hat{x}^{(4)}(h)\,p_1((h-t)/h)\} + O(h^6)\,,$$

from which follows the desired result.

In an exactly analogous manner as in Theorem 3, we obtain

THEOREM 5. *If $r \ge 7$, there exists a sufficiently smooth $\phi_k(t)$ such that at the mesh point t_i*

(15) $$e_k^{(m)}(t) = (-1)^k\,(h^2/12)\,\psi^{(m)}(t) + h^4\phi^{(m)}(t) + h^6\phi_k^{(m)}(t) + O(h^{\min(8,r)})$$
$$(m=0,1)\,.$$

3. Asymptotic expansion of error function $e_k(t)$ $(k=3,\ 4)$

In case of quintic spline, we shall require the following:

LEMMA 4. *At all the mesh point t_i, we have*

$$(P_4g)(t_i) = g_i + (h^4/180)\,g_i^{(4)} - (h^6/2160)\,g_i^{(6)} + O(h^8)$$
$$(P_4g)'(t_i) = g_i' + (h^6/5040)g_i^{(7)} + O(h^7)$$
$$(P_4g)''(t_i) = g_i'' - (h^2/12)\,g_i^{(4)} + (h^4/120)\,g_i^{(6)} + O(h^6)\,.$$

PROOF. Since $s(t) = (P_4g)(t)$ is cubic on each interval $[t_i,\ t_{i+1}]$, we obtain

$$(s_i,L_i) = (h/6)\,(s_{i+1}+4s_i+s_{i-1})-(h^3/360)\,(7s''_{i+1}+16s''_i+7s''_{i-1})\,.$$

From the definition of P_4, we have

$$(P_4g, L_i) = (h/12)\,(g_{i+1}+10g_i+g_{i-1})$$

from which follows

$$(1/6)\,(s_{i+1}+4s_i+s_{i-1})-(h^2/360)\,(7s''_{i+1}+16s''_i+7s''_{i-1}) = (1/12)\,(g_{i+1}+10g_i+g_{i-1})\,.$$

Noting the well-known consistency relation:

$$(1/6)\,(s''_{i+1}+4s''_i+s''_{i-1}) = (1/h^2)\,(s_{i+1}-2s_i+s_{i-1})\,,$$

we have the system of equations with respect to $(s''_0, s''_1,\cdots, s''_n)$:

$$(1/120)\,(s''_{i+2}+26s''_{i+1}+66s''_i+26s''_{i-1}+s''_{i-2})$$
$$= (1/12h^2)\,(g_{i+2}+8g_{i+1}-18g_i+8g_{i+1}+g_{i+2})\,.$$

Since $\Delta^r s''_0=\Delta^{r+1}s''_0=\nabla^r s''_n=\nabla^{r+1}s''_n=0$ $(r\geq 6)$, we obtain

$$s''_i = g''_i-(h^2/12)\,g_i^{(4)}+(h^4/120)\,g_i^{(6)}+O(h^6) \qquad ([6])\,.$$

For s_i, similarly we have

$$(1/120)\,(s_{i+2}+26s_{i+1}+66s_i+26s_{i-1}+s_{i-2})$$
$$= (1/72)\,(g_{i+2}+14g_{i+1}+42g_i+14g_{i-1}+g_{i-2}) \qquad (i = 2, 3,\cdots, n-2)\,.$$

Using the above consistency relation, we have

$$s_2-2s_1+s_0 = (h^2/6)\,(s''_2+4s''_1+s''_0)=(h^2/6)\,(g''_2+4g''_1+g''_0)$$
$$-(h^4/72)\,(g_2^{(4)}+4g_1^{(4)}+g_0^{(4)})+(h^6/720)\,(g_2^{(6)}+4g_1^{(6)}+g_0^{(6)})+O(h^8)$$
$$= h^2g''_1+(h^4/12)\,g_1^{(4)}+(h^6/120)\,g_1^{(6)}+O(h^6)\,,\ s_3-2s_2+s_1 = h^2g''_2+(h^4/12)\,g_2^{(4)}$$
$$+(h^6/120)\,g_2^{(6)}+O(h^8)\,,\ s_n-2s_{n-1}+s_{n-2} = h^2g''_{n-1}+(h^4/12)\,g_{n-1}^{(4)}+(h^6/120)\,g_{n-1}^{(6)}$$
$$+O(h^8)\,,\cdots.$$

Thus we have

$$s_i = g_i+(h^4/180)\,g_i^{(4)}-(h^6/2160)\,g_i^{(6)}+O(h^8)\,.$$

The next Lemma, whose proof is based on this lemma and Taylor series argument, is a key to our asymptotic expansion.

LEMMA 5. *At the mesh point t_i, we have*

$$I(t;(I-P_k)\,g) = d_kh^4I(t;g^{(4)}) + h^6\nu_k(t)+O(h^8)$$
$$(d_3, d_4) = (1/720\,,\ -1/240)$$

for sufficiently smooth $\nu_k(t)$ $(k=3,\ 4)$.

PROOF. Using the cubic Hermite polynomial $p_3(t;g)$ interpolating to g, w we obtain

$$(I-P_3)\,g = g-p_3(t;g)+(h^5/180)\,[g_0^{(5)}p_1(t/h)-g_1^{(5)}p_1((h-t)/h)\,]+O(h^8)$$
$$(I-P_4)\,g = g-p_3(t;g)-(h^4/180)\,[g_0^{(4)}p_0(t/h)+g_1^{(4)}p_0((h-t)/h)\,]$$

$$+ (h^6/2160)\,[g_0^{(6)}p_0(t/h) + g_1^{(6)}p_0((h-t)/h)\,] - (h^7/5040)\,[g_0^{(7)}p_1(t/h)$$
$$- g_1^{(7)}p_1((h-t)/h)\,] + O(h^8) \qquad (0 \le t \le h)\,.$$

Since $g(t)-p_3(t;\,g)=(1/24)\,t^2(t-h)^2\,g^{(4)}(h/2)+(1/120)t^2(t-h)^2(t-h/2)\,g^{(5)}(h/2)+(1/720)t^2(t-h)^2(t^2-ht+3h^2/4)g^{(6)}(h/2)+\cdots$, we obtain

$$I(t;g-p_3(\cdot\,;g)\,) = (h^5/720)\,\Sigma\, H(t,c_i)\,g^{(4)}(c_i)+(h^7/40320)\,\Sigma\,[H(t,c_i)\,g^{(6)}(c_i)$$
$$+(2/5)\,H'(t,c_i)\,g^{(5)}(c_i)+H''(t,c_i)\,g^{(4)}(c_i)\,] + O(h^8)\,.$$

Noting that $p_1(t)+p_1(1-t)=t(1-t)$ and $p_1(t)-p_1(1-t)=t(t-1)(2t-1)$, we find

$$I(t;(I-P_3)\,g)-I\,(t;g-p_3(\cdot\,;g)\,) = (h^5/180)\,\Sigma\,[2g^{(5)}(c_i)\int H(t,s)\,(s/h)$$
$$(s/h-1)\,(s/h-1/2)\,ds+(h/2)\,g^{(6)}(c_i)\int H(t,s)\,(s/h)(s/h-1)\,ds+\cdots]+O(h^8)$$
$$= (h^5/180)\,\Sigma\,[-(h^2/60)\,g^{(5)}(c_i)\,H'(t,c_i) - (h^2/12)\,g^{(6)}(c_i)\,H(t,c_i)+\cdots]$$
$$+O(h^8) = -(h^6/10800)\,\{I_1(t;g^{(5)})+5I(t;g^{(6)})\,\}+O(h^8)\,.$$

Since $g_0^{(4)}p_0(t/h) + g_1^{(4)}p_0((h-t)/h) = g^{(4)}(h/2)+(h^2/8)\,g^{(6)}(h/2) - (h/2)\,\{4(t/h)^2 - 6(t/h)+1\}\,g^{(5)}(h/2)+\cdots \quad (0 \le t \le h)$, we obtain

$$I(t;(I-P_4)\,g)-I(t;g-P_3(\cdot\,;g)\,) = -(h^4/180)\,\Sigma\,[g^{(4)}(c_i)\int H(t,s)\,ds$$
$$+(h^2/8)\,g^{(6)}(c_i)\int H(t,s)ds-(h/2)\,hg^{(5)}(c_i)\int H(t,s)\,\{4(s/h)^2-6(s/h)+1\}\,ds]$$
$$+(h^6/2160)\,I(t;g^{(6)})+O(h^8) = -(h^4/180)\,\{I(t;g^{(4)})-(h^2/12)\,I(t;g^{(6)})\,\}$$
$$-(h^6/1440)\,I(t;g^{(6)})+(h^6/2160)\,I(t;g^{(6)})-(h^6/1800)\,I_1(t;g^{(5)})+O(h^8)$$
$$= -(h^4/180)\,I(t;g^{(4)})+(h^6/4320)\,I(t;g^{(6)})-(h^6/1800)\,I_1(t;g^{(5)})+O(h^8)\,.$$

This completes the proof of this lemma.

Combining Theorem 2 and Lemma 5 yields:

Theorem 6. *At the mesh point t_i, we have*

(16) $$e_k^{(m)}(t) = d_k h^4\theta^{(m)}(t)+O(h^6) \qquad (m=0,1)$$

(17) $$e_k^{(m)}(t) = d_k h^4\,\{\theta^{(m)}(t)-\hat{x}^{(4+m)}(t)\,\} + C_{k,m}h^{6-m}\hat{x}^{(6)}(t)+O(h^{8-m}) \qquad (m=2,4)$$

(18) $$e_k^{(3)}(t) = d_k h^4\,\{\theta^{(3)}(t)-\hat{x}^{(7)}(t)\} + C_{k,3}h^4\hat{x}^{(7)}(t)+O(h^5)$$

$$(C_{3,2}=0\,,\ C_{3,3}=1/180\,,\ C_{3,4}=1/12;\,C_{4,2}=-1/180\,,\ C_{4,3}=0\,,\ C_{4,4}=1/12)\,.$$

Proof. From the definition of operator P_4, we have

$$(1/120)\,(x_{i+2}^{(4)}+26x_{i+1}^{(4)}+66x_i^{(4)}+26x_{i-1}^{(4)}+x_{i-2}^{(4)}) = (1/12h^2)\,\{f(t_{i+2},x_{i+2},x'_{i+2})$$
$$+8f(t_{i+1},x_{i+1},x'_{i+1})-18f(t_i,x_i,x'_i)+8f(t_{i-1},x_{i-1},x'_{i-1})+f(t_{i-2},x_{i-2},x'_{i-2})\,\}$$
$$= (1/12h^2)(\hat{x}''_{i+2}+8\hat{x}''_{i+1}-18\hat{x}''_i+8\hat{x}''_{i-1}+\hat{x}''_{i-2})-(d_4h^2/12)\,\{(\theta''-\hat{x}^{(6)})_{i+2}+8$$
$$(\theta''-\hat{x}^{(6)})_{i+1}-18(\theta''-\hat{x}^{(6)})_i+8(\theta''-\hat{x}^{(6)})_{i-1}+(\theta''-\hat{x}^{(6)})_{i-2}\} + O(h^4)\,,$$

with $\Delta^r x_0^{(4)} = \Delta^{r+1}x_0^{(4)} = \nabla^r x_n^{(4)} = \nabla^{r+1}x_n^{(4)} = 0\,,$

where $x_i^{(4)} = x_4^{(4)}(t_i)$.

Therefore we obtain

$$e_4^{(4)}(t_i) = (h^2/12)\,\hat{x}_i^{(6)} + O(h^4).$$

In an exactly analogous manner as in Lemma 4, we have

$$e_4''(t_i) = d_4 h^4\,\{\theta''(t_i) - \hat{x}^{(6)}(t_i)\} - (h^4/180)\,\hat{x}^{(6)}(t_i) + O(h^6).$$

For $k=3$, we obtain

$$(1/6)\,\{x_3^{(4)}(t_{i+1}) + 4x_3^{(4)}(t_i) + x_3^{(4)}(t_{i-1})\} = (1/h^2)\,\{x_3''(t_{i+1}) - 2x_3''(t_i)$$
$$+ x_3''(t_{i-1})\} = (1/h^2)(\hat{x}_{i+1}'' - 2\hat{x}_i'' + \hat{x}_{i-1}'') - (d_3 h^2)\,\{(\theta'' - \hat{x}^{(6)})_{i+1} - 2(\theta'' - \hat{x}^{(6)})_i$$
$$+ (\theta'' - \hat{x}^{(6)})_{i-1}\} + O(h^4),\ \Delta^r x_0^{(4)} = \nabla^r x_n^{(4)} = 0,$$

from which follows

$$e_3^{(4)}(t_i) = (1/12)\,h^2 \hat{x}_i^{(6)} + O(h^4).$$

COROLLORY. *At the mid point c_i, we obtain*

$$e_k^{(m)}(t) = d_k h^4 \theta^{(m)}(t) + O(h^6) \qquad (m = 0, 1)$$
$$e_k''(t) = d_k h^4\,\{\theta''(t) - \hat{x}^{(6)}(t)\} + (1/384 + C_{k,2})\,h^4 \hat{x}^{(6)}(t) + O(h^6)$$
$$e_k^{(3)}(t) = d_k h^4\,\{\theta^{(3)}(t) - \hat{x}^{(7)}(t)\} + (1/384 + C_{k,3})\,h^4 \hat{x}^{(7)}(t) - (C_{k,4}/8) h^4 \hat{x}^{(7)}(t) + O(h^5).$$

PROOF. Since x_k is quintic on $[0, h]$, we have

$$e_k(t) = \hat{x}(t) - p_5(t;\hat{x}) + d_k h^4 p_5(t;\theta) + O(h^6) = d_k h^4 \theta(t) + O(h^6),$$

where $p_5(t; g)$ denotes the quintic Hermite polynomial interpolating to g. Further we have

$$e_k'(t) = \hat{x}'(t) - p_5(t;\hat{x}') + [e_k'(0)\,q_0(t/h) + e_k'(h)\,q_0((h-t)/h)$$
$$+ h\{e_k''(0)\,q_1(t/h) - e_k''(h)\,q_1((h-t)/h)\} + h^2\{e_k^{(3)}(0)\,q_2(t/h)$$
$$+ e_k^{(3)}(h)\,q_2((h-t)/h)\}] = \hat{x}'(t) - p_5(t;\hat{x}') + d_k h^4 p_5(t;\theta') + (C_{k,2} - d_k)\cdot$$
$$h^5[\hat{x}_0^{(6)} q_1(t/h) - \hat{x}_1^{(6)}(h)\,q_1((h-t)/h)] + O(h^6),$$

where $q_k(t)$ is a quintic polynomial with the property $q_i^{(j)}(0) = \delta_{i,j}$, $q_i^{(j)}(1) = 0$ $(0 \le i, j \le 2)$. Hence we have

$$e_k'(h/2) = d_k h^4 \theta'(h/2) + O(h^6).$$

Since x_k'' is cubic on $[0, h]$, we obtain

$$e_k''(t) = \hat{x}''(t) - p_3(t;\hat{x}'') + d_k h^4 p_3(t;\theta'') + (C_{k,2} - d_k)\,h^4[\hat{x}_0^{(6)} p_0(t/h)$$
$$+ \hat{x}_1^{(6)} p_0((h-t)/h)] + (C_{k,3} - d_k)\,h^5[\hat{x}_0^{(7)} p_1(t/h) - \hat{x}_1^{(7)} p_1((h-t)/h)] + O(h^6).$$

from which follows

$$e_k''(h/2) = d_k h^4\{\theta''(h/2) - \hat{x}^{(6)}(h/2)\} + (1/384 + C_{k,2})\,h^4 \hat{x}^{(6)}(h/2) + O(h^6),$$

Similarly we have

$$e_k^{(3)}(t) = \hat{x}^{(3)}(t) - p_3(t;\hat{x}^3) + d_k h^4 p_3(t;\theta^{(3)}) + (C_{k,3} - d_k)\, h^4[\hat{x}_0^{(7)} p_0(t/h) + \hat{x}_1^{(7)} p_0((h-t)/h)] + C_{k,4} h^3 [\hat{x}_0^{(6)} p_1(t/h) - \hat{x}_1^{(6)} p_1((h-t)/h)] + O(h^5)$$

from which follows the desired result.

Combining this Corollary and Lemma 5 yields:

THEOREM 7. *At the mesh point t_i, we have*

$$e_k^{(m)}(t) = d_k h^4 \theta^{(m)}(t) + h^6 \theta_k^{(m)}(t) + O(h^8) \qquad (m = 0, 1). \tag{19}$$

for sufficiently smooth $\theta_k(t)$ (k=3, 4).

4. Numerical Illustration

In this section we shall consider the application of the above stated method to the sample problems. Both in the cubic and quintic spline approximations, we have taken h=1/16 and r=6. All the computations were performed on FACOM M–200, a computer of Kyushu University Computer Center.

Example 1. As our first example, we consider the linear problem:

$$x'' = 100x, \qquad x(0) = x(1) = 1.$$

The exact solution is $\hat{x}(t)$=cosh(10t–5)/cosh(5).

Table 1.1

	x_1-$\hat{x}$	x_2-$\hat{x}$	z_1-$\hat{x}$	z_2-$\hat{x}$	w_1-$\hat{x}$
1/8	−6.05(−3)	5.66(−3)	−1.95(−4)	4.87(−5)	−6.87(−8)
2/8	−3.49(−3)	3.33(−3)	−7.68(−5)	1.91(−5)	−4.40(−8)
3/8	−1.65(−3)	1.61(−3)	−1.75(−5)	4.33(−6)	−2.39(−8)
4/8	−1.10(−3)	1.09(−3)	−3.25(−6)	7.90(−7)	−1.67(−8)

Table 1.2

	x_3-$\hat{x}$	x_4-$\hat{x}$	z_3-$\hat{x}$	z_4-$\hat{x}$	w_2-$\hat{x}$
1/8	3.45(−5)	−1.13(−4)	−3.15(−8)	1.17(−7)	3.78(− 9)
2/8	2.02(−5)	−6.56(−5)	−1.80(−8)	3.63(−8)	1.40(− 9)
3/8	9.70(−6)	−3.14(−5)	−8.57(−9)	2.92(−8)	4.31(−10)
4/8	6.52(−6)	−2.11(−5)	−5.74(−9)	1.93(−8)	2.15(−10)

Here $z_i(t)$ and $w_i(t)$ are defined as follows:

$$z_1(t) = [x_1(t;h) + x_2(t;h)]/2, \qquad z_3(t) = [3x_3(t;h/2) + x_4(t;h/2)]/4$$

$$z_2(t) = [4x_1(t;h/2) - x_1(t;h) + 4x_2(t;h/2) - x_2(t;h)]/6$$

$$z_4(t) = [3\{16x_3(t;h/2) - x_3(t;h)\} + 16x_4(t;h/2) - x_4(t;h)]/60$$

$$w_1(t) = [z_1(t) + 4z_2(t)]/5, \qquad w_2(t) = [16z_3(t) + 5z_4(t)]/21.$$

Example 2 ([9]). Next we consider the nonlinear problem:

$$x'' = (x^2+x'^2)/2e^t, \quad x(0)-x'(0)=0, \quad x(1)+x'(1)=2e.$$

The solution is $\hat{x}(t)=e^t$.

Table 2.1

	x_1-$\hat{x}$	x_2-$\hat{x}$	z_1-$\hat{x}$	z_2-$\hat{x}$	w_1-$\hat{x}$
0	−1.41(−4)	1.41(−4)	−1.03(−7)	2.58(−8)	−7.50(−12)
1/8	−1.59(−4)	1.59(−4)	−1.06(−7)	2.64(−8)	−7.02(−12)
2/8	−1.75(−4)	1.75(−4)	−1.08(−7)	2.71(−8)	−6.61(−12)
3/8	−1.89(−4)	1.89(−4)	−1.10(−7)	2.75(−8)	−6.29(−12)
4/8	−2.00(−4)	2.00(−4)	−1.11(−7)	2.76(−8)	−6.07(−12)
5/8	−2.07(−4)	2.07(−4)	−1.08(−7)	2.69(−8)	−5.96(−12)
6/8	−2.09(−4)	2.09(−4)	−9.98(−8)	2.49(−8)	−5.98(−12)
7/8	−2.02(−4)	2.02(−4)	−8.38(−8)	2.09(−8)	−6.16(−12)
1	−1.86(−4)	1.85(−4)	−5.57(−8)	1.39(−8)	−6.54(−12)

Table 2.2

	x_3-$\hat{x}$	x_4-$\hat{x}$	z_3-$\hat{x}$	z_4-$\hat{x}$	w_2-$\hat{x}$
0	9.96(−9)	−2.76(−8)	−7.58(−14)	3.10(−13)	1.60(−14)
1/8	1.03(−8)	−3.10(−8)	−1.13(−13)	4.20(−13)	1.43(−14)
2/8	1.14(−8)	−3.41(−8)	−1.49(−13)	5.32(−13)	1.31(−14)
3/8	1.23(−8)	−3.69(−8)	−1.85(−13)	6.43(−13)	1.23(−14)
4/8	1.30(−8)	−3.91(−8)	−2.19(−13)	7.53(−13)	1.20(−14)
5/8	1.35(−8)	−4.05(−8)	−2.53(−13)	8.58(−13)	1.18(−14)
6/8	1.36(−8)	−4.08(−8)	−2.83(−13)	9.57(−13)	1.19(−14)
7/8	1.31(−8)	−3.95(−8)	−3.11(−13)	1.04(−12)	1.20(−14)
1	1.20(−8)	−3.62(−8)	−3.32(−13)	1.12(−12)	1.31(−14)

Example 3 ([9]). We consider the following:

$x''=(exp(2x)+x'^2)/2$, $x(0)$-$x'(0)=1$, $x(1)+x'(1)=$-$\ln(2)$-$1/2$. The solution is $\hat{x}(t)=\ln(1/(1+t))$.

Table 3.1

	x_1-$\hat{x}$	x_2-$\hat{x}$	z_1-$\hat{x}$	z_2-$\hat{x}$	w_1-$\hat{x}$
0	−2.44(−4)	2.44(−4)	1.82(−7)	−4.59(−8)	−3.19(− 9)
1/8	−2.62(−4)	2.62(−4)	1.18(−8)	−6.45(−9)	−6.68(−10)
2/8	−2.63(−4)	2.63(−4)	−6.50(−8)	1.31(−8)	−2.49(− 9)
3/8	−2.54(−4)	2.54(−4)	−9.87(−8)	2.19(−8)	−2.23(− 9)
4/8	−2.40(−4)	2.40(−4)	−1.12(−7)	2.54(−8)	−2.01(− 9)
5/8	−2.22(−4)	2.22(−4)	−1.15(−7)	2.64(−8)	−1.82(− 9)
6/8	−2.03(−4)	2.03(−4)	−1.13(−7)	2.61(−8)	−1.65(− 9)
7/8	−1.83(−4)	1.83(−4)	−1.08(−7)	2.52(−8)	−1.50(− 5)
1	−1.63(−4)	1.63(−4)	−1.03(−7)	2.40(−8)	−1.37(− 9)

Table 3.2

	$x_3-\hat{x}$	$x_4-\hat{x}$	$z_3-\hat{x}$	$z_4-\hat{x}$	$w_2-\hat{x}$
0	2.02(−7)	−6.26(−7)	−1.04(−10)	1.70(−10)	−3.85(−11)
1/8	2.15(−7)	−6.63(−7)	−1.00(−10)	1.82(−10)	−3.31(−11)
2/8	2.10(−7)	−6.47(−7)	−9.25(−11)	1.73(−10)	−2.93(−11)
3/8	1.98(−7)	−6.08(−7)	−8.36(−11)	1.57(−10)	−2.61(−11)
4/8	1.82(−7)	−5.59(−7)	−7.49(−11)	1.41(−10)	−2.34(−11)
5/8	1.66(−7)	−5.09(−7)	−6.68(−11)	1.25(−10)	−2.11(−11)
6/8	1.49(−7)	−4.58(−7)	−5.94(−11)	1.10(−10)	−1.90(−11)
7/8	1.33(−7)	−4.09(−7)	−5.26(−11)	9.60(−11)	−1.72(−11)
1	1.18(−7)	−3.62(−7)	−4.64(−11)	8.28(−11)	−1.56(−11)

Example 4 ([9]). The problem solved was:

$x''=(x+tx')/(1+t)$, $x(0)-2x'(0)=-1, x(1)+2x'(1)=3\mathbf{e}$. The solution is $\hat{x}(t)=e^t$.

Table 4.1

	$x_1-\hat{x}$	$x_2-\hat{x}$	$z_1-\hat{x}$	$z_2-\hat{x}$	$w_1-\hat{x}$
0	−2.95(−4)	2.95(−4)	−1.22(−7)	3.04(−8)	−1.32(−11)
1/8	−3.13(−4)	3.13(−4)	−1.19(−7)	2.97(−8)	−1.27(−11)
2/8	−3.30(−4)	3.30(−4)	−1.15(−7)	2.88(−8)	−1.23(−11)
3/8	−3.45(−4)	3.44(−4)	−1.11(−7)	2.77(−8)	−1.21(−11)
4/8	−3.57(−4)	3.56(−4)	−1.04(−7)	2.61(−8)	−1.20(−11)
5/8	−3.64(−4)	3.64(−4)	−9.47(−8)	2.37(−8)	−1.21(−11)
6/8	−3.66(−4)	3.66(−4)	−8.09(−8)	2.02(−8)	−1.23(−11)
7/8	−3.60(−4)	3.60(−4)	−6.15(−8)	1.54(−8)	−1.26(−11)
1	−3.44(−4)	3.44(−4)	−3.48(−8)	8.68(−9)	−1.31(−11)

Table 4.2

	$x_3-\hat{x}$	$x_4-\hat{x}$	$z_3-\hat{x}$	$z_4-\hat{x}$	$w_1-\hat{x}$
0	1.92(−8)	−5.76(−8)	−2.47(−13)	6.82(−13)	−2.61(−14)
1/8	2.03(−8)	−6.11(−8)	−2.84(−13)	8.05(−13)	−2.46(−14)
2/8	2.14(−8)	−6.44(−8)	−3.21(−13)	9.30(−13)	−2.29(−14)
3/8	2.24(−8)	−6.73(−8)	−3.58(−13)	1.06(−12)	−2.13(−14)
4/8	2.32(−8)	−6.96(−8)	−3.96(−13)	1.18(−12)	−1.95(−14)
5/8	2.37(−8)	−7.11(−8)	−4.33(−13)	1.31(−12)	−1.78(−14)
6/8	2.38(−8)	−7.14(−8)	−4.69(−13)	1.43(−12)	−1.61(−14)
7/8	2.43(−8)	−7.03(−8)	−5.03(−13)	1.55(−12)	−1.49(−14)
1	2.24(−8)	−6.72(−8)	−5.33(−13)	1.65(−12)	−1.28(−14)

Example 5. We consider the following van der Pol's equation:

$$x''=(1-x^2)x'/2-x/4,\ x(0)=0,\ x(1)=2.$$

Table 5.1

	z_1	z_2	w_1
1/8	0.2500 4470	0.2500 4485	0.2500 4473
2/8	0.5139 7139	69	45
3/8	0.7880 3037	74	44
4/8	1.0653 1507	36	13
5/8	1.3359 6745	51	46
6/8	1.5885 3760	43	57
7/8	1.8123 5625	06	21

Table 5.2

	z_3	z_4	w_2
1/8	0.2500 447339	0.2500 447338	0.2500 447339
2/8	0.5139 714521	09	19
3/8	0.7880 304438	18	34
4/8	1.0653 151289	68	85
5/8	1.3359 674648	37	46
6/8	1.5885 375651	1	1
7/8	1.8123 562137	40	38

Example 6. Let us consider the following nonlinear problem;

$$x''=1.5x^2, \qquad x(0)=4, \qquad x(1)=1.$$

This problem has two isolated solutions such that $\hat{x}(0.5)=16/9$ and $\hat{x}(0.5)\fallingdotseq-10.5362$.

Table 6.1 $(x(0.5)=16/9)$

	$x_1-\hat{x}$	$x_2-\hat{x}$	$z_1-\hat{x}$	$z_2-\hat{x}$	$w_1-\hat{x}$
1/8	−1.47(−3)	1.44(−3)	−1.23(−5)	3.06(−6)	−3.50(− 9)
2/8	−1.87(−3)	1.84(−3)	−1.35(−5)	3.36(−6)	−3.79(− 9)
3/8	−1.52(−3)	1.79(−3)	−1.16(−5)	2.90(−6)	−3.26(− 9)
4/8	−1.57(−3)	1.55(−3)	−9.10(−6)	2.27(−6)	−2.56(− 9)
5/8	−1.24(−3)	1.23(−3)	−6.60(−6)	1.65(−6)	−1.87(− 9)
6/8	−8.59(−3)	8.51(−4)	−4.27(−6)	1.07(−6)	−1.14(− 9)
7/8	−4.46(−4)	4.42(−4)	−2.10(−6)	5.23(−7)	−6.04(−10)

Table 6.2 $(x(0.5)=16/9)$

	$x_3-\hat{x}$	$x_4-\hat{x}$	$z_3-\hat{x}$	$z_4-\hat{x}$	$w_2-\hat{x}$
1/8	2.74(−6)	−8.60(−6)	−1.28(− 9)	4.75(− 9)	1.52(−10)
2/8	3.22(−6)	−1.00(−5)	−1.37(− 9)	4.90(− 9)	1.21(−10)
3/8	2.93(−6)	−9.11(−6)	−1.17(− 9)	5.36(− 9)	8.77(−11)
4/8	2.40(−6)	−7.45(−6)	−9.16(−10)	3.20(− 9)	6.32(−11)
5/8	1.81(−6)	−5.61(−6)	−6.68(−10)	2.32(− 9)	4.42(−11)
6/8	1.22(−6)	−3.75(−6)	−4.36(−10)	1.51(− 9)	2.86(−11)
7/8	6.11(−7)	−1.89(−6)	−2.16(−10)	9.82(−10)	1.86(−11)

Table 6.3 $(x(0.5) \doteqdot -10.53)$

	z_1	z_2	w_1
1/8	− 0.3948 0250	− 0.3950 0272	0.3949 6268
2/8	− 4.7114 3147	− 4.7120 5668	− 4.7119 3164
3/8	− 8.4517 1162	− 8.4523 0279	− 8.4521 8460
4/8	−10.5361 5338	−10.5362 4491	−10.5362 2603
5/8	−10.1182 7137	−10.1183 1982	−10.1183 1011
6/8	− 7.3812 7323	− 7.3816 4259	− 7.3815 6872
7/8	− 3.3562 7630	− 3.3563 4859	− 3.3562 9075

Table 6.4 $(x(0.5) \doteqdot -10.53)$

	z_3	z_4	w_2
1/8	− 0.3949 6258	−0.3949 6223	−0.3949 6250
2/8	− 4.7119 3147	096	135
3/8	− 8.4521 8498	66	91
4/8	−10.5362 2615	34	19
5/8	−10.1183 1020	41	25
6/8	− 7.3815 6862	54	60
7/8	− 3.3562 9067	0	5

Example 7. The next example, sometimes referred as Troesch's equation, is given by

$$x'' = 10 \sinh(10x), \qquad x(0) = 0, \qquad x(1) = 1.$$

The results, using Newton's method and identically zero starting guess, are shown in Tables 7.1 and 7.2.

Table 7.1 $(x'(0))$

	$z_1'(0)$	$z_2'(0)$	$w_1'(0)$
$n=$ 256	3.528484(−4)	3.596624(−4)	3.582996(−4)
512	3.579589(−4)	3.584231(−4)	3.583303(−4)
1024	3.583070(−4)	3.583450(−4)	3.583374(−4)

	$z_3'(0)$	$z_4'(0)$	$w_2'(0)$
$n=$ 256	3.581797(−4)	3.583061(−4)	3.582100(−4)
512	3.583291(−4)	3.583390(−4)	3.583314(−4)
1024	3.583375(−4)	3.583378(−4)	3.583376(−4)

Table 7.2 $(x'(1))$

	$z_1'(1)$	$z_2'(1)$	$w_1'(1)$
$n=$ 256	171.5186	144.6854	150.0520
512	151.3937	147.2639	148.0900
1024	148.2963	148.3316	148.3246

	$z_3'(1)$	$z_4'(1)$	$w_2'(1)$
$n=$ 256	153.4681	152.0266	153.1249
512	148.9191	148.6158	148.8469
1024	148.4267	148.3938	148.4188

Example 8. The above stated methods are also applicable to the following problem subject to the nonlinear boundary conditions:

$$x'' = x^3-(1+\cos t)^3-\cos t\,,$$

$$x'(0)=0\,,\qquad x'(1)=-x^3(0)\sin 1/(1+\sin 1)^3\,.$$

Table 8.1

	x_1-$\hat{x}$	x_2-$\hat{x}$	z_1-$\hat{x}$	z_2-$\hat{x}$	w_1-$\hat{x}$
0	−2.63(−5)	2.63(−5)	−7.57(−9)	2.52(−9)	−2.29(−12)
1/8	−2.62(−5)	2.62(−5)	−7.65(−9)	1.91(−9)	−1.54(−12)
2/8	−2.60(−5)	2.60(−5)	−7.87(−9)	1.97(−9)	−1.06(−12)
3/8	−2.55(−5)	2.55(−5)	−8.24(−9)	2.06(−9)	−7.81(−13)
4/8	−2.48(−5)	2.47(−5)	−8.82(−9)	2.20(−9)	−6.28(−13)
5/8	−2.36(−5)	2.36(−5)	−9.73(−9)	2.43(−9)	−5.73(−13)
6/8	−2.20(−5)	2.20(−5)	−1.12(−9)	2.80(−9)	−6.00(−13)
7/8	−1.98(−5)	1.98(−5)	−1.37(−9)	3.43(−9)	−7.05(−13)
1	−1.68(−5)	1.68(−5)	−1.79(−9)	4.47(−9)	−8.95(−13)

Table 8.2

	x_1-$\hat{x}$	x_4-$\hat{x}$	z_3-$\hat{x}$	z_4-$\hat{x}$	w_2-$\hat{x}$
0	−1.71(−9)	5.14(−9)	−2.15(−14)	5.00(−14)	−4.49(−15)
1/8	−1.71(−9)	5.12(−9)	−2.04(−14)	5.09(−14)	−3.44(−15)
2/8	−1.69(−6)	5.07(−9)	−2.02(−14)	5.32(−14)	−2.72(−15)
3/8	−1.66(−9)	4.98(−9)	−2.09(−14)	5.73(−14)	−2.26(−11)
4/8	−1.61(−9)	4.83(−9)	−2.22(−14)	6.41(−14)	−1.67(−15)
5/8	−1.54(−9)	4.61(−9)	−2.51(−14)	7.38(−14)	−1.55(−15)
6/8	−1.44(−9)	4.30(−9)	−2.91(−14)	8.86(−14)	−1.06(−15)
7/8	−1.29(−9)	3.86(−9)	−3.51(−14)	1.10(−13)	−5.46(−16)
1	−1.09(−9)	3.27(−9)	−4.40(−14)	1.40(−13)	−2.50(−16)

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