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By

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Abstract

In the present paper we treat an almost symplectic Finsler structure, defined as an alternate, non-degenerate Finsler tensor field of type (0, 2), and especially consider the problem of its integrability.

§ 0. Introduction.

On a differentiable manifold there exist many remarkable geometrical structures, as metrical, conformal, almost complex, almost symplectic, conformal almost symplectic, almost cosymplectic, conformal almost cosymplectic etc. ([2], [7], [12], [13]), whose corresponding Finsler structures have been studied from various standpoints ([9], [11], [3], [1]). It seems to be important to clarify their special geometrical properties much more. For example, if someone wants to study the concept of Finsler analytical dynamics, he will have need of the theory of almost symplectic Finsler structures.

In their recent papers [10, 11], the authors have investigated the metrical Finsler connections and the conformal Finsler connections, as respective compatible ones with a Finsler metric and a conformal Finsler structure. Continued from them, in the present paper we shall treat an almost symplectic Finsler structure, defined as an alternate, non-degenerate Finsler tensor field of type (0, 2).

We first introduce the notion of almost symplectic Finsler structure (§1), and define the notion of almost symplectic Finsler connection on a geometrical way, and study the properties of these notions (§2). And, the structure of the set of all almost symplectic Finsler connections [1] is discussed (§3), and the group of their transformations preserving a non-linear connection gives us the various important invariants (§4). For a 2-form on the tangent bundle $T(M)$ of the base manifold M, we characterize the case when it is closed, using only the Finsler tensor fields (§5), and finally solve the problem of integrability of an almost symplectic Finsler structure, by lifting it to a 2-form on $T(M)$ (§6).

As to the terminology and notations we retain those in our previous joint papers [10, 11], which are essentially based on M. Matsumoto [5, 6].

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§1. The notion of almost symplectic Finsler structure.

Let M be a differentiable manifold of dimension $2n$. $x=(x^i)$ and $y=(y^i)$ denote a point of M and a supporting element respectively. As a geometrical Finsler object on M, we give

Definition 1.1. A Finsler tensor field a_{ij} of type (0, 2) on a differentiable manifold M is called an *almost symplectic Finsler structure on M*, if it is alternate and non-degenerate:

$$a_{ij} = -a_{ji}\,, \tag{1.1}$$

$$\det(a_{ij}) \neq 0\,. \tag{1.2}$$

Example 1. Let ω_i be a Finsler covariant vector field on M. The Finsler tensor field $\omega_{ij}=\partial\omega_i/\partial y^j-\partial\omega_j/\partial y^i$ defines an almost symplectic Finsler structure on M, if $\det(\omega_{ij})\neq 0$.

Example 2. If a Finsler space M is an almost Hermitian space, it admits an almost symplectic Finsler structure. In fact, let f^i_j be an almost complex structure such that

$$f^i_r f^r_j = -\delta^i_j\,, \qquad g_{rs}f^r_i f^s_j = g_{ij}\,, \tag{1.3}$$

where g_{ij} is the fundamental tensor field. Then, the Finsler tensor field $a_{ij}=g_{ir}f^r_j$ satisfies $a_{ij}=-a_{ji}$, $\det(a_{ij})\neq 0$.

The latter interesting example was communicated by Y. Ichijyō.

Given an almost symplectic Finsler structure a_{ij}, we may associate Obata's operators:

$$\Theta^{ir}_{sj} = \frac{1}{2}(\delta^i_s\delta^r_j - a_{sj}a^{ir})\,, \qquad \Theta^{*ir}_{sj} = \frac{1}{2}(\delta^i_s\delta^r_j + a_{sj}a^{ir})\,, \tag{1.4}$$

where (a^{ij}) is the inverse matrix of (a_{ij}):

$$a_{ij}a^{jk} = \delta^k_i\,. \tag{1.5}$$

a^{ij} is also alternate. Obata's operators have the same properties as ones associated with the Finsler space [10].

§2. Almost symplectic Finsler connections.

A Finsler connection $F\Gamma$ on a differentiable manifold M is, by the third definition of M. Matsumoto [5, 6], a triad of a V-connection Γ_V in the linear frame bundle $L(M)$, a non-linear connection N in the tangent bundle $T(M)$, and a vertical connection Γ^v in the Finsler bundle $F(M)$. Let F^i_{jk}, N^i_k and C^i_{jk} be the respective coefficients of Γ_V, N and Γ^v.

For a Finsler tensor field, e.g. K^i_j, the h- and v-covariant derivatives with respect to $F\Gamma$ are given by

(2.1) $K^i_{j|k} = \delta K^i_j/\delta x^k + K^m_j F^i_{mk} - K^i_m F^m_{jk}, \qquad K^i_j|_k = \partial K^i_j/\partial y^k + K^m_j C^i_{mk} - K^i_m C^m_{jk},$

where $\delta/\delta x^k = \partial/\partial x^k - N^m_k \partial/\partial y^m$.

The Ricci identities, applied to a Finsler tensor field a_{ij}, are

$$(2.2)\quad \begin{cases} a_{ij|k|l} - a_{ij|l|k} = -a_{mj}R^m_{ikl} - a_{im}R^m_{jkl} - a_{ij|m}T^m_{kl} - a_{ij}|_m R^m_{kl}, \\ a_{ij|k}|_l - a_{ij}|_{l|k} = -a_{mj}P^m_{ikl} - a_{im}P^m_{jkl} - a_{ij|m}C^m_{kl} - a_{ij}|_m P^m_{kl}, \\ a_{ij}|_k|_l - a_{ij}|_l|_k = -a_{mj}S^m_{ikl} - a_{im}S^m_{jkl} - a_{ij}|_m S^m_{kl}, \end{cases}$$

where five torsion tensor fields T^i_{jk}, R^i_{jk}, C^i_{jk}, P^i_{jk}, S^i_{jk} and three curvature tensor fields R^i_{jkl}, P^i_{jkl}, S^i_{jkl} appear.

Let $C(x^i(t))$ be a differentiable curve in M and $\tilde{C}(x^i(t), y^i(t))$ a differentiable curve in $T(M)$ mapped on C by the canonical projection of $T(M)$. For a given Finsler connection $F\Gamma$, a tangent vector field $X^i(t)$ along C is called *parallel* along C with respect to $\tilde{C}$, if

$$(2.3)\qquad dX^i/dt + F^i_{jk}X^j(dx^k/dt) + C^i_{jk}X^j(\delta y^k/dt) = 0,$$

where $$\delta y^k = dy^k + N^k_m dx^m.$$

Now and in the following, let an almost symplectic Finsler structure a_{ij} be given on M. For two Finsler vector fields X, Y,

$$(2.4)\qquad a(X,Y) = a_{ij}X^iY^j$$

is a Finsler scalar field. Then we have

Definition 2.1. A Finsler connection $F\Gamma$ is called *almost symplectic* with respect to a_{ij}, if $a(X, Y)$ is preserved when $X(t)$, $Y(t)$ are parallel along any C with respect to any $\tilde{C}$.

Theorem 2.1. *The necessary and sufficient conditions that a Finsler connection $F\Gamma$ be almost symplectic with respect to a_{ij} are*

$$(2.5)\qquad a_{ij|k} = 0, \qquad a_{ij}|_k = 0.$$

Theorem 2.2. *The Finsler tensor fields $\Theta^{*ir}_{sj}R^s_{rkl}$, $\Theta^{*ir}_{sj}P^s_{rkl}$, $\Theta^{*ir}_{sj}S^s_{rkl}$ and their h- and v-covariant derivatives of every order vanish, for every almost symplectic Finsler connection $F\Gamma$ with respect to a_{ij}.*

Proof. Applying the Ricci identities (2.2) to a_{ij}, and remarking that Obata's operators are covariantly constant, we get the statement.

§3. The set of almost symplectic Finsler connections.

Starting from a fixed Finsler connection $\overset{0}{F\Gamma}$, all almost symplectic Finsler connections are obtained in the same manner as in the previous papers [10, 11].

Theorem 3.1. *The set of all almost symplectic Finsler connections is given by*

$$
(3.1)\quad
\begin{cases}
N^i_k = \overset{0}{N}{}^i_k - X^i_k ,\\[2mm]
F^{\ i}_{j\ k} = \overset{0}{F}{}^{\ i}_{j\ k} + \overset{0}{C}{}^{\ i}_{j\ m} X^m_k + \dfrac{1}{2} a^{im}(a_{mj\overset{0}{|}k} + a_{mj}\overset{0}{|}_p X^p_k) + \Theta^{ir}_{sj} X^{\ s}_{r\ k} ,\\[2mm]
C^{\ i}_{j\ k} = \overset{0}{C}{}^{\ i}_{j\ k} + \dfrac{1}{2} a^{im} a_{mj}\overset{0}{|}_k + \Theta^{ir}_{sj} Y^{\ s}_{r\ k} ,
\end{cases}
$$

where $\overset{0}{F\Gamma}$ is a fixed Finsler connection, $\overset{0}{|}$ and $\overset{0}{|}$ denote the h- and v-covariant differentiations with respect to $\overset{0}{F\Gamma}$, and X^i_k, $X^{\ i}_{j\ k}$, $Y^{\ i}_{j\ k}$ are arbitrary Finsler tensor fields.

This result is due to Gh. Atanasiu and I. Ghinea [1], where the inverse matrix (a^{ij}) of (a_{ij}) is defined by

$$
(1.5')\qquad a_{ij}a^{ik} = \delta^k_j ,
$$

and so the above a^{ik} means a^{ki} in the present paper.

Putting $X^i_k = X^{\ i}_{j\ k} = Y^{\ i}_{j\ k} = 0$ in Theorem 3.1 we have an example of an almost symplectic Finsler connection, which corresponds to the Kawaguchi metrical Finsler connection derived from $\overset{0}{F\Gamma}$ in a Finsler space ([10]).

Theorem 3.2. *Let $\overset{0}{F\Gamma}$ be a fixed Finsler connection. Then, the following Finsler connection $F\Gamma$ is almost symplectic:*

$$
(3.2)\qquad N^i_k = \overset{0}{N}{}^i_k ,\ F^{\ i}_{j\ k} = \overset{0}{F}{}^{\ i}_{j\ k} + \frac{1}{2} a^{im} a_{mj\overset{0}{|}k} ,\ C^{\ i}_{j\ k} = \overset{0}{C}{}^{\ i}_{j\ k} + \frac{1}{2} a^{im} a_{mj}\overset{0}{|}_k .
$$

On the other hand, if we take an almost symplectic Finsler connection as $\overset{0}{F\Gamma}$ in Theorem 3.1, we have

Theorem 3.3. *Let $\overset{0}{F\Gamma}$ be a fixed almost symplectic Finsler connection. Then, the set of all almost symplectic Finsler connections is given by*

$$
(3.3)\quad
\begin{cases}
N^i_k = \overset{0}{N}{}^i_k - X^i_k ,\\
F^{\ i}_{j\ k} = \overset{0}{F}{}^{\ i}_{j\ k} + \overset{0}{C}{}^{\ i}_{j\ m} X^m_k + \Theta^{ir}_{sj} X^{\ s}_{r\ k} ,\\
C^{\ i}_{j\ k} = \overset{0}{C}{}^{\ i}_{j\ k} + \Theta^{ir}_{sj} Y^{\ s}_{r\ k} ,
\end{cases}
$$

where X^i_k, X^i_{jk}, Y^i_{jk} are arbitrary Finsler tensor fields.

The set in Theorem 3.3 has the following subset. We denote by $F\Gamma(\overset{0}{N})$ a Finsler connection having $\overset{0}{N}$ as the non-linear connection.

Theorem 3.4. *Let $\overset{0}{F\Gamma}$ be a fixed almost symplectic Finsler connection. Then, the set of all almost symplectic Finsler connections $F\Gamma(\overset{0}{N})$ is given by*

$$
(3.4)\qquad N^i_k = \overset{0}{N}{}^i_k ,\quad F^{\ i}_{j\ k} = \overset{0}{F}{}^{\ i}_{j\ k} + \Theta^{ir}_{sj} X^{\ s}_{r\ k} ,\quad C^{\ i}_{j\ k} = \overset{0}{C}{}^{\ i}_{j\ k} + \Theta^{ir}_{sj} Y^{\ s}_{r\ k} ,
$$

where $X^{\ i}_{j\ k}$, $Y^{\ i}_{j\ k}$ are arbitrary Finsler tensor fields.

§ 4. The group of transformations of almost symplectic Finsler connections.

Let us consider the transformations $F\Gamma(N) \to F\bar{\Gamma}(N)$ of almost symplectic Finsler connections [8], which preserve the non-linear connection N. Owing to Theorem 3.4 they are given by

$$(4.1) \qquad \bar{N}^i_k = N^i_k, \qquad \bar{F}^i_{jk} = F^i_{jk} + \Theta^{ir}_{sj} X^s_{rk}, \qquad \bar{C}^i_{jk} = C^i_{jk} + \Theta^{ir}_{sj} Y^s_{rk},$$

where X^i_{jk}, Y^i_{jk} are arbitrarily given Finsler tensor fields.

Evidently we have

Theorem 4.1. *The set of all transformations* (4.1) *and the mapping product form an abelian group* G_{as}, *which is isomorphic to the additive group of the pairs of Finsler tensor fields* $(\Theta^{ir}_{sj} X^s_{rk}, \Theta^{ir}_{sj} Y^s_{rk})$.

We shall pay attention to the invariants of the group G_{as}. The torsion tensor fields T^i_{jk}, S^i_{jk}, R^i_{jk}, P^i_{jk} are expressed as follows:

$$(4.2) \qquad \begin{cases} T^i_{jk} = \mathfrak{A}_{jk}\{F^i_{jk}\}, & S^i_{jk} = \mathfrak{A}_{jk}\{C^i_{jk}\}, \\ R^i_{jk} = \mathfrak{A}_{jk}\{\delta N^i_j/\delta x^k\}, & P^i_{jk} = \partial N^i_j/\partial y^k - F^i_{kj}, \end{cases}$$

where $\mathfrak{A}_{jk}\{\cdots\}$ denotes the alternate summation: $\mathfrak{A}_{jk}\{A_{jk}\} = A_{jk} - A_{kj}$. The torsion tensor field R^i_{jk} and the Finsler tensor field t^i_{jk} defined by

$$(4.3) \qquad t^i_{jk} = \mathfrak{A}_{jk}\{\partial N^i_j/\partial y^k\}$$

are called the *curvature* and *torsion tensor fields* of the non-linear connection N respectively ([3]). Since they depend on N only, they are invariants of G_{as}.

We make here some notations:

$$(4.4) \qquad \begin{cases} t^*_{ijk} = \mathfrak{S}_{ijk}\{a_{im} t^m_{jk}\}, & R^*_{ijk} = \mathfrak{S}_{ijk}\{a_{im} R^m_{jk}\}, \\ T^*_{ijk} = \mathfrak{S}_{ijk}\{a_{im} T^m_{jk}\}, & S^*_{ijk} = \mathfrak{S}_{ijk}\{a_{im} S^m_{jk}\}, \end{cases}$$

where $\mathfrak{S}_{ijk}\{\cdots\}$ denotes the cyclic summation: $\mathfrak{S}_{ijk}\{A_{ijk}\} = A_{ijk} + A_{jki} + A_{kij}$, and

$$(4.5) \qquad \begin{cases} \overset{1}{\kappa}_{ijk} = a_{km} T^m_{ij} + \mathfrak{A}_{ij}\{a_{im} P^m_{jk}\}, & \overset{2}{\kappa}_{ijk} = a_{im} S^m_{jk} + \mathfrak{A}_{jk}\{a_{km} C^m_{ij}\}, \\ \overset{3}{\kappa}_{ijk} = \mathfrak{A}_{jk}\{a_{km} P^m_{ij}\}, & \overset{4}{\kappa}_{ijk} = \mathfrak{A}_{ij}\{a_{im} C^m_{jk}\}. \end{cases}$$

It is noted that t^*_{ijk}, R^*_{ijk}, T^*_{ijk}, S^*_{ijk} are alternate, and $\overset{a}{\kappa}_{ijk}$ for $a=1, 4$ (resp. $a=2, 3$) are alternate with respect to i, j (resp, j, k).

By direct calculations we have

Theorem 4.2. *The Finsler tensor fields* t^i_{jk}, R^i_{jk}, t^*_{ijk}, R^*_{ijk}, T^*_{ijk}, S^*_{ijk} *and* $\overset{a}{\kappa}_{ijk}$ $(a=1, 2, 3, 4)$ *are invariants of the group* G_{as}.

Proposition 4.1. *Between the invariants in Theorem 4.2 there exist the following relations:*

$$(4.6)\quad \begin{cases} \mathfrak{S}_{ijk}\{\overset{1}{\kappa}_{ijk}\} = 2T^*_{ijk}+t^*_{ijk}, & \mathfrak{S}_{ijk}\{\overset{2}{\kappa}_{ijk}\} = 2S^*_{ijk}, \\ \mathfrak{S}_{ijk}\{\overset{3}{\kappa}_{ijk}\} = T^*_{ijk}+t^*_{ijk}, & \mathfrak{S}_{ijk}\{\overset{4}{\kappa}_{ijk}\} = S^*_{ijk}, \\ \overset{2}{\kappa}_{ijk}+\overset{4}{\kappa}_{jki} = S^*_{ijk}, & \overset{1}{\kappa}_{ijk}+\overset{4}{\kappa}_{kij}=t^*_{ijk}+T^*_{ijk}-a_{km}t^m_{ij}. \end{cases}$$

Theorem 4.3. *Let N be a non-linear connection in the tangent bundle $T(M)$.*

(1) *The invariant T^*_{ijk} (resp. S^*_{ijk}) vanishes if and only if there exists an almost symplectic Finsler connection $F\Gamma(N)$ with $T^i_{jk}=0$ (resp. $S^i_{jk}=0$).*

(2) *The invariants T^*_{ijk} and S^*_{ijk} vanish if and only if there exists an almost symplectic Finsler connection $F\Gamma(N)$ with $T^i_{jk}=S^i_{jk}=0$.*

Proof. If we put $X^i_{jk}=\alpha T^i_{jk}$ in (4.1), where α is a real number, we have

$$\bar{T}^i_{jk} = T^i_{jk}+\mathfrak{A}_{jk}\{\Theta^{ir}_{sj}X^s_{rk}\} = \left(1+\frac{3}{2}\alpha\right)T^i_{jk} + \frac{\alpha}{2}a^{ir}T^*_{rjk}.$$

Taking $\alpha=-2/3$, $T^*_{ijk}=0$ implies $\bar{T}^i_{jk}=0$. The converse is evident. The statement about S^*_{ijk} is proved in the same way. (2) follows from the independence of two procedures in (1).

Paying attention to $a_{ij}|_k=\partial a_{ij}/\partial y^k-\overset{4}{\kappa}_{ijk}=0$, Proposition 4.1 tells us the condition that a_{ij} be a usual almost symplectic structure.

Theorem 4.4. *An almost symplectic Finsler structure a_{ij} does not depend on the supporting element y, if and only if $\overset{4}{\kappa}_{ijk}=0$, which is equivalent to $\overset{2}{\kappa}_{ijk}=0$. In this case it holds $S^*_{ijk}=0$.*

For the later use we have

Proposition 4.2. (1) *If $\overset{1}{\kappa}_{ijk}=0$ then $2T^*_{ijk}=-t^*_{ijk}$.*

(2) *Assume that $\overset{1}{\kappa}_{ijk}+a_{km}R^m_{ij}=0$, $\overset{2}{\kappa}_{ijk}+\overset{3}{\kappa}_{ijk}=0$, and $S^*_{ijk}=0$. Then, $R^*_{ijk}=0$ is equivalent to $T^*_{ijk}=0$.*

(3) *Assume that $\overset{4}{\kappa}_{ijk}+\alpha\overset{1}{\kappa}_{ijk}+a_{km}R^m_{ij}=0$, $\alpha\overset{2}{\kappa}_{ijk}+\overset{3}{\kappa}_{ijk}=0$, and $S^*_{ijk}=0$, where $\alpha\neq\pm1$ is a real number. Then, $T^*_{ijk}+\alpha R^*_{ijk}=0$ is equivalent to $T^*_{ijk}=0$.*

Forming the cyclic summations of each of the assumed formulas, the proof follows from Proposition 4.1.

In the following paragraphs we shall study the cases when some invariants in Theorem 4.2 vanish, related with the integrability of the structure a_{ij}.

§ 5. 2-forms on the tangent bundle.

Let $\Lambda^k(T(M))$ be the $\mathfrak{F}$-module of all k-forms on the tangent bundle $T(M)$, where $\mathfrak{F}$ is the ring of all differentiable functions on $T(M)$. If a non-linear connection N is given in $T(M)$, then $(dx^i, \delta y^i)$ makes a local basis of $\Lambda^1(T(M))$, which is dual to $(\delta/\delta x^i, \partial/\partial y^i)$, where

$$\delta y^i = dy^i + N^i_m dx^m, \tag{5.1}$$

$$\delta/\delta x^i = \partial/\partial x^i - N^m_i \partial/\partial y^m. \tag{5.2}$$

The differential of $f \in \mathfrak{F}$ is written as

$$df = (\delta f/\delta x^i)\, dx^i + (\partial f/\partial y^i)\, \delta y^i, \tag{5.3}$$

and the exterior differential of δy^i is given by

$$d(\delta y^i) = \frac{1}{2} R^i_{jk} dx^k \wedge dx^j + (\partial N^i_j/\partial y^k)\, \delta y^k \wedge dx^j. \tag{5.4}$$

If we express $\omega \in \Lambda^1(T(M))$ in the form

$$\omega = \varpi_i dx^i + \dot{\omega}_i \delta y^i, \tag{5.5}$$

the exterior differential $d\omega$ is given by

$$d\omega = \frac{1}{2} \varpi_{ij} dx^j \wedge dx^i + \omega_{ij} \delta y^j \wedge dx^i + \frac{1}{2} \dot{\omega}_{ij} \delta y^j \wedge \delta y^i, \tag{5.6}$$

where

$$\begin{cases} \varpi_{ij} = \delta\varpi_i/\delta x^j - \delta\varpi_j/\delta x^i + R^m_{ij}\, \dot{\omega}_m, \\ \omega_{ij} = \partial\varpi_i/\partial y^j - \delta\dot{\omega}_j/\delta x^i + (\partial N^m_i/\partial y^j)\, \dot{\omega}_m, \\ \dot{\omega}_{ij} = \partial\dot{\omega}_i/\partial y^j - \partial\dot{\omega}_j/\partial y^i. \end{cases} \tag{5.7}$$

ϖ_{ij}, ω_{ij}, $\dot{\omega}_{ij}$ are Finsler tensor fields. In fact, as the tensorial expressions we have

Proposition 5.1. *If a Finsler connection $F\Gamma(N)$ is given, ϖ_{ij}, ω_{ij}, $\dot{\omega}_{ij}$ have the expressions*

$$\begin{cases} \varpi_{ij} = \varpi_{i|j} - \varpi_{j|i} + T^m_{ij} \varpi_m + R^m_{ij}\, \dot{\omega}_m, \\ \omega_{ij} = \varpi_i|_j - \dot{\omega}_{j|i} + C^m_{\cdot j} \varpi_m + P^m_{ij}\, \dot{\omega}_m, \\ \dot{\omega}_{ij} = \dot{\omega}_i|_j - \dot{\omega}_j|_i + S^m_{ij} \dot{\omega}_m. \end{cases} \tag{5.7'}$$

In general, $\omega \in \Lambda^2(T(M))$ is written in the form

$$\omega = \frac{1}{2} \tilde{a}_{ij} dx^i \wedge dx^j + \tilde{b}_{ij} dx^i \wedge \delta y^j + \frac{1}{2} \tilde{c}_{ij} \delta y^i \wedge \delta y^j, \tag{5.8}$$

where $\tilde{a}_{ij} = -\tilde{a}_{ji}$, $\tilde{c}_{ij} = -\tilde{c}_{ji}$. The exterior differential $d\omega$ is given by

$$\begin{aligned} d\omega = {} & \frac{1}{6} \overset{1}{\omega}_{ijk} dx^i \wedge dx^j \wedge dx^k + \frac{1}{2} \overset{2}{\omega}_{ijk} dx^i \wedge dx^j \wedge \delta y^k \\ & + \frac{1}{2} \overset{3}{\omega}_{ijk} dx^i \wedge \delta y^j \wedge \delta y^k + \frac{1}{6} \overset{4}{\omega}_{ijk} \delta y^i \wedge \delta y^j \wedge \delta y^k, \end{aligned} \tag{5.9}$$

where

$$
(5.10)\quad
\begin{cases}
\overset{1}{\omega}_{ijk} = \mathfrak{S}_{ijk}\{\delta\tilde{a}_{ij}/\delta x^k + \tilde{b}_{im}R^{\ m}_{j\ k}\}\ , \\
\overset{2}{\omega}_{ijk} = \partial\tilde{a}_{ij}/\partial y^k + \tilde{c}_{km}R^{\ m}_{i\ j} + \mathfrak{A}_{ij}\{\delta\tilde{b}_{jk}/\delta x^i + \tilde{b}_{im}\partial N^{\ m}_{j}/\partial y^k\}\ , \\
\overset{3}{\omega}_{ijk} = \delta\tilde{c}_{jk}/\delta x^i + \mathfrak{A}_{jk}\{\partial\tilde{b}_{ij}/\partial y^k + \tilde{c}_{km}\partial N^{\ m}_{i}/\partial y^j\}\ , \\
\overset{4}{\omega}_{ijk} = \mathfrak{S}_{ijk}\{\partial\tilde{c}_{ij}/\partial y^k\}\ .
\end{cases}
$$

If we calculate $d^2\omega=0$ from (5.9), we have

Theorem 5.1. *The coefficients* $\overset{a}{\omega}_{ijk}(a=1, 2, 3, 4)$ *of* (5.9) *satisfy the following identities*:

$$
(5.11)\quad
\begin{cases}
\mathfrak{A}_{ij}\{\delta\overset{1}{\omega}_{jkl}/\delta x^i\} - \mathfrak{A}_{kl}\{\delta\overset{1}{\omega}_{ijk}/\delta x^l\} - \mathfrak{S}_{ijk}\{\overset{2}{\omega}_{ijm}R^{\ m}_{k\ l} - \overset{2}{\omega}_{lim}R^{\ m}_{j\ k}\} = 0\ , \\
-\partial\overset{1}{\omega}_{ijk}/\partial y^l + \mathfrak{S}_{ijk}\{\delta\overset{2}{\omega}_{ijl}/\delta x^k - \overset{2}{\omega}_{ijm}\partial N^{\ m}_{k}/\partial y^l + \overset{3}{\omega}_{iml}R^{\ m}_{j\ k}\} = 0\ , \\
\mathfrak{A}_{jk}\{\partial\overset{2}{\omega}_{ijl}/\partial y^k\} + \mathfrak{A}_{ij}\{\delta\overset{3}{\omega}_{jkl}/\delta x^i - \overset{3}{\omega}_{imk}\partial N^{\ m}_{j}/\partial y^l + \overset{3}{\omega}_{iml}\partial N^{\ m}_{j}/\partial y^k\} + \overset{4}{\omega}_{mkl}R^{\ m}_{j\ i} = 0\ , \\
\delta\overset{4}{\omega}_{ljk}/\delta x^i - \mathfrak{S}_{jkl}\{\partial\overset{3}{\omega}_{ijk}/\partial y^l + \overset{4}{\omega}_{mjk}\partial N^{\ m}_{i}/\partial y^l\} = 0\ , \\
\mathfrak{A}_{ij}\{\partial\overset{4}{\omega}_{jkl}/\partial y^i\} - \mathfrak{A}_{kl}\{\partial\overset{4}{\omega}_{ijk}/\partial y^l\} = 0\ .
\end{cases}
$$

$\overset{a}{\omega}_{ijk}$ are Finsler tensor fields. In fact, as the tensorial expressions we have

Proposition 5.2. *If a Finsler connection* $F\Gamma(N)$ *is given,* $\overset{a}{\omega}_{ijk}(a=1, 2, 3, 4)$ *have the expressions*

$$
(5.10')\quad
\begin{cases}
\overset{1}{\omega}_{ijk} = \mathfrak{S}_{ijk}\{\tilde{a}_{ij|k} + \tilde{a}_{im}T^{\ m}_{j\ k} + \tilde{b}_{im}R^{\ m}_{j\ k}\}\ , \\
\overset{2}{\omega}_{ijk} = \tilde{a}_{ij}|_k + \tilde{b}_{mk}T^{\ m}_{j\ i} + \tilde{c}_{km}R^{\ m}_{i\ j} + \mathfrak{A}_{ij}\{\tilde{b}_{jk|i} + \tilde{a}_{im}C^{\ m}_{j\ k} + \tilde{b}_{im}P^{\ m}_{j\ k}\}\ , \\
\overset{3}{\omega}_{ijk} = \tilde{b}_{im}S^{\ m}_{j\ k} + \tilde{c}_{jk|i} + \mathfrak{A}_{jk}\{\tilde{b}_{ij}|_k + \tilde{b}_{mj}C^{\ m}_{i\ k} + \tilde{c}_{mj}P^{\ m}_{i\ k}\}\ , \\
\overset{4}{\omega}_{ijk} = \mathfrak{S}_{ijk}\{\tilde{c}_{ij}|_k + \tilde{c}_{im}S^{\ m}_{j\ k}\}\ .
\end{cases}
$$

For $\omega\in\Lambda^2(T(M))$ written in the form (5.8) we put

$$
(5.12)\quad A = \begin{bmatrix} \tilde{a}_{ij} & \tilde{b}_{ij} \\ -\tilde{b}_{ji} & \tilde{c}_{ij} \end{bmatrix}.
$$

Definition 5.1. A 2-form $\omega\in\Lambda^2(T(M))$, for which the matrix A is non-degenerate, is called *integrable* if $d\omega=0$.

One knows [4] that, in this case, $d\omega=0$ characterizes the fact that $\omega\in\Lambda^2(T(M))$ has the property that there exists a local coordinate system in $T(M)$ in which, in the natural basis, the coefficients of w are all constant.

Theorem 5.2. *A 2-form* $\omega\in\Lambda^2(T(\mathrm{M}))$, *for which the matrix* A *is non-degenerate, is integrable if and only if the Finsler tensor fields* $\overset{a}{\omega}_{ijk}$ $(a=1, 2, 3, 4)$ *vanish.*

It is easily seen that for $\omega\in\Lambda^2(T(M))$ the property det $A\neq0$ does not on the choice of the local basis. A 2-form $\omega\in\Lambda^2(T(M))$ with $\det A\neq0$ is called *non-degenerate*, and determines an almost symplectic structure on $T(M)$.

When $\tilde{b}_{ij}=0$, then $\tilde{a}_{ij}$ and $\tilde{c}_{ij}$ give two almost symplectic Finsler structures on M. When $\tilde{a}_{ij}=0$ or $\tilde{c}_{ij}=0$, and $\tilde{b}_{ij}=-\tilde{b}_{ji}$, then $\tilde{b}_{ij}$ gives an almost symplectic Finsler structure on M.

Conversely, let a_{ij} be a given almost symplectic Finsler structure on M. Then the 2-forms on $T(M)$ defined by $\omega=1/2\,a_{ij}dx^i\wedge dx^j+1/2\,a_{ij}\delta y^i\wedge\delta y^j$, $\omega=a_{ij}dx^i\wedge\delta y^j$, etc. determine almost symplectic structures on $T(M)$. The integrability of each of these 2-forms gives some type of integrability for the given almost symplectic Finsler structure a_{ij}. We discuss these cases in the following last section.

§6. Integrabilities of an almost symplectic Finsler structure.

Assume that a non-linear connection N be given in the tangent bundle $T(M)$. Then, an almost symplectic Finsler structure a_{ij} on the base manifold M is lifted to a 2-form w on $T(M)$ in various ways. We consider the following w of three single types I, II, III and four combined types I+II, I+III, II+III, I+αII+III, where $\alpha\neq\pm1$ is a real number:

$$\omega=\frac{1}{2}\tilde{a}_{ij}dx^i\wedge dx^j+\tilde{b}_{ij}dx^i\wedge\delta y^j+\frac{1}{2}\tilde{c}_{ij}\delta y^i\wedge\delta y^j\,,$$

where

	$\tilde{a}_{ij}$	$\tilde{b}_{ij}$	$\tilde{c}_{ij}$
I	a_{ij}	0	0
II	0	a_{ij}	0
III	0	0	a_{ij}
I+II	a_{ij}	a_{ij}	0
I+III	a_{ij}	0	a_{ij}
II+III	0	a_{ij}	a_{ij}
I+αII+III	a_{ij}	αa_{ij}	a_{ij}

Proposition 6.1. *Each 2-form w of types* II, I+II, I+III, II+III *and* I+αII+III *is non-degenerate, and defines an almost symplectic structure on* $T(M)$.

Proposition 6.2. *The coefficients* $\overset{a}{\omega}_{ijk}$ $(a=1, 2, 3, 4)$ *of the exterior differentials of the 2-forms given in Proposition* 6.1 *are invariants of the group* G_{as}, *and are given in the following table*:

	$\overset{1}{\omega}_{ijk}$	$\overset{2}{\omega}_{ijk}$	$\overset{3}{\omega}_{ijk}$	$\overset{4}{\omega}_{ijk}$
II	$R^*{}_{ijk}$	$\overset{1}{\kappa}_{ijk}$	$\overset{2}{\kappa}_{ijk}$	0
I+II	$T^*{}_{ijk}+R^*{}_{ijk}$	$\overset{1}{\kappa}_{ijk}+\overset{4}{\kappa}_{ijk}$	$\overset{2}{\kappa}_{ijk}$	0
I+III	$T^*{}_{ijk}$	$\overset{4}{\kappa}_{ijk}+a_{km}R^m_{ij}$	$\overset{3}{\kappa}_{ijk}$	$S^*{}_{ijk}$
II+III	$R^*{}_{ijk}$	$\overset{1}{\kappa}_{ijk}+a_{km}R^m_{ij}$	$\overset{2}{\kappa}_{ijk}+\overset{3}{\kappa}_{ijk}$	$S^*{}_{ijk}$
I+αII+III	$T^*{}_{ijk}+\alpha R^*{}_{ijk}$	$\overset{4}{\kappa}_{ijj}+\alpha\overset{1}{\kappa}_{ijk}+a_{km}R^m_{ij}$	$\alpha\overset{2}{\kappa}_{ijk}+\overset{3}{\kappa}_{ijk}$	$S^*{}_{ijk}$

Proof. Calculating directly from Proposition 5.2 we have

	$\overset{1}{\omega}_{ijk}$	$\overset{2}{\omega}_{ijk}$	$\overset{3}{\omega}_{ijk}$	$\overset{4}{\omega}_{ijk}$
I	$T^*{}_{ijk}$	$\overset{4}{\kappa}_{ijk}$	0	0
II	$R^*{}_{ijk}$	$\overset{1}{\kappa}_{ijk}$	$\overset{2}{\kappa}_{ijk}$	0
III	0	$a_{km}R^m_{ij}$	$\overset{3}{\kappa}_{ijk}$	$S^*{}_{ijk}$

Since $\overset{a}{\omega}_{ijk}$ are linear combinations of $\tilde{a}_{ij}$, $\tilde{b}_{ij}$, $\tilde{c}_{ij}$, the expressions for the combined types are obtained as the linear combinations of the ones for I, II, III.

Theorem 6.1. *The Finsler tensor fields* $\overset{a}{\omega}_{ijk}$ $(a=1, 2, 3, 4)$ *given in Proposition* 6.2 *satisfy the equation* (5.11).

Now, corresponding to Definition 5.1 we have

Definition 6.1. An almost symplectic Finsler structure a_{ij} on a differentiable manifold M is called *integrable* of the type II, I+II, I+III, II+III or I+αII+III, if there exists an almost symplectic Finsler connection $F\Gamma(N)$ such that the corresponding lifted 2-form on $T(M)$ is integrable.

Then, from Theorems 4.3 and 4.4 and Proposition 4.2 we have

Theorem 6.2. *An almost symplectic Finsler structure* a_{ij} *on a differentiable manifold* M *is integrable of the type* II, I+II, I+III, II+III *or* I+αII+III, *if and only if there exists an almost symplectic Finsler connection* $F\Gamma(N)$ *satisfying the following conditions in each type*:

II : $R^*{}_{ijk}=0$, $\overset{1}{\kappa}_{ijk}=0$, a_{ij} *does not depend on the supporting element* y.

I+II : $-2R^*{}_{ijk}+t^*{}_{ijk}=0$, $\overset{1}{\kappa}_{ijk}=0$, a_{ij} *does not depend on the supporting element* y.

I+III : $T^i_{jk}=S^i_{jk}=0$, $\overset{4}{\kappa}_{ijk}+a_{km}R^m_{ij}=0$, $\overset{3}{\kappa}_{ijk}=0$.

II+III : $T^i_{jk}=S^i_{jk}=0$, $\overset{1}{\kappa}_{ijk}+a_{km}R^m_{ij}=0$, $\overset{2}{\kappa}_{ijk}+\overset{3}{\kappa}_{ijk}=0$.

I+αII+III : $T^i_{jk}=S^i_{jk}=0$, $\overset{4}{\kappa}_{ijk}+\alpha\overset{1}{\kappa}_{ijk}+a_{km}R^m_{ij}=0$, $\alpha\overset{2}{\kappa}_{ijk}+\overset{3}{\kappa}_{ijk}=0$.

Finally, we note that the above integrabilities are reduced to two types II, εI+III, where $\varepsilon\neq 0$ is a constant. In fact, the transformation $N^i_k\to\bar{N}^i_k=N^i_k-X^i_k$ of non-linear connections implies $\bar{\delta}y^i=\delta y^i-X^i_j dx^j$. Taking $X^i_k=-1/2\,\delta^i_k$, $X^i_k=-\delta^i_k$ or $X^i_k=-\alpha\delta^i_k$, a 2-form of the type I+II, II+III or I+αII+III is reduced to a 2-form of the type II, I+III or $(1-\alpha^2)/2$ I+III respectively.

If an almost symplectic Finsler structure a_{ij} really depends on the supporting element, there does not exist the integrability of the type II, which shows the importance of the lift of the type εI+III. For the case we have

Theorem 6.3. *Let an almost symplectic Finsler structure* a_{ij} *be integrable of the* *type* εI+III. a_{ij} *does not depend on the supporting element, if and only if the concerned non-linear connection is integrable*: $R^{i}_{j\,k}=0$.

The proof follows from $\varepsilon\overset{4}{\kappa}_{ijk}+a_{km}R^{m}_{i\,j}=0$, and Theorem 4.4.

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