NORMAL SUBGROUPS OF MULTIPLY TRANSITIVE PERMUTATION GROUPS

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1. Introduction

Let G be a t-fold transitive group on $\Omega=\{1, 2,\cdots, n\}$ with $t\geqq 2$, $H(\neq 1)$ a normal subgroup of G and assume that $n>t+1$. The following is a classical result of Jordan.

Proposition 1 (Jordan, [8]). *Under the above assumptions H must be $(t-1)$-fold transitive.*

There are several results on t-told transitivity of H by Wagner [7] for $t=3$, Ito [4] and Saxl [6] for $t=4$ and Bannai [3] for $t\geqq 4$, and it has been conjectured that if $t\geqq 4$ then H must be t-told transitive.

The purpose of this paper is to prove the following

Theorem. *Let G and H be as above, and assume that $t\geqq 4$ and $n>t+1$.* Let Δ_1, $\Delta_2, \cdots, \Delta_s$ *be the orbits of $H_{1,2,\cdots,t-1}$ on $\Omega-\{1, 2, \cdots, t-1\}$. Assume that q is an odd prime which divides $(t-1)$ and $n\equiv r(\mathrm{mod}\ q)$, $0<r<q$.*
Then s divides r, and if $r\geqq 2$ then s is less than r. In particular if $r=1$ or a prime then H is t-fold transitive.

Notation. For a set X, let $|X|$ denote the number of elements of X. For a subset X of a group G, we denote by $N_G(X)$ the normalizer of X in G. For a permutation group G on Ω, let $G_{i,j,\cdots,k}$ denote the stabilizer of the points $i, j,\cdots, k$ in G. Let Δ be a subset of Ω. We denote by $G_{(\Delta)}$ the setwise stabilizer of Δ. For a set X of permutations the totality of the points left fixed by X is denoted by $I(X)$. If a subset Δ of Ω is a fixed block of X, i.e. if $\Delta^X=\Delta$, the restriction of X on Δ is a set of permutations on Δ. We denote it by X^Δ.

2. Preliminary results

We list here the results which are needed for the proof of our theorem.

Proposition 2. *Let G be t-fold transitive on Ω, and let $\Gamma\subseteqq\Omega$ with $|\Gamma|=t$. Let K be a normal subgroup of G and let P be a Sylow p-subgroup of K_Γ for some prime p. Then $N_G(P)$ is t-fold transitive on $I(P)$.*

Proposition 3 [2]. *Let G be a t-fold transitive permutation group on a set Ω for $t\geqq 4$ and let $H\neq 1$ be a normal subgroup of G. Then for all $\Delta\subseteqq\Omega$ with $|\Delta|=t$, $H_{(\Delta)}{}^\Delta=S_t$.*

Proposition 4 [6]. *Let H be a t-fold transitive permutation group on a set Ω $(t\geqq 2)$ such that $H_{(\Gamma)}{}^{\Gamma}=S_{t+1}$ for all $\Gamma\subseteqq\Omega$ with $|\Gamma|=t+1$. Then H_{Δ} and $H_{(\Delta)}$ have the same orbits on $\Omega-\Delta$ for all $\Delta\subseteqq\Omega$ with $|\Delta|=t$.*

Proposition 5 [5]. *Let G be a triply transitive permutation group of odd degree n such that*
(1) *G is a normal subgroup of a quadruply transitive group, and*
(2) *any involution in G fixes at most three points. Then n is 5, 7, or 11, and G is A_5, S_5, A_7 or M_{11}.*

Proposition 6 [1]. *Let p be an odd prime. Let G be a $2p$-fold transitive permutation group such that $G_{1,2,\cdots,2p}$ is of order prime to p. Then G is one of S_n $(2p\leqq n\leqq 3p-1)$ and A_n $(2p+2\leqq n<3p-1)$.*

3. Proof of Theorem

Let (G, H) be a counter example of the smallest degree n to our theorem. Then under the assumption in Theorem $s\nmid r$ or $2\leqq r\leqq s$, in particular $s>1$. Since $G_{1,2,\cdots,t-1}$ is transitive on $\Omega-\{1, 2,\cdots, t-1\}$ and $H_{1,2,\cdots,t-1}$ is a normal subgroup of $G_{1,2,\cdots,t-1}$, $|\Delta_1|=|\Delta_2|=\cdots|\Delta_s|$ and hence

$$n-(t-1)=s|\Delta_1|\equiv r\ (\mathrm{mod}\ q)\,.$$

Let $t\in\Delta_1$ and let S be a Sylow q-subgroup of $H_{1,2,\cdots,t}$. Then, since $|\Delta_1|=|H_{1,2,\cdots,t-1}:H_{1,2,\cdots,t}|$ is prime to q, S is a Sylow q-subgroup of $H_{1,2,\cdots,t-1}$. Now $H_{1,2,\cdots,t-1}$ is a normal subgroup of $G_{1,2,\cdots,t-1}$, and S is a Sylow q-subgroup, $G=N_G(S)H_{1,2,\cdots,t-1}$. Thus we have that $N_G(S)\subsetneqq H$. Also $N_G(S)\cap H\neq 1$ because $|H|=n(n-1)\cdots(n-t+2)|H_{1,2,\cdots,t-1}|$.

Next we shall show that the number of orbits of $(N_H(S))_{1,2,\cdots,t-1}$ on $I(S)-\{1,2,\cdots,t-1\}$ is s. Since $(|\Delta_i|, q)=1$, $\Delta_i\cap I(S)\neq\phi$ (i.e. there are at least s orbits). $I(S)\cap\Delta_i$ is an orbit for all i. For let α, $\beta\in I(S)\cap\Delta_i$. Since Δ_i is an orbit of $H_{1,2,\cdots,t-1}$ on $\Omega-\{1,2,\cdots,t-1\}$, there exists an element h in $H_{1,2,\cdots,t-1}$ such that $\alpha^h=\beta$. Both S^h and S are Sylow q-subgroups of $H_{1,2,\cdots,t-1,\beta}$. Thus there exists an element l in $H_{1,2,\cdots,t-1,\beta}$, such that $S^h=S^l$. We have that $hl^{-1}\in N_{H_{1,2,\cdots,t-1}}(S)$ and $\alpha^{hl^{-1}}=\beta$. We are done.

Therefore, if $S\neq 1$, then by induction, we have that s divides r and $2\leqq s<r$, or $|I(S)|\leqq t+1$. If the first case holds, then this is a contradiction. If the second case holds, then $|I(S)|=t+1$ because $(|\Delta_i|, q)=1$. So $N_H(S)^{I(S)}\geqq A^{I(S)}$, where $I(S)=\{1,2,\cdots, t, t'\}$, and $A^{I(S)}$ is an alternating group of degree $t+1$ on $I(S)$. There exists an element x in $H_{\{1,2,\cdots,t-1\}}$ such that $x=\cdots(t\ t')\cdots$: the existence of such an element is given by our knowledge of $N_H(S)^{I(S)}$. By Proposition 3 and Proposition 4 we obtain that $t'\in\Delta_1$. This is a contradiction. Therefore $S=1$.

From now on we shall divide the proof of Theorem into the following two cases:

Case 1: $t-1$ is not a prime number.

Case 2: t–1 is a prime number.

Case 1: Suppose that t–1 is not a prime number. (That is, $t-1=kq$, where q is a prime and $k\geqq 2$). In this case H is kq-fold transitive on Ω and $H_{1,2,\cdots,kq}$ is of order prime to q. Therefore using Proposition 6 H is one of $S_n(kq\leqq n\leqq kq+q-1)$ and $A_n(kq+2\leqq n<kq+q-1)$. Since $n>t+1$, H is a t-fold transitive permutation group on Ω. Thus $s=1$, which is a contradiction.

Case 2: Suppose that t–1 is q, a prime number. Let $t\in\Delta_1$ and let T be a Sylow 2-subgroup of $H_{1,2,\cdots,t}$. By Proposition 2 $N_G(T)^{I(T)}$ is t-fold transitive on $I(T)$, and By Proposition 1 $N_H(T)^{I(T)}$ is $(t-1)$-fold transitive on $I(T)$. Hence Proposition 5 implies that $N_H(T)^{I(T)}=A_{t+1}$, S_{t+1} or A_{t+3} when $t\geqq 6$, and $N_H(T)^{I(T)}=A_5$, S_5, A_7 or M_{11} when $t=4$. Let $\varepsilon\in I(T)$ and $\varepsilon\notin\{1,2,\cdots,t\}$. If $|I(T)|=t+1$ then $\varepsilon\in\Delta_1$ since $|\Delta_i|\equiv 0$ (mod 2) and T is a 2-group. Also in the other cases $\varepsilon\in\Delta_1$, since then $N_H(T)^{I(T)}$ is t-told transitive. Hence $I(T)\subseteqq\Delta_1\cup\{1,2,\cdots,t\}$.

Let x be a q-element of $N_H(T)$ involving the q-cycle $(1,2,\cdots,q)$ and fixing at least 2 points of $I(T)$; the existence of such a q-element follows from our knowledge of $N_H(T)^{I(T)}$. Then $x\in H\{_{1,2,\cdots,t-1}\}$, and by Proposition 3 and Proposition 4 x preserves the $H_{1,2,\cdots,t-1}$-orbits. Hence if $|\Delta_1|\equiv 1$ (mod q) then a Sylow q-subgroup of $H_{1,2,\cdots,t-1}\neq 1$. This is a contradiction. Thus $|\Delta_1|\equiv l$ (mod q), $1<l<q$. Therefore $n=sl$ (mod q). This is also a contradiction. For since $q=t-1>|I(x)|\geqq sl$, $sl=r$.

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